Existence and Uniqueness of Solutions

$$\boxed{\frac{d}{dt}X = F(X)\,; \qquad X(0) = X_0}$$

$$\boxed{X(t) = X_0 + \int_0^t F(X(s))\,ds}$$

Existence and Uniqueness of Solutions

$$\boxed{\frac{d}{dt}X = F(X)\,; \qquad X(0) = X_0}$$

$$\boxed{X(t) = X_0 + \int_0^t F(X(s))\,ds}$$

(For simplicity)

- F is bounded: $\|F(X)\| \leq M$
- F is Lipschitz: $\|F(X) - F(Y)\| \leq K\|X - Y\|$

(Both global: M & K do not depend on, X Y.)

Existence and Uniqueness of Solutions

$$\boxed{\frac{d}{dt}X = F(X) \; ; \qquad X(0) = X_0}$$

$$\boxed{X(t) = X_0 + \int_0^t F(X(s))\,ds}$$

$$\left\| \frac{F(x)-F(y)}{x-y} \right\| \leq K$$

(For simplicity)

- F is bounded: $\|F(X)\| \leq M$ $\quad \sim \|DF\| \leq K$
- F is Lipschitz: $\|F(X) - F(Y)\| \leq K \|X - Y\|$

(Both global: M & K do not depend on X, Y.)

Existence and Uniqueness of Solutions

Uniqueness: **Lipschitz** condition of F

Existence:

- Picard's Iteration
- Banach Fixed Point Theorem
- Time Stepping (Euler) Scheme
- Schauder Fixed Point Theorem

Existence and Uniqueness of Solutions

Uniqueness: Lipschitz condition of F

Existence:

- Picard's Iteration
- Banach Fixed Point Theorem
- Time Stepping (Euler) Scheme
- Schauder Fixed Point Theorem

constructive

non-constructive

Existence and Uniqueness of Solutions

Uniqueness: Lipschitz condition of F

Existence:

- Picard's Iteration
- Banach Fixed Point Theorem

require F is Lipschitz

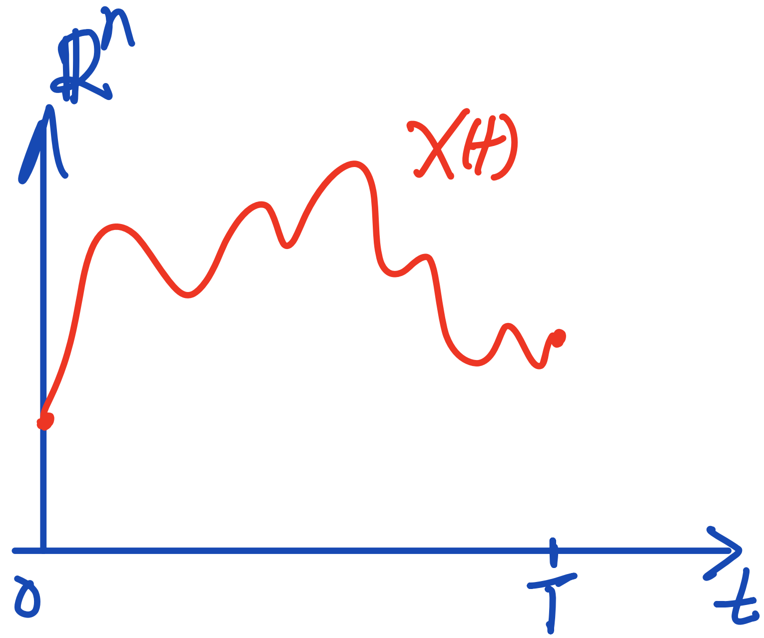
- Time Stepping (Euler) Scheme
- Schauder Fixed Point Theorem

just need continuity of F

Function Spaces

$$X : [0,T] \longrightarrow \mathbb{R}^n$$

$$(t \in [0,T] \longrightarrow X(t) \in \mathbb{R}^n)$$



$\boxed{C^0([0,T];\mathbb{R}^n) : \text{space of continuous functions}}$

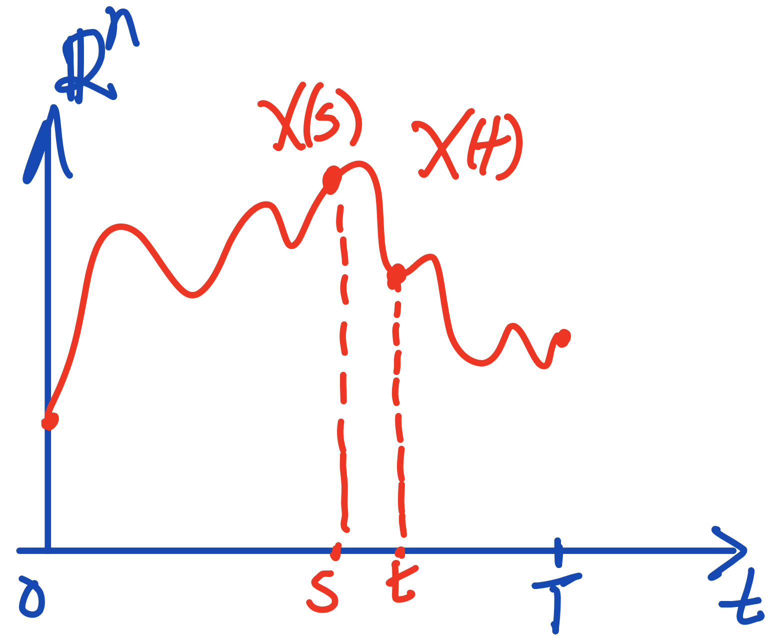
$$X \in C^0([0,T];\mathbb{R}^n) : \quad \|X\|_{C^0} = \sup_{t\in[0,T]} \|X(t)\|$$

$$X, Y \in C^0([0,T];\mathbb{R}^n) : \quad \|X-Y\|_{C^0} = \sup_{t\in[0,T]} \|X(t)-Y(t)\|$$

Function Spaces

$X: [0,T] \longrightarrow \mathbb{R}^n$

$(t \in [0,T] \longrightarrow X(t) \in \mathbb{R}^n)$



$\boxed{C^0([0,T]; \mathbb{R}^n) : \text{space of continuous functions}}$

X is continuous at t if $\forall \varepsilon > 0$, $\exists\, \delta(t) > 0$

such that $\forall s$ satisfying $|t-s| \le \delta(t)$,

then $\|X(t) - X(s)\| \le \varepsilon$

Existence of Solutions — Iterations

$$X(t) = X_0 + \int_0^t F(X(s))\,ds$$

$$\begin{cases} X^{(0)}(t) \equiv X_0 \\ X^{(i+1)}(t) = X_0 + \int_0^t F(X^{(i)}(s))\,ds \qquad i = 0, 1, 2, \cdots \end{cases}$$

$\Downarrow$

$$\begin{cases} \{X^{(i)}\}_{i=0,1,2,3,\ldots} \text{ converges, } X^{(i)} \xrightarrow[C^0]{i\to\infty} X \\ X \text{ satisfies: } \quad X(t) = X_0 + \int_0^t F(X(s))\,ds \end{cases}$$

Existence of Solutions — Iterations

$$X(t) = X_0 + \int_0^t F(X(s))\,ds$$

$$\begin{cases} X^{(0)}(t) \equiv X_0 \\ X^{(i+1)}(t) = X_0 + \int_0^t F(X^{(i)}(s))\,ds \qquad i = 0, 1, 2, \dots \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \|X^{(i+1)} - X^{(i)}\|_{C^0} \leq \dfrac{MK^i T^i}{i!} \\ \displaystyle\sum_{i=0}^{\infty} \|X^{(i+1)} - X^{(i)}\|_{C^0} \leq \sum_{i=0}^{\infty} \frac{MK^i T^i}{i!} \leq Me^{KT} < \infty \end{cases}$$

Existence of Solutions – Banach Fixed Pt. Thm.

Consider $\mathcal{T} : C^0([0,T];\mathbb{R}^n) \longrightarrow C^0([0,T];\mathbb{R}^n)$

$$\mathcal{T}(X)(t) = X_0 + \int_0^t F(X(s))\,ds, \quad t \in [0,T]$$

① X is a fixed point of $\mathcal{T}$ if $\mathcal{T}(X) = X$

i.e. $X(t) = X_0 + \int_0^t F(X(s))\,ds, \quad t \in [0,T]$

② $\mathcal{T}$ is a contraction map if $\exists\ C < 1$ s.t.

$$\|\mathcal{T}(X) - \mathcal{T}(Y)\|_{C^0} \leq C\,\|X - Y\|_{C^0}$$

Existence of Solutions – Banach Fixed Pt. Thm.

Consider $\mathcal{T} : C^0([0,T];\mathbb{R}^n) \longrightarrow C^0([0,T];\mathbb{R}^n)$

$$\mathcal{T}(X)(t) = X_0 + \int_0^t F(X(s))\,ds, \qquad t \in [0,T]$$

$$\boxed{\begin{array}{l}\underline{\text{Banach Fixed Point Thm}}: \\ \quad \text{For any } \underline{\text{contraction map}} \text{ on a complete} \\ \quad \text{metric space, there is a } \underline{\text{unique fixed pt.}}\end{array}}$$

Existence of Solutions – Banach Fixed Pt. Thm.

Consider $\mathcal{J} : C^0([0,T];\mathbb{R}^n) \longrightarrow C^0([0,T];\mathbb{R}^n)$

$$\mathcal{J}(X)(t) = X_0 + \int_0^t F(X(s))\,ds, \qquad t \in [0,T]$$

Let $X, Y \in C^0([0,T];\mathbb{R}^n)$.

$$\mathcal{J}(X)(t) - \mathcal{J}(Y)(t) = \int_0^t \big(F(X(s)) - F(Y(s))\big)\,ds$$

$$\|\mathcal{J}(X)(t) - \mathcal{J}(Y)(t)\| \leq \int_0^t L\,\|X(s) - Y(s)\|\,ds$$

Existence of Solutions - Banach Fixed Pt. Thm.

Consider $\mathcal{J} : C^0([0,T];\mathbb{R}^n) \longrightarrow C^0([0,T];\mathbb{R}^n)$

$$\mathcal{J}(X)(t) = X_0 + \int_0^t F(X(s))\,ds, \quad t \in [0,T]$$

Let $X, Y \in C^0([0,T];\mathbb{R}^n)$.

$$\boxed{\|\mathcal{J}(X) - \mathcal{J}(Y)\|_{C^0} \leq \underbrace{LT}_{LT<1 \text{ if } T<\frac{1}{L}.} \|X - Y\|_{C^0}}$$

Existence of Solutions - Time Stepping Scheme

(1) $$\frac{dX(t)}{dt} = F(X(t))$$

(2) $$\frac{X(t+\Delta t) - X(t)}{\Delta t} \simeq F(X(t))$$

(3) $$X(t+\Delta t) \simeq X(t) + F(X(t))\Delta t$$

Existence of Solutions – Time Stepping Scheme

Let $\Delta t > 0$. Define $\{X^{(\Delta t)}(i\Delta t)\}_{i=0,1,\cdots,\frac{T}{\Delta t}}$ as:

$$X^{(\Delta t)}(0) = X_0$$

$$X^{(\Delta t)}(\Delta t) = X^{(\Delta t)}(0) + F(X^{(\Delta t)}(0))\Delta t$$

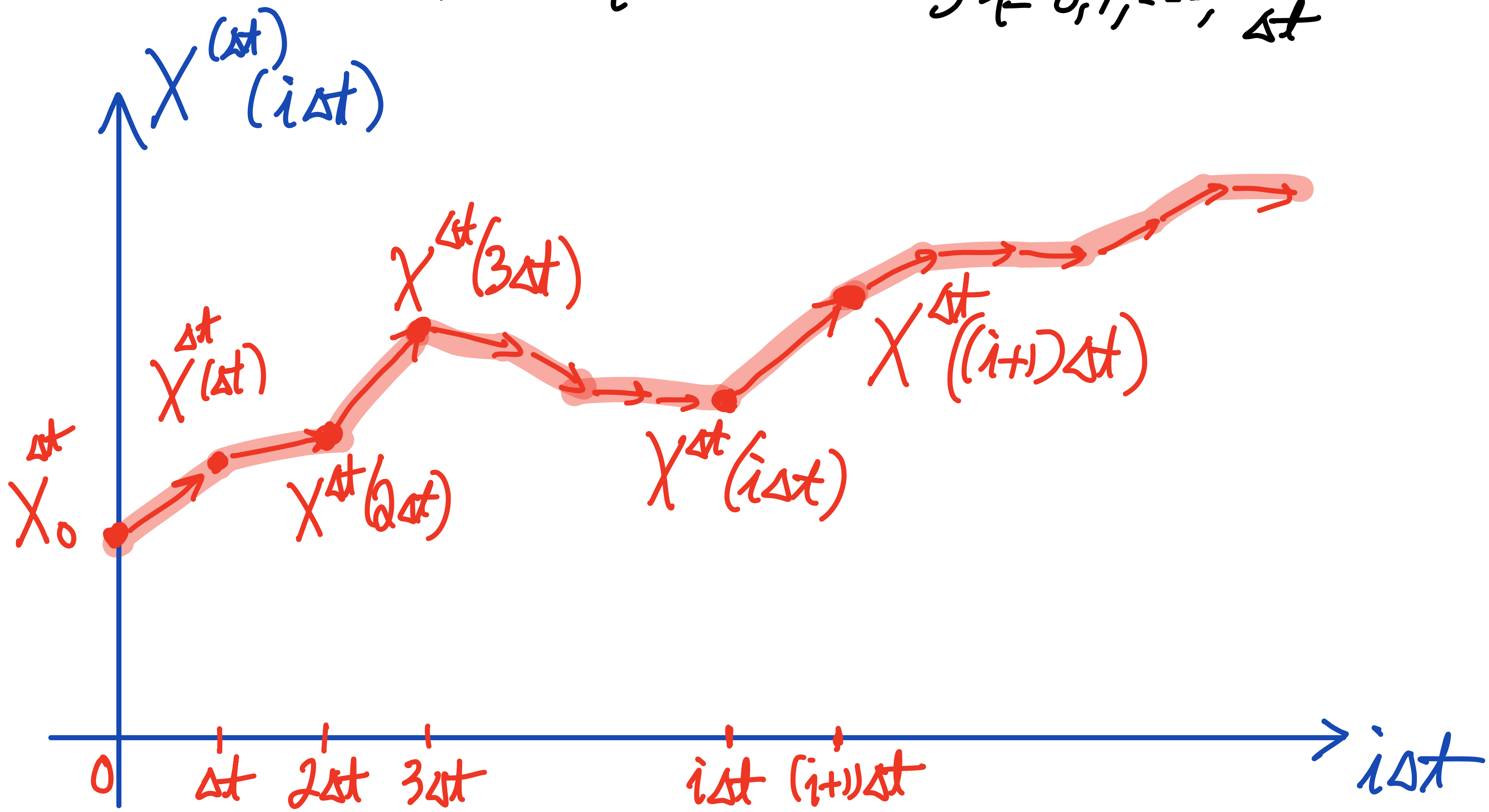
$$X^{(\Delta t)}(2\Delta t) = X^{(\Delta t)}(\Delta t) + F(X^{(\Delta t)}(\Delta t))\Delta t$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$X^{(\Delta t)}((i+1)\Delta t) = X^{(\Delta t)}(i\Delta t) + F(X^{(\Delta t)}(i\Delta t))\Delta t$$

Existence of Solutions – Time Stepping Scheme

Let $\Delta t > 0$. Define $\{X^{(\Delta t)}(i\Delta t)\}_{i=0,1,\cdots,\frac{T}{\Delta t}}$ as:



Existence of Solutions – Time Stepping Scheme

Let $\Delta t > 0$. Define $\{X^{(\Delta t)}(i\Delta t)\}_{i=0,1,\dots,\frac{T}{\Delta t}}$ as:

(1) Then $\{X^{(\Delta t)}(i\Delta t)\}$ satisfies:

$$X^{(\Delta t)}(i\Delta t) = X_0 + \sum_{j=0}^{i-1} F(X^{(\Delta t)}(j\Delta t))\,\Delta t$$

(2) $\{X^{(\Delta t)}(i\Delta t)\}$ is equi-continuous: (M–Lipschitz)

$$\| X^{\Delta t}(k\Delta t) - X^{(\Delta t)}(l\Delta t) \| \leq M \| k\Delta t - l\Delta t \|$$

Existence of Solutions – Time Stepping Scheme

Let $\Delta t > 0$. Define $\{X^{(\Delta t)}(i\Delta t)\}_{i=0,1,\dots,\frac{T}{\Delta t}}$ as:

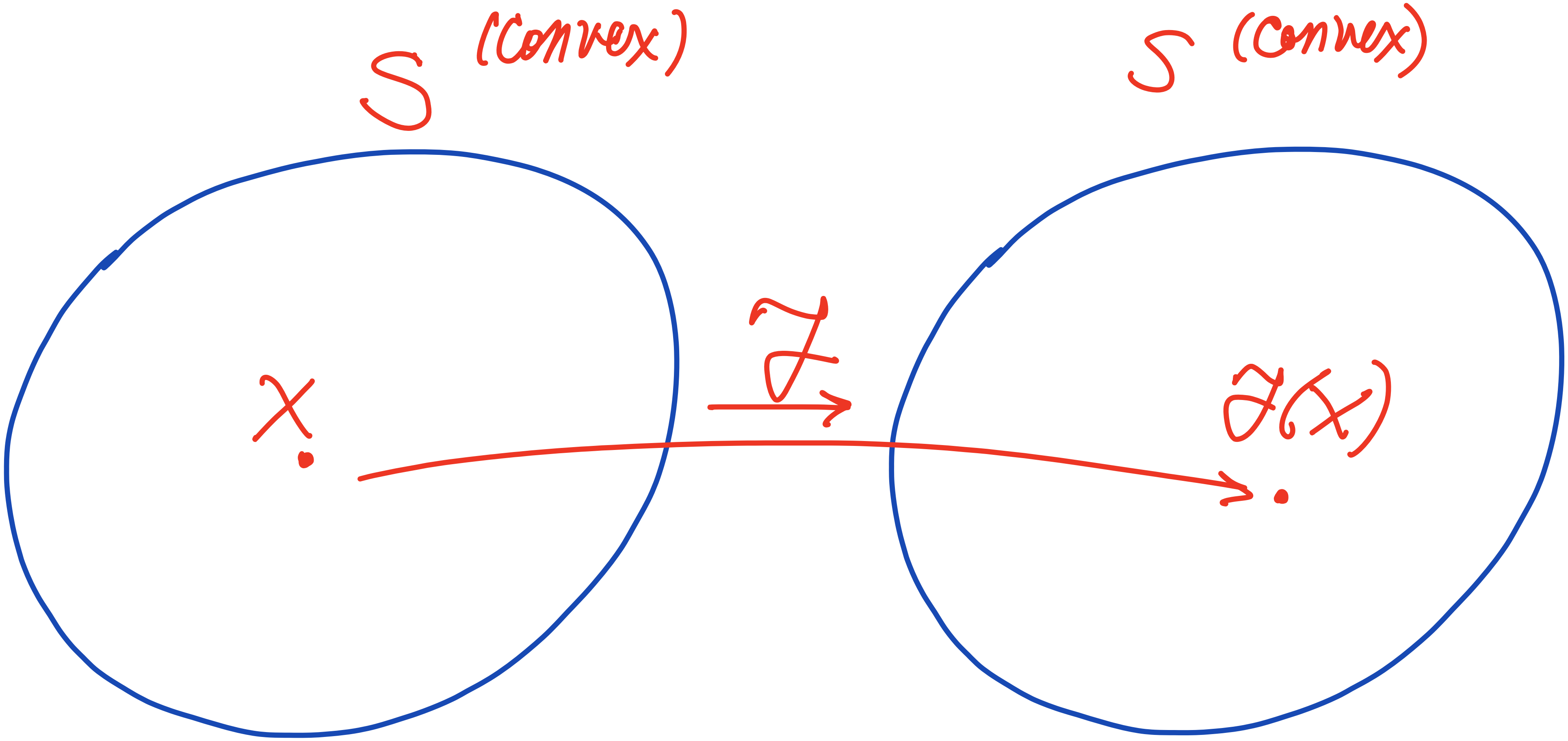
(3) $\exists$ a subsequence of $\{X^{\Delta t}\}_{\Delta t>0} \xrightarrow{\Delta t \to 0} X^*$

$$\boxed{X^{(\Delta t)}(i\Delta t) = X_0 + \sum_{j=0}^{i-1} F(X^{(\Delta t)}(j\Delta t))\Delta t}$$

$$\Downarrow$$

$$\boxed{X^*(t) = X_0 + \int_0^t F(X^*(s))\,ds}$$

Existence of Solutions - Schauder Fixed Pt. Thm



$\boxed{\text{If } \mathcal{F} \text{ is continuous map with compact image then } \mathcal{F} \text{ has a fixed pt: } \mathcal{F}(X^*) = X^*}$

Existence of Solutions – Schauder Fixed Pt. Thm

① Let $S \subseteq C^0([0,T]; \mathbb{R}^n)$ be a convex subset of $C^0([0,T]; \mathbb{R}^n)$

② Let $\mathcal{T}: S \to S$ be a continuous & compact map,

(i) $X^i \to X \Rightarrow \mathcal{T}(X^{(i)}) \longrightarrow \mathcal{T}(X)$

(ii) $\mathcal{T}(S)$ has compact closure

③ Then $\mathcal{T}$ has a fixed point in S,

i.e. $\exists\, X \in S$ s.t. $\mathcal{T}(X) = X$

Existence of Solutions – Schauder Fixed Pt. Thm

① Let $\mathcal{T} : C^0([0,T];\mathbb{R}^n) \longrightarrow C^0([0,T];\mathbb{R}^n)$

$$\mathcal{T}(X)(t) = X_0 + \int_0^t F(X(s))\,ds$$

② Let $S = \{X : \|X - X_0\|_{C^0} \leq MT\}$ ← a convex set

③ Then $\mathcal{T} : S \longrightarrow S$, is cont. & compact.

$\Longrightarrow$ $\boxed{\exists \text{ a fixed pt. } X \text{ of } \mathcal{T}, \text{ i.e. } \mathcal{T}(X) = X.}$