MA 421 Fall 2024 (Aaron N. K. Yip)
Homework 7, due on Thursday, Nov 13th, 11:59pm, in Gradescope
(Abstract of your project is due on Friday, Nov 14th, 11:59pm, in Brightspace)

1. [V] p.207, Exercises: 12.1 (L^2-), 12.2 (L^1-), *and* L^∞-regression line of the data points, i.e., given the data points $\{(x_i, y_i) : i = 1, 2 \ldots\}$, find m and c so as to *minimize* the following L^2-, L^1- and L^∞-errors:

$$\sum_i (y_i - mx_i - c)^2, \quad \sum_i |y_i - mx_i - c|, \quad \max_i |y_i - mx_i - c|.$$

   Use any of your favorite computer program/software to plot the points and the regression lines you have found, *all in one plot*. Compare and contrast your results.

2. Consider the following labelled points (all lying on the x-axis):

$$\mathrm{P} = \{1, 2, 4, 5\}, \quad \mathrm{N} = \{-4, -3, -2\}.$$

   Rigorously find the binary classification point, i.e. *with proof*, find x_* that solve the following problem:

$$\max_x \left\{ \min_{i:x_i \in \mathrm{P}} \{x_i - x\}, \min_{i:x_i \in \mathrm{N}} \{x - x_i\} \right\}$$

   Do it again but with the following new set of "corrupted" points:

$$\mathrm{P} = \{-2.5, -1, 0, 1, 2, 4, 5\}, \quad \mathrm{N} = \{-4, -3, -2, 0.5, 1.5\}.$$

3. This question is for you to implement Caratheodory Theorem [V] Theorem 10.3. Consider the following convex combination of points:

$$\begin{pmatrix} 1.2 \\ 0.5 \\ 0.3 \end{pmatrix} = 0.1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 0.2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 0.2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 0.2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + 0.1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + 0.1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + 0.1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

   Use LP to express the left-hand-side as a convex combination of $4 (= 3 + 1)$ points out of the 7 points from the right-hand-side. (Well, for such a small size problem, you can certain try brute force search: you will have ${}_7C_4 = 35$ so many choices. The "advantage" of the brute force search is that you can indeed find all the possible solutions.)

   Note. The proof of the Caratheodory Theorem is "constructive" in the sense that it gives an algorithm to find the solution. With this said, the initial dictionary for this problem is not feasible. Probably the easiest way is to introduce three *auxiliary* variables (not slack variables) $w_1, w_2, w_3 \geq 0$ and use simplex to maximize $-w_1 - w_2 - w_3$. The end of this phase automatically gives you a solution. There is no Phase II for this problem because all you need is simply a feasible dictionary.

4. (a) Consider the following two polyhedrons:

$$\begin{aligned} P_1 &= \{x_1 \geq 0;\ \ x_2 \geq \frac{x_1}{2} + 5;\ \ x_1 + x_2 \geq 10\}; \\ P_2 &= \{x_2 \geq 0;\ \ x_2 \leq x_1;\ \ x_2 \leq -3x_1 + 30\}. \end{aligned}$$

Use Farkas' Lemma to find two disjoint half spaces H_1 and H_2 such that $P_1 \subseteq H_1$ and $P_2 \subseteq H_2$. After you have found your answer, use any of your favorite computer program/software to plot P_1, P_2, H_1 and H_2 in a single x_1x_2-plane.

(b) Consider the following two polyhedrons:

$$P_1 = \left\{ \begin{aligned} 2x_1 + 3x_2 + x_3 &\leq 5; \\ 3x_1 + 4x_2 + 2x_3 &\leq 8; \\ x_1,\ \ x_2,\ \ x_3 &\geq 0 \end{aligned} \right. \quad \text{and} \quad P_2 = \left\{ \begin{aligned} 5x_1 + 4x_2 + 3x_3 &\geq 14; \\ 4x_1 + x_2 + 2x_3 &\leq 11; \\ x_1,\ \ x_2,\ \ x_3 &\geq 0 \end{aligned} \right.$$

Use Farkas' Lemma to find two disjoint half spaces H_1 and H_2 such that $P_1 \subseteq H_1$ and $P_2 \subseteq H_2$.

5. Find the biggest circle that can fit inside the following set:

$$x_2 \leq \frac{x_1}{2} + 2,\ \ x_1 + x_2 \leq 5,\ \ x_1 \geq 0,\ \ x_2 \geq 0.$$

use any of your favorite computer program/software to draw the above set and also the circle you have found.

(Note: the above problem can be formulated as an LP – see [MG] p.23, Section 2.6.)