Behavior Near An Equilibrium Point

\[ \frac{dX}{dt} = F(X) \]

\[ F(X) = X^* \]

\[ F(X) = F(X^*) + [DF(X^*)](X - X^*) + \frac{1}{2} [D^2F(X^*)(X - X^*)^2] \]

\[ O(||X - X^*||^2) \]

(Let \( Y(t) = X(t) - X^* \))

\[ \frac{dY}{dt} = AY(t) + g(Y(t)), \]

\[ ||g(Y)|| \sim O(||Y||^p) \]

\[ \ll ||Y|| \ll \]

for \( ||Y|| \ll 1 \)
Behavior Near An Equilibrium Point

\[ \frac{dx}{dt} = F(x) \]

\[ F(x) = \frac{dx}{dx} \]

\[ x^* = \begin{cases} 
0 & \text{if } (x^* - x)^2 \leq 0 \\
A & \text{for } |x^* - x| < 1 \\
1 & \text{for } |x^* - x| \geq 1 
\end{cases} \]

\[ x = \frac{AX + g(x)}{h(x)} \]

\[ \frac{dx}{dt} = \frac{+X}{Y} \]

\[ (x=0 \text{ is an equilibrium pt.}) \]
Invariant Subspaces of $A^{n \times n}$

More generally, \[ p(\lambda) = \det(A - \lambda I) \] is the characteristic polynomial.

\[ = c(\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_k)^{n_k} \]

- \[ E_i = \text{Null} \{(A - \lambda_i I)^{n_k}\} \]
  \[ = \left\{ u : (A - \lambda_i I)^{n_k} u = 0 \right\} \]
  \[ = \text{Generalized eigenspace} \]
Invariant Subspaces of $A^{n \times n}$

More generally,

- $\mathbb{R}^n = E_1 \oplus E_2 \oplus \ldots \oplus E_k$

- $E_i$ is invariant under $A$,
  i.e. $u \in E_i$ then $Au \in E_i$

- $E_i$ is invariant by $e^{At}$,
  i.e. $u \in E_i$ then $e^{At}u \in E_i$
Invariant Subspaces of $A$
Invariant Subspaces of $A^{n \times n}$

\[ R^n = E_1 \oplus E_2 \oplus \cdots \oplus E_k \]

\[ = E_s \oplus \overline{E_c} \oplus E_u \]

where

\[ E_s = \bigoplus_{\text{Re}(\lambda_i) < 0} E_{\lambda_i} \quad \text{stable subspace} \]

\[ E_u = \bigoplus_{\text{Re}(\lambda_i) > 0} E_{\lambda_i} \quad \text{unstable subspace} \]

\[ E_c = \bigoplus_{\text{Re}(\lambda_i) = 0} E_i \quad \text{center subspace} \]
Linear Stability

A is called hyperbolic if

\[ \text{Re}(\lambda_i(A)) \neq 0 \]

i.e. \( E^c = \{ 0 \} \) and

\[ \mathbb{R}^n = E^s \oplus E^u \]
Linear Stability

- \( X_0 \subseteq E_s \iff \exists K > 0 \text{ s.t. } \| e^{At}X_0 \| \leq C e^{-Kt} \| X_0 \| \text{ for all } t > 0 \)

  i.e. \( e^{At}X_0 \xrightarrow{t \to +\infty} 0 \) exponentially fast in \( t \)

- \( X_0 \subseteq E_u \iff \exists K > 0 \text{ s.t. } \| e^{At}X_0 \| \leq C e^{Kt} \| X_0 \| \text{ for all } t < 0 \)

  i.e. \( e^{At}X_0 \xrightarrow{t \to -\infty} 0 \) exponentially fast in \( t \)
Linear Stability

$\mathbb{R}^n = E_0 \oplus E \oplus E_u$
Nonlinear Stability

Lyapunov Stability (M, p.116) (of $X_*$)

For any $\varepsilon > 0$, there is $\delta > 0$ such that if $||X(0) - X_*|| < \delta$, then $||X(t) - X_*|| < \varepsilon$
Nonlinear Stability
Asymptotic Stability (M, p 118) (of $X_*$)

There is a $\epsilon > 0$ such that
if $\|X(0) - X_*\| < \epsilon$, then $\|X(t) - X_*\| \xrightarrow{t \to \infty} 0$
Nonlinear Stability [M, Thm 4.6, p.121]

Linear asymptotic stability implies (nonlinear) asymptotic stability

\[ \frac{d}{dt} X = AX + g(X), \quad \|g(X)\| \ll \|X\| \text{ for } \|X\| \ll 1 \]

Assume \( \text{Re} \lambda_i(A) < 0 \), (i.e. \( R^n = E^5 \))

Then \( X_0 = 0 \) is asymptotically stable.

(There is \( \varepsilon > 0 \), s.t. if \( \|X_0\| < \varepsilon \), then \( \|X(t)\| \xrightarrow{t \to +\infty} 0 \) (exponentially fast))
Invariant Manifolds $W^s(0), W^u(0), W^c(0)$

There are 3 manifolds $W^s(0), W^u(0), W^c(0)$ (in a neighborhood of $x_*=0$):

1. invariant under the flow:

   \[ \text{if } x \in W^s(0), \text{ then } \phi_t(x) \in W^s(0) \]

   \[ \phi_t(W^s) \subseteq W^s \]  

   (Similarly for $W^u$ and $W^c$)
Invariant Manifolds $W^{s}(0)$, $W^{u}(0)$, $W^{c}(0)$

There are 3 manifolds $W^{s}(0)$, $W^{u}(0)$, $W^{c}(0)$ (in a neighborhood of $X_*=0$):

1. $W^{s}$, $W^{u}$, $W^{c}$ pass through $X_*=0$ and tangent to $E^{s}$, $E^{u}$, $E^{c}$.

$$\dim (W^{s}) = \dim E^{s}$$

(Similarly for $W^{u}$, $W^{c}$.)
Invariant Manifolds \(W^s(0), W^u(0), W^c(0)\)

\((M, \text{Thm 5.3 p. 175, Thm 5.8 p. 180})\)

There are 3 manifolds \(W^s(0), W^u(0), W^c(0)\) (in a neighborhood of \(x_*=0\)):

1. \(\forall x \in W^s, \ \phi^t(x) \in W^s \xrightarrow{t \to +\infty} 0\)
2. \(\forall x \in W^u, \ \phi^t(x) \in W^u \xrightarrow{t \to -\infty} 0\)

(both with exponentially rates.)
Invariant Manifolds: $W^s(0), W^u(0), W^c(0)$

There are 3 manifolds $W^s(0), W^u(0), W^c(0)$ (in a neighborhood of $X^* = 0$):

4. $W^s \neq W^u$ are unique

(while $W^c$ might not be.)

5. If $F$ is $C^k$ (has continuous $k$-derivatives) then so are $W^s, W^u, W^c$. 
Invariant Manifolds \( W^s(\sigma), W^u(\phi), W^c(\chi) \)

\((M, \text{Thm 5.3 p. 75, Thm 5.8 p. 186})\)
Invariant Manifolds $W^s(0), W^u(0), W^c(0)$

$(M, \text{Thm 5.3 p. 175, Thm 5.8 p. 186)}$
Invariant Manifolds $W^s(0), W^u(0), W^c(0)$

$(M, \text{Thm 5.3 p. 175, Thm 5.8 p. 180})$
Invariant Manifolds $W^s(o), W^u(o), W^c(o)$

$(M, \text{Thm 5.3 p. 175, Thm 5.8 p. 180)}$