Local vs. Maximal Solution

Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be locally Lipschitz, i.e., $\forall x \in \mathbb{R}^n, \exists K_x, \delta_x > 0$ s.t., $\forall y, z \in B_{\delta_x}(x)$, we have $\|F(y) - F(z)\| \leq K_x \|y - z\|$.
Local vs. Maximal Solution

Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be locally Lip.

Then if $y \in \mathbb{R}^n$, $\exists (-T(y), T(y)) \subseteq \mathbb{R}$ s.t.

the equation

$$\frac{dX}{dt} = F(X), \quad X(0) = y$$

has a solution for $t \in (-T(y), T(y))$

$[M, \text{Thm 3.10}]$ (p. 86)
Local vs. Maximal Solution

Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be locally Lip.

Then if \( y \in \mathbb{R}^n \), \( \exists (-T(y), T(y)) \subseteq \mathbb{R} \) s.t. the equation

\[
\frac{dX}{dt} = F(X), \quad X(0) = y
\]

has a solution for \( t \in (-T(y), T(y)) \).

\[ [M, \text{Thm 3.10}] \ (p. 86) \]
Local vs. Maximal Solution

Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be locally Lip.

Extension of Solution in time
Local vs. Maximal Solution

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz.

Maximal Interval of Existence ([M, Thm 3.17], p. 99)

There is a maximal (time) interval $(\alpha^*, \beta^*) \subseteq \mathbb{R}$

$(-\infty \leq \alpha^* < \beta^* \leq +\infty)$

of existence for the solution of

$$\frac{dX}{dt} = F(X); \quad X(\delta) = y$$
Local vs. Maximal Solution

Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be locally Lip.

Maximal Interval of Existence ([M, Thm 3.17] p. 99)

$(\alpha^*, \beta^*)$ is maximal in the sense that

If $Z(t)$ satisfies $\frac{dZ}{dt} = F(Z)$, $Z(0) = y$

for $t \in (\alpha, \beta)$,

then $(\alpha, \beta) \subseteq (\alpha^*, \beta^*)$ i.e. $\alpha \leq \alpha < \beta \leq \beta^*$
Let $F: E = \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz in $E$. 

$F$ is smooth by assumption.

[ML, Thm 3.18, Cor 3.19, p. 100]
Let $F: E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz in $E$.

\[ \dot{x} = f(x), \quad x(0) = y \]
Behavior of $X(t)$ as $t \to \beta^*$ (or $\alpha^*$)

Let $F: E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be locally lip in $E$

If $\beta^* < \infty$, then as $t \to \beta^*$,

$X(t)$ must approach the boundary of $E$ ($\partial E$)

(or $|X(t)| \to \infty$ if $E = \mathbb{R}^n$)
Behavior of $X(t)$ as $t \to \beta^*$ (or $\alpha^*$)

Let $F : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be locally lipschitz in $E$

If $\beta^* < \infty$, then as $t \to \beta^*$,

(If not, ...
Behavior of $X(t)$ as $t \to \beta^*$ (or $\alpha^*$)

Let $F: E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz in $E$.

If $\beta^* < \infty$, then as $t \to \beta^*$,

(If not,

$\exists t_i \to \beta^*$

s.t.

$X(t_i) \to X^* \cap E$)
Behavior of $X(t)$ as $t \to \beta^*$ (or $\alpha^*$)

Let $F : E \leq \mathbb{R}^n \to \mathbb{R}^n$ be locally lipschitz in $E$

If $\beta^* < \infty$, then as $t \to \beta^*$

(If not,)

By constructing the solution starting at $x_0$, can increase $\beta^*$.
Behavior of $X(t)$ as $t \to \beta^*$ ($\alpha < \beta$)

Let $F: E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz in $E$ and $0 \to \beta^*$, $X(t)$ converges to $x$.

If $\beta^* < \infty$, then as $X(t)$.
Behavior of $X(t) \rightarrow \beta^*$ (or $\alpha^*$)

Let $F: E \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ be locally Lipschitz in $E$.

If $\beta^* < \infty$, it is possible that $\alpha \rightarrow \beta^*$, $X(t)$ does not have a limit, but still $X(t)$ must approach $\partial E$. 
Behavior of \( X(t) \) as \( t \to t^* \) (or \( x^* \))

Let \( F : E \subseteq \mathbb{R}^n \to \mathbb{R}^n \) be locally Lipschitz in \( E \).

If \( \beta^* < \infty \), it is possible that \( x(t) \to x(t^*) \) does not have a limit, but still \( X(t) \) must approach \( \partial E \).
Behavior of $X(t)$ as $t \to \beta^*$ (or $\alpha^*$)

Let $F: E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be locally Lip in $E$

If $\beta^* = \infty$, as $t \to \beta^*$, $X(t)$ can approach $\partial E$: 
Behavior of $X(t)$ as $t \to \beta^*$ (or $\alpha^*$)

Let $F: E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz in $E$.

If $\beta^* = \infty$, as $t \to \beta^*$,
or $X(t)$ can remain inside $E$. 

\[ E \subseteq \mathbb{R}^n \]
\[ \partial E \]
\[ 2E \]
Behavior of $X(t)$ as $t \to \beta^*$ (or $\alpha^*$)

Let $F : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz in $E$.

If $\beta^* = \infty$, as $t \to \beta^*$, or $X(t)$ can remain inside $E$.

(Periodic orbit)
How to Ensure $\beta^*$ (and $x^*$) = $\infty$?

\[ \frac{dx}{dt} = F(x); \quad x(0) = x_0. \]

If $F$ is (globally) bounded, then $x(t)$ is bounded as $t \to \infty$.

If $F(x)$ has (at most) linear growth, then $x(t)$ grows linearly.

Do a change of time scale $T = \frac{t}{t_0}$.

[Thm 4.3]

[Thm 4.5]

[Thm 4.4]
How to Ensure $\beta^* (\text{and } \alpha^*) = \infty$?

\[ \frac{dX}{dt} = F(X); \quad X(0) = X_0 \]

[Thm 4.4]

Do a change of time scale

\[ \frac{dX}{dt} = F(X) \]

\[ \frac{dY}{d\tau} = \frac{F(Y)}{1 + ||F(Y)||} \]

\[ \tau = \tau(t) \]