Regularity (Smoothness) Property of Functions

\[ f : \mathbb{R}^n \rightarrow \mathbb{R}^n \]

\( f \) is continuous at \( x \), if \( y \rightarrow x \), then \( f(y) \rightarrow f(x) \)

Mathematically,

\[ \forall \varepsilon > 0, \exists \delta(x) > 0, \text{ such that } \]

if \( |y - x| < \delta(x) \), then \( |f(y) - f(x)| \leq \varepsilon \)
Regularity (Smoothness) Property of Functions

\[ f : \mathbb{R}^n \rightarrow \mathbb{R}^n \]

\( f \) is uniformly continuous

If the rate of convergence of \( f(y) \rightarrow f(x) \)

does not depend on \( x \).

Mathematically,

\[ \forall \varepsilon > 0, \exists \delta > 0, \text{ such that } \]

if \( \|y-x\| < \delta \), then \( |f(y) - f(x)| \leq \varepsilon \)
Regularity (Smoothness) Property of Functions

\[ f : \mathbb{R}^n \rightarrow \mathbb{R}^m \]

\( f \) is \underline{locally} Lipschitz at \( x \),
if there is a \( K_x \) and \( \delta_x \) s.t.

\[
\text{if } |y-x| \leq \delta_x, \text{ then } |f(y) - f(x)| \leq K_x |x-y|
\]

\( f \) is \underline{globally} Lipschitz, if there is \( K > 0 \)

s.t.

\[
|f(y) - f(x)| \leq K |y-x|
\]
Regularity (Smoothness) Property of Functions

\[ f : \mathbb{R}^n \to \mathbb{R}^m \]

Slope or rate of change of \( f \) is bounded (by a constant)

\[ \| f(x) - f(y) \| \leq K \frac{\| x - y \|}{\| \nabla f \|} \]
Regularity (Smoothness) Property of Functions

\[ f : \mathbb{R}^n \rightarrow \mathbb{R}^m \]

**Note:** Lipschitz with constant \( K \) 

Then for all \( \| y - x \| \leq \frac{\varepsilon}{K} \),

\[ \| f(y) - f(x) \| \leq K \| y - x \| \leq \frac{\varepsilon}{K} \]

Given, \( 0 < \varepsilon \) and \( \varepsilon < 3A \)
Regularity (Smoothness) Property of Functions

\[ f : \mathbb{R}^n \rightarrow \mathbb{R}^m \]

\( f \) is differentiable at \( x \), if there is a matrix \( A^{n \times m} \) such that

\[
\lim_{y \to x} \frac{\| f(y) - f(x) - A(y-x) \|}{\| y - x \|} = 0
\]
Regularity (Smoothness) Property of Functions

\[ f: \mathbb{R}^n \rightarrow \mathbb{R}^m \]

\( f \) is differentiable at \( x \), if there is a matrix \( A^{n \times m} \) such that
Regularity (Smoothness) Property of Functions

\[ f : \mathbb{R}^n \to \mathbb{R}^m \]

If \( f \) is differentiable at \( x \), if there is a matrix \( A_{nxm} \) such that

\[ f(y) - f(x) = A_{nxm} (y - x) + o(||y - x||) \]

as \( y \to x \), then

\[ ||x - h|| \leq \delta \Rightarrow ||y - h|| \leq \delta \Rightarrow o(||y - x||) \to 0 \quad \text{as} \quad y \to x \]
Regularity (Smoothness) Property of Functions

\[ f : \mathbb{R}^n \rightarrow \mathbb{R}^m \]

\[ f \text{ is differentiable at } x, \text{ if there is a matrix } A \text{ such that} \]

\[ \lim_{h \to 0} \frac{\| f(x+h) - f(x) - Ah \|}{\| h \|} = 0 \]
Regularity (Smoothness) Property of Functions

\[ f : \mathbb{R}^n \rightarrow \mathbb{R}^m \]

\( f \) is differentiable at \( x \), if there is a matrix \( A^{n \times m} \) such that

\[ \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. if } \| h \| \leq \delta, \text{ then } \| f(x+h) - f(x) - Ah \| \leq \varepsilon \| h \| \]
Regularity (Smoothness) Property of Functions

\[ f: \mathbb{R}^n \rightarrow \mathbb{R}^m \]

\( f \) is differentiable at \( x \), if there is a matrix \( A^{n \times m} \) such that

\[ f(y) = f(x) + A(y-x) + o(||y-x||) \]

\[ \lim_{a \to 0} \frac{o(a)}{a} = 0 \quad \text{or} \quad o(a) \ll a \]
Regularity (Smoothness) Property of Functions

\[ f : \mathbb{R}^n \rightarrow \mathbb{R}^m \]

\[ f \text{ is differentiable at } x, \text{ if there is a matrix } A^{m \times n} \text{ such that} \]

\[ f(x+h) = f(x) + Ah + o(||h||) \]

\[ \lim_{a \to 0} \frac{o(a)}{a} = 0 \text{ or } o(a) \ll a \]
Regularity (Smoothness) Property of Functions

\[ f : \mathbb{R}^n \rightarrow \mathbb{R}^m \]

\[ A = [Df](x) \]

\[ f(x) = (f_1(x_1, \ldots, x_n), f_2(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) \]
Regularity (Smoothness) Property of Functions

\[ f : \mathbb{R}^n \rightarrow \mathbb{R} \]

\[ A = [Df](x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \]
Regularity (Smoothness) Property of Functions

\[ f : \mathbb{R}^n \rightarrow \mathbb{R}^m \]

\[ A = [Df](\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \mathbf{x} & \cdots & \frac{\partial f_1}{\partial x_n} \mathbf{x} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} \mathbf{x} & \cdots & \frac{\partial f_m}{\partial x_n} \mathbf{x} \end{bmatrix} \]