Purdue University MA 520
Fourier Analysis and Boundary Value Problems
Spring 2003, Final Examination
(Instructor: Aaron N. K. Yip)

Name: ______________________ (Department/Company: ____________)

- This test booklet has EIGHT QUESTIONS totaling 100 points for the whole test. You have 120 minutes to do this test. **Plan your time well. Read the questions carefully. You do not need to attempt the questions in sequence.**
- This test is **open note** but **closed book**. No photocopy of any book pages or chapter. All the note should be prepared by yourself. **No calculator** is allowed.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- You can use both sides of the papers to write your answers. But please indicate so if you do.

Answer Key

Question	Score
1.(10 pts)	
2.(10 pts)	
3.(10 pts)	
4.(10 pts)	
5.(10 pts)	
6.(10 pts)	
7.(20 pts)	
8.(20 pts)	
Total (100 pts)	

1. Consider the following 1D wave equation:

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx}, \quad x \in (-\infty, \infty) \\
u(x,0) &= f(x) \\
u_t(x,0) &= g(x)
\end{aligned}
$$

Prove that the following is a solution of the above equation:

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

i.e. verify that the given $u(x,t)$ satisfies the wave equation and the initial conditions.

$$u_t = \frac{1}{2}\left[-c f'(x-ct) + c f'(x+ct)\right]$$

$$+ \frac{1}{2c}\left[g(x+ct)c - g(x-ct)(-c)\right]$$

$$\to u_{tt} = \frac{1}{2}\left[c^2 f''(x-ct) + c^2 f''(x+ct)\right]$$

$$+ \frac{1}{2c}\left[c^2 g'(x+ct) - c^2 g'(x-ct)\right]$$

c^2

$$u_x = \frac{1}{2}\left[f'(x-ct) + f'(x+ct)\right] + \frac{1}{2c}\left[g(x+ct) - g(x-ct)\right]$$

$$\to u_{xx} = \frac{1}{2}\left[f''(x-ct) + f''(x+ct)\right] + \frac{1}{2c}\left[g'(x+ct) - g'(x-ct)\right]$$

Hence $\boxed{u_{tt} = c^2 u_{xx}}$

$$u(x,0) = \frac{1}{2}\left[f(x-0) + f(x+0)\right] + \cancelto{0}{\frac{1}{2c}\int_x^x g(y)\,dy}$$

$$= f(x)$$

This is a scrap paper.

$$u_t(x,0) = \frac{1}{2}\left[-c f'(x-0) + c f'(x+0)\right]$$
$$+ \frac{1}{2c}\left[g(x+0)c + g(x-0)\right]$$
$$= g(x).$$

So $u(x,t)$ is a solution to the wave eq.

2. Consider the following 1D heat equation:

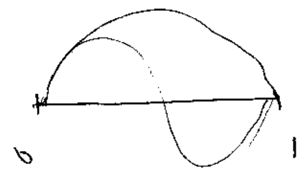
$$
\begin{aligned}
u_t &= u_{xx} + x, \quad x \in (0,1) \\
u(0,t) = 0, &\qquad u(1,t) = 0, \\
u(x,0) &= 0.
\end{aligned}
$$

Find $u(x,t)$ and $\lim_{t\to+\infty} u(x,t)$.

$u_t = u_{xx} + x$, Dir B.C

Eigenvalues & eigenfcts are:

$\lambda_n = -(n\pi)^2$, $\quad \varphi_n(x) = \sin(n\pi x)$



Express $x = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$

$$b_n = \frac{\int_0^1 x \sin n\pi x \, dx}{\int_0^1 \sin^2 n\pi x \, dx}$$

$$= \frac{-\frac{x \cos n\pi x}{n\pi}\Big|_0^1 + \cancel{\int_0^1 \cos n\pi x \, dx}}{\frac{1}{2}\int_0^1 (1 - \cancel{\cos 2n\pi x})\, dx}$$

$$= 2\left[-\frac{\cos n\pi}{n\pi}\right] = \frac{2}{n\pi}(-1)^{n+1}$$

So $\boxed{x = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} \sin n\pi x}$

This is a scrap paper.

Let $u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin n\pi x$

$$\Rightarrow \boxed{\dot{a}_n(t) = -(n\pi)^2 a_n(t) + b_n}$$

So $a_n(t) = e^{-(n\pi)^2 t} a_n(0) + e^{-(n\pi)^2 t} \int_0^t e^{(n\pi)^2 s} b_n \, ds$

$\boxed{\text{By initial condition, } a_n(0) = 0 \text{ for all } n}$

So $a_n(t) = e^{-(n\pi)^2 t} \int_0^t e^{(n\pi)^2 s} b_n \, ds$

$$= b_n e^{-(n\pi)^2 t} \frac{e^{(n\pi)^2 t} - 1}{(n\pi)^2}$$

$$= b_n \left[\frac{1 - e^{-(n\pi)^2 t}}{(n\pi)^2} \right]$$

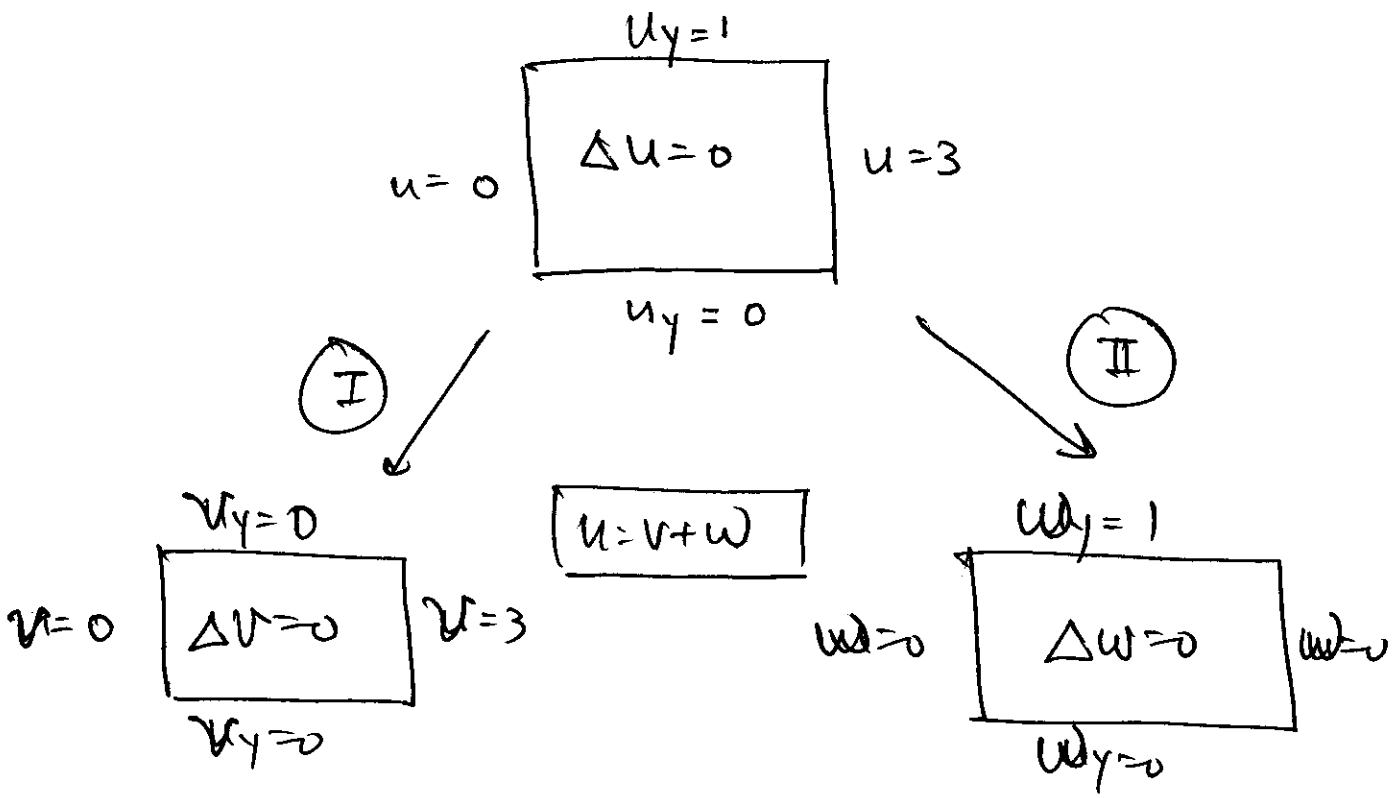
So $\boxed{u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2(-1)^{n+1}}{(n\pi)^3} \right) \left(1 - e^{-(n\pi)^2 t} \right) \sin(n\pi x)}$

As $t \to \infty$, $u(x,t) \longrightarrow \sum_{n=1}^{\infty} \left(\frac{2(-1)^{n+1}}{(n\pi)^3} \right) \sin(n\pi x)$

3. Given the domain $\Omega = \{0 \le x \le 2,\ 0 \le y \le 1\}$. Consider the following boundary value problem:

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega \\ u(0,y) = 0 \quad \text{and} \quad & u(2,y) = 3 \text{ for } y \in (0,1) \\ u_y(x,0) = 0 \quad \text{and} \quad & u_y(x,1) = 1 \text{ for } x \in (0,2) \end{aligned}$$

Is the above equation solvable? If so, solve it. If not, explain why.



I: $v(x,y) = X(x)Y(y)$

· Neuman B.C. for $Y(y)$

$$v(x,y) = \boxed{(A_0 + B_0 x)}^{\,!} + \sum_{n=1}^{\infty} \left(A_n e^{+n\pi x} + B_n e^{-n\pi x}\right) \cos(n\pi y).$$

$\boxed{X=0}$ $\quad 0 = A_0 + \sum_{n=1}^{\infty} (A_n + B_n) \cos n\pi y$

$\boxed{x=2}$ $\quad 3 = (A_0 + 2B_0) + \sum_{n=1}^{\infty} \left(A_n e^{n\pi 2} + B_n e^{-n\pi 2}\right) \cos n\pi y$

This is a scrap paper.

$$\Longrightarrow \quad A_0 = 0, \qquad A_n + B_n = 0$$

$$\Longrightarrow \quad A_0 + 2B_0 = 3, \qquad A_n e^{n\pi 2} + B_n e^{-n\pi/2} = 0$$

$$\Longrightarrow \quad B_0 = 3/2, \quad A_n = B_n = 0, \quad A_0 = 0$$

So $\boxed{v(x,y) = \frac{3}{2}x}$ (It works ! for I)

II $\quad w(x,y) = X(x)Y(x)$

Dir for $X(x)$

$$w(x,y) = \sum_{n=1}^{\infty} \left(A_n e^{+\frac{n\pi}{2}y} + B_n e^{-\frac{n\pi}{2}y} \right) \sin\left(\frac{n\pi x}{2}\right)$$

$$w_y = \sum_{n=1}^{\infty} \left(\frac{n\pi}{2}\right) \left[A_n e^{\frac{n\pi y}{2}} - B_n e^{-\frac{n\pi}{2}y} \right] \sin\left(\frac{n\pi x}{2}\right)$$

$$y=0 \Longrightarrow \quad 0 = \sum_{n=1}^{\infty} \left(\frac{n\pi}{2}\right) \left[A_n - B_n \right] \sin\left(\frac{n\pi x}{2}\right)$$

$$y=1 \Rightarrow \quad 1 = \sum_{n=1}^{\infty} \left(\frac{n\pi}{2}\right) \left[A_n e^{\frac{n\pi}{2}} - B_n e^{-\frac{n\pi}{2}} \right] \sin\left(\frac{n\pi x}{2}\right).$$

This is a scrap paper.

So $A_n = B_n \Longrightarrow$

$$1 = \sum_{n=1}^{\infty} \underbrace{\left(\frac{n\pi}{2}\right)\left[e^{\frac{n\pi}{2}} - e^{-\frac{n\pi}{2}}\right] A_n}_{C_n} \sin\left(\frac{n\pi x}{2}\right)$$

$$C_n = \frac{\int_0^2 \sin\left(\frac{n\pi x}{2}\right) dx}{\int_0^2 \sin^2\left(\frac{n\pi x}{2}\right) dx} = \frac{-\frac{2}{n\pi}\left(\cos\frac{n\pi x}{2}\right)_0^2}{\frac{1}{2}\int_0^2 1 - \cos(n\pi x)\, dx}$$

$$= -\frac{4}{n\pi}\left[\cos n\pi - 1\right]$$

$$\boxed{u(x,y) = v + w}$$

$$= \left(-\frac{4}{n\pi}\right)\left[(-1)^n - 1\right]$$

$$A_n = \frac{C_n}{\left(\frac{n\pi}{2}\right)\left[e^{n\pi/2} - e^{-n\pi/2}\right]} = -\frac{8\left[(-1)^n - 1\right]}{(n\pi)^2\left(e^{n\pi/2} - e^{-n\pi/2}\right)}$$

$$\boxed{w(x,y) = \sum_{n=1}^{\infty} \frac{8\left(1 + (-1)^{n+1}\right)}{(n\pi)^2\left(e^{n\pi/2} - e^{-n\pi/2}\right)}\left(e^{\frac{n\pi y}{2}} + e^{-\frac{n\pi y}{2}}\right)\sin\left(\frac{n\pi x}{2}\right)}$$

4. Given the domain (described in polar coordinates): $\Omega = \{0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$. Solve the Laplace equation $\Delta u = 0$ in Ω with the following boundary conditions:

$$2u + \frac{\partial u}{\partial n}\bigg|_{r=1} = \sin\theta$$

where $\frac{\partial}{\partial n}$ refers to the outward unit normal.

General solution: $(0 \le r \le 1,\ 0 \le \theta \le 2\pi)$.

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n r^n \cos n\theta + B_n r^n \sin n\theta.$$

$$u(1,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\theta + B_n \sin n\theta$$

$$u_r(1,\theta) = \sum_{n=1}^{\infty} n A_n \cos n\theta + n B_n \sin n\theta$$

So $2u + u_r\big|_{r=1} = \sin\theta$

$$\Rightarrow \quad A_0 + \sum_{n=1}^{\infty} (2A_n + nA_n)\cos n\theta + (2B_n + nB_n)\sin n\theta = \sin\theta$$

(Compare coefficients)

So $B_1 = \frac{1}{3}$, all other coeff. are zero!

$$\boxed{u(r,\theta) = \frac{1}{3} r \sin\theta}$$

5. Let f and g be two arbitrary functions. Recall the following definition:

$$f * g(x) = \int_{-\infty}^{+\infty} f(y) g(x-y)\, dy$$

(a) Let $g_t(x) = \frac{1}{\sqrt{4D\pi t}} e^{-\frac{x^2}{4Dt}}$. Prove that for any $t, s > 0$, $g_t * g_s(x) = g_{t+s}(x)$.

(b) Given that $p_t(x) = \frac{t}{\pi(x^2+t^2)}$. Prove that for any $t, s > 0$, $p_t * p_s(x) = p_{t+s}(x)$.

(Note: The subscript t is just a notation to specify the dependence of the function on t. It does *not* mean $\frac{d}{dt}$.)

(a) Recall:

$$e^{-ax^2/2} \overset{\mathcal{F}}{\longleftrightarrow} \sqrt{\frac{2\pi}{a}}\, e^{-\xi^2/2a}$$

$$a = \frac{1}{2Dt} \Rightarrow e^{-\frac{x^2}{4Dt}} \overset{\mathcal{F}}{\longleftrightarrow} \sqrt{4D\pi t}\, e^{-\xi^2 Dt}$$

i.e.

$$\frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4D\pi t}} \overset{\mathcal{F}}{\longleftrightarrow} e^{-Dt\xi^2}$$

So

$$g_t(x) = \frac{e^{-x^2/4Dt}}{\sqrt{4D\pi t}} \overset{\mathcal{F}}{\longleftrightarrow} e^{-Dt\xi^2}$$

$$g_s(x) = \frac{e^{-x^2/4Ds}}{\sqrt{4D\pi s}} \overset{\mathcal{F}}{\longleftrightarrow} e^{-Ds\xi^2}$$

$$(g_t * g_s)(x) \overset{\mathcal{F}}{\longleftrightarrow} e^{-Dt\xi^2} \cdot e^{-Ds\xi^2}$$

$$\|$$

$$g_{t+s}(x) = \frac{e^{-x^2/4D(t+s)}}{\sqrt{4D\pi(t+s)}} \xleftarrow{\mathcal{F}^{-1}} e^{-D(t+s)\xi^2}$$

This is a scrap paper.

(b) $$\frac{1}{x^2+a^2} \xleftrightarrow{\mathcal{F}} \frac{\pi}{a} e^{-a|\xi|}$$

i.e. $$\frac{a}{\pi(x^2+a^2)} \xleftrightarrow{\mathcal{F}} e^{-a|\xi|}$$

So $$p_t(x) = \frac{t}{\pi(x^2+t^2)} \xleftrightarrow{\mathcal{F}} e^{-t|\xi|}$$

$$p_s(x) = \frac{s}{\pi(x^2+s^2)} \xleftrightarrow{\mathcal{F}} e^{-s|\xi|}$$

So $$(p_t * p_s)(x) \xleftrightarrow{\mathcal{F}} e^{-t|\xi|} \cdot e^{-s|\xi|}$$

$$\shortparallel$$

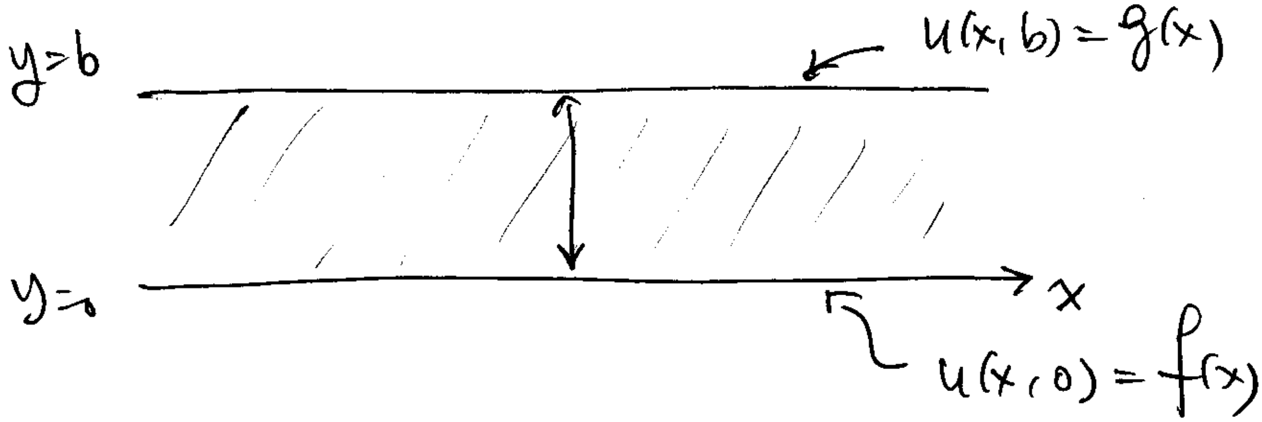
$$p_{t+s}(x) \xleftarrow{\mathcal{F}^{-1}} e^{-(t+s)|\xi|}$$

So $$(p_t * p_s)(x) = p_{t+s}(x) = \frac{t+s}{\pi(x^2+(t+s)^2)}$$

6. Consider the infinite strip domain $\Omega = \{-\infty < x < \infty, 0 \le y \le b\}$. Let $u(x,y)$ satisfy the Laplace equation $\Delta u = 0$ in Ω with the boundary condition:

$$u(x,0) = f(x) \text{ and } u(x,b) = g(x).$$

Find explicitly the Fourier transform $\hat{u}(\xi, y)$ of $u(x,y)$ with respect to the x-variable.



$\mathcal{F}$ in x

$$u_{xx}(x,y) + u_{yy}(x,y) = 0$$

$$\mathcal{F}(u_{xx}) + \mathcal{F}(u_{yy}) = 0$$

$$(i\xi)^2 \mathcal{F}(u) + (\mathcal{F}(u))_{yy} = 0$$

$$-\xi^2 \hat{u}(\xi,y) + \hat{u}(\xi,y)_{yy} = 0$$

Or $\quad \hat{u}(\xi,y)_{yy} - \xi^2 \hat{u}(\xi,y) = 0$

(ξ - parameter constant; y - variable)

Recall: $\quad \ddot{x} - a^2 x = 0 \Rightarrow x(t) = Ae^{at} + Be^{-at}.$

This is a scrap paper.

So $$\hat{u}(\xi, y) = A(\xi) e^{\xi y} + B(\xi) e^{-\xi y}$$

$$y = 0 \Rightarrow \hat{u}(\xi, 0) = \hat{f}(\xi) = A(\xi) + B(\xi)$$

$$y = b \Rightarrow \hat{u}(\xi, b) = \hat{g}(\xi) = A e^{b\xi} + B e^{-b\xi}$$

So $$A(\xi) = \frac{\hat{g}(\xi) - \hat{f}(\xi) e^{-b\xi}}{e^{b\xi} - e^{-b\xi}}$$

$$B(\xi) = \frac{\hat{f}(\xi) e^{b\xi} - \hat{g}(\xi)}{e^{b\xi} - e^{-b\xi}}$$

So $$\hat{u}(\xi, y) = \left(\frac{\hat{g}(\xi) - \hat{f}(\xi) e^{-b\xi}}{e^{b\xi} - e^{-b\xi}} \right) e^{\xi y}$$

$$+ \left(\frac{\hat{f}(\xi) e^{b\xi} - \hat{g}(\xi)}{e^{b\xi} - e^{-b\xi}} \right) e^{-\xi y}$$

7. You are given the following information:

$$\begin{pmatrix} -5 & 2 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = -4\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -5 & 2 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = -9\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Solve the following systems:

$$\begin{aligned} x_t(t) &= -5x(t) + 2y(t) + t, \\ y_t(t) &= 2x(t) - 8y(t) + 1 \end{aligned}$$

such that $x(0) = 4$ and $y(t) = 1$.

$$\frac{d}{dt}\vec{X} = A\vec{X} + \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

Note

A is symmetric, $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \perp \begin{pmatrix} -1 \\ 2 \end{pmatrix}$!

Consider

$$\begin{pmatrix} t \\ 1 \end{pmatrix} = b_1(t)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + b_2(t)\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

$$b_1(t) = \frac{\begin{pmatrix} t \\ 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}} = \frac{2t+1}{5}.$$

$$b_2(t) = \frac{\begin{pmatrix} t \\ 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}}{\begin{pmatrix} -1 \\ 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}} = \frac{-t+2}{5}.$$

Let $\vec{X}(t) = a_1(t)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + a_2(t)\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

So
$$\begin{cases} \dot{a}_1(t) = -4a_1(t) + b_1(t) \\ \dot{a}_2(t) = -9\,a_2(t) + b_2(t) \end{cases}$$

This is a scrap paper.

$$a_1(t) = e^{-4t} a_1(0) + \frac{e^{-4t}}{5} \int_0^t e^{4s} (2s+1)\, ds$$

$$= e^{-4t} a_1(0) + \frac{1}{5} e^{-4t} \left[\int_0^t e^{4s} ds + 2\int_0^t e^{4s} s\, ds \right]$$

$$= e^{-4t} a_1(0) + \frac{1}{5} e^{-4t} \left[\frac{e^{4s}}{4} \Big|_0^t + 2 \left(\frac{e^{4s} s}{4} \Big|_0^t - \int_0^t \frac{e^{+4s}}{4} ds \right) \right]$$

$$= e^{-4t} a_1(0) + \frac{1}{5} e^{-4t} \left[\frac{e^{4t}-1}{4} + 2 \left[\frac{e^{4t} t}{4} \right] - \frac{2}{16} e^{4s} \Big|_0^t \right]$$

$$= e^{-4t} a_1(0) + \frac{1}{5} e^{-4t} \left\{ \frac{e^{4t}-1}{4} + \frac{1}{2} e^{4t} t - \frac{1}{8} e^{4t} + \frac{1}{8} \right\}$$

$$= e^{-4t} a_1(0) + \frac{1}{5} \left\{ \frac{1-e^{-4t}}{4} + \frac{1}{2} t - \frac{1}{8} + \frac{e^{-4t}}{8} \right\}$$

$$= \boxed{e^{-4t} a_1(0) + \frac{1}{5} \left\{ -\frac{1}{8} + \frac{1}{2} t - \frac{1}{8} e^{-4t} \right\}}$$

This is a scrap paper.

$$a_2(t) = e^{-9t}a_2(0) + e^{-9t}\int_0^t e^{9s}\, b_2(s)\, ds$$

$$= e^{-9t}a_2(0) + \frac{e^{-9t}}{5}\int_0^t e^{9s}(-s+2)\, ds$$

$$\int_0^t (-e^{9s}s + \cancel{2}e^{9s})\, ds = -\frac{e^{9s}s}{9}\Big|_0^t + \int_0^t \frac{e^{9s}}{9}\, ds + \int_0^t e^{9s}\, ds$$

$$= -\frac{e^{9t}t}{9} + \frac{10}{9}\int_0^t e^{9s}\, ds$$

$$= \frac{e^{+9t}t}{9} + \frac{10}{81}\left(e^{9t}-1\right)$$

So $\boxed{a_2(t) = e^{-9t}a_2(0) + \frac{1}{5}\left\{\frac{10}{81}(1-e^{-9t}) + \frac{t}{9}\right\}}$

$$\boxed{\vec{X}(t) = a_1(t)\begin{pmatrix}2\\1\end{pmatrix} + a_2(t)\begin{pmatrix}-1\\2\end{pmatrix}}$$

Solution!

$t=0$

$$\begin{pmatrix}4\\1\end{pmatrix} = a_1(0)\begin{pmatrix}2\\1\end{pmatrix} + a_2(0)\begin{pmatrix}-1\\2\end{pmatrix}$$

$\boxed{a_1(0) = \frac{\begin{pmatrix}4\\1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}}{\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}} = \frac{9}{5},}$ $\boxed{a_2(0) = \frac{\begin{pmatrix}4\\1\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix}}{\begin{pmatrix}-1\\2\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix}} = \frac{-2}{5}}$

8. Consider the following 2π-periodic functions and their Fourier series expansions:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx)$$
$$g(x) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(nx) + D_n \sin(nx)$$

Consider the new function $h(x)$ obtained from f and g by $h(x) = \int_{-\pi}^{\pi} f(y)g(x-y)\,dy$. Let the Fourier series of $h(x)$ be:

$$h(x) = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos(nx) + F_n \sin(nx)$$

Derive an expression for E_n and F_n in terms of A_n, B_n, C_n and D_n.

The following formulas might be useful:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$E_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(x) \cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(y) g(x-y) \cos nx \, dy\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(y) g(x-y) \left[\cos(n(x-y)+ny)\right] dy\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(y) g(x-y) \left[\cos n(x-y) \cos ny - \sin n(x-y) \sin ny\right] dy\, dx$$

This is a scrap paper.

$$= \frac{1}{\pi}\int_{y=-\pi}^{\pi} f(y)\cos ny \underbrace{\left(\int_{-\pi}^{\pi} g(x-y)\cos n(x-y)\,dx\right)}_{=\pi C_n} dy$$

$$- \frac{1}{\pi}\int_{y=-\pi}^{\pi} f(y)\sin ny \underbrace{\left(\int_{-\pi}^{\pi} g(x-y)\sin n(x-y)\,dx\right)}_{=\pi D_n} dy$$

$$= \underbrace{\int_{y=-\pi}^{y=\pi} (f(y)\cos ny)}_{\pi A_n} C_n\,dy - \underbrace{\int_{-\pi}^{\pi} f(y)\sin ny}_{\pi B_n} D_n\,dy$$

$$\boxed{E_n = \pi\left[A_nC_n - B_nD_n\right]}$$

$$F_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\sin nx\,dy\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\sin\left(n(x-y)+ny\right)dy\,dx$$

This is a scrap paper.

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\left[\sin n(x-y)\cos ny + \cos n(x-y)\sin ny\right]dy\,dx$$

$$= \frac{1}{\pi}\int_y f(y)\cos ny\left(\int_x g(x-y)\sin n(x-y)\,dx\right)$$

$$+ \frac{1}{\pi}\int_y f(y)\sin ny\left(\int_x g(x-y)\cos n(x-y)\,dx\right)$$

$$\boxed{F_n = \pi\left(A_n D_n + B_n C_n\right)}$$