Behavior Near An Equilibrium Point

$$\frac{dX}{dt} = F(X) \qquad \boxed{F(X_*) = X_*}$$

$$F(X) = \underbrace{F(X_*)}_{0} + \underbrace{[DF(X_*)]}_{A}(X - X_*) + \underbrace{\frac{1}{2}[D^2F(X_*)(X-X_*)^2]}_{O(\|X-X^*\|^2)}$$

(Let $Y(t) = X(t) - X_*$)

$$\boxed{\frac{d}{dt}Y(t) = AY(t) + g(Y(t)),}$$

$\|g(Y)\| \sim O(\|Y\|^2)$

$\ll \|Y\|$

for $\|Y\| \ll 1$

Behavior Near An Equilibrium Point

$$\frac{dX}{dt} = F(X) \qquad \boxed{F(X_*) = X_*}$$

$$F(X) = \underbrace{F(X_*)}_{0} + \underbrace{[DF(X_*)]}_{A}(X - X_*) + \underbrace{\tfrac{1}{2}[D^2F(X_*)(X-X_*)^2]}_{O(\|X-X^*\|^2)}$$

$$\boxed{\frac{dX}{dt} = AX + g(X), \quad \text{with } \|g(X)\| \sim O(\|X\|^2) \ll \|X\| \text{ for } \|X\| \ll 1}$$

($X=0$ is an equilibrium pt.)

Invariant Subspaces of $A^{n\times n}$

More generally, [M, Sec. 2.6]

- $p(\lambda) = \det(A-\lambda I)$ characteristic poly.

$$= c(\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2} \cdots (\lambda-\lambda_k)^{n_k}$$

- $E_i = \mathrm{Null}\{(A-\lambda_i I)^{n_k}\}$

$$= \{u : (A-\lambda_i I)^{n_k} u = 0\}$$

= Generalized eigenspace

Invariant Subspaces of $A^{n \times n}$

More generally,

- $\boxed{\mathbb{R}^n = E_1 \oplus E_2 \oplus \cdots \oplus E_k}$

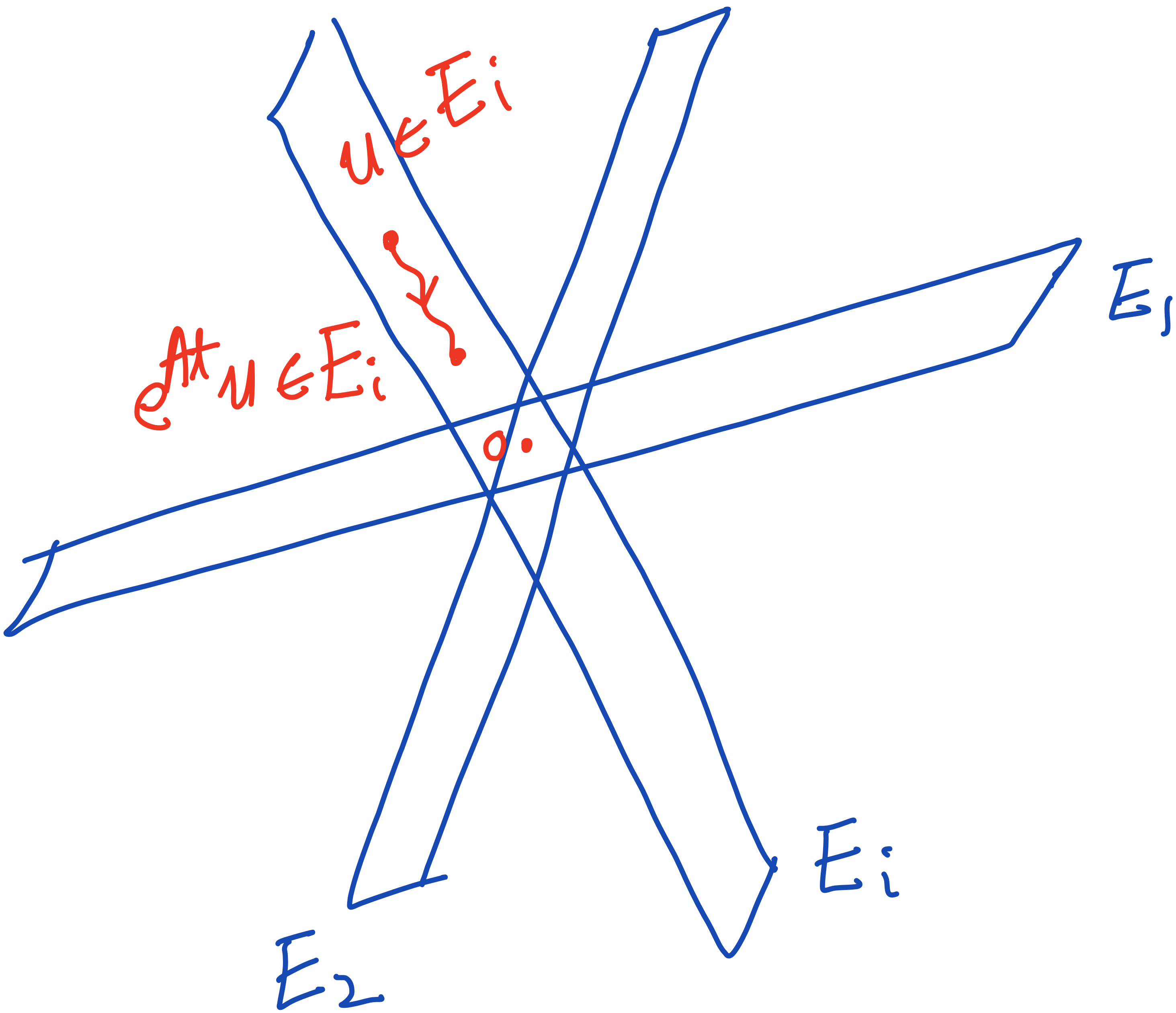
- E_i is invariant under A,

  i.e. $u \in E_i$ then $Au \in E_i$

- E_i is invariant by e^{At},

  i.e. $u \in E_i$, then $e^{At}u \in E_i$

Invariant Subspaces of $A^{n \times n}$



Invariant Subspaces of $A^{n\times n}$

[M, Sec 2.7]

$$\mathbb{R}^n = E_1 \oplus E_2 \oplus \cdots\cdots \oplus E_k$$

$$= \underline{E_s \oplus E_c \oplus E_u}$$

where

$$E_s = \bigoplus_{Re(\lambda_i)<0} E_{\lambda_i} = \text{\underline{stable subspace}}$$

$$E_u = \bigoplus_{Re(\lambda_i)>0} E_{\lambda_i} = \text{\underline{unstable subspace}}$$

$$E_c = \bigoplus_{Re(\lambda_i)=0} E_i = \text{\underline{center subspace}}$$

Linear Stability

A is called hyperbolic if

$$\underline{Re(\lambda_i(A)) \neq 0}$$

ie. $E^c = \{\vec{0}\}$ and

$$\underline{\mathbb{R}^n = E^s \oplus E^u}$$

Linear Stability

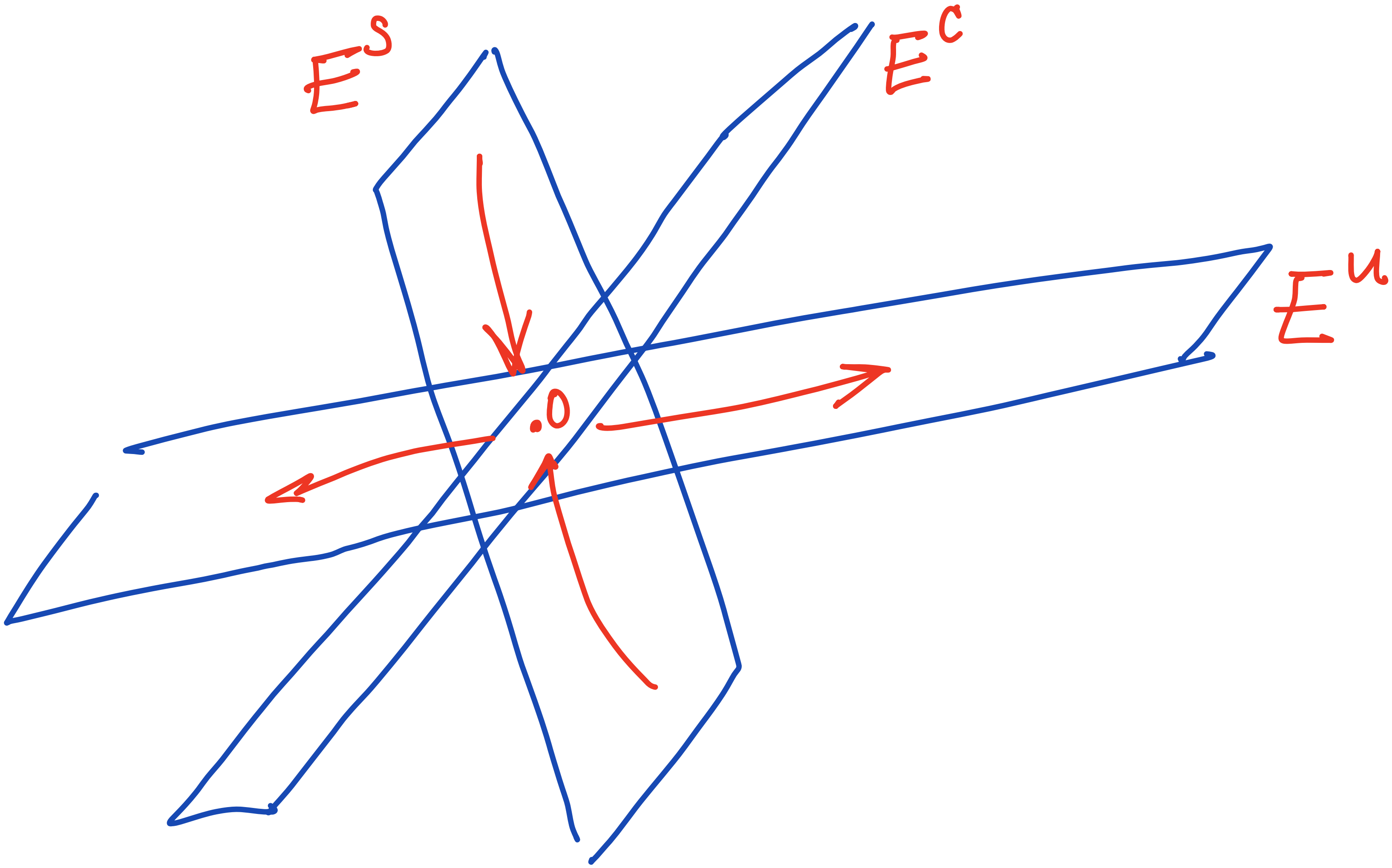
[M, Sec 2.7]
[Bellman, p.25]

- $x_0 \in E_s \iff$ there is $K>0$ s.t
  $$\|e^{At}x_0\| \leq Ce^{-Kt}\|x_0\| \text{ for all } t>0$$
  i.e. $e^{At}x_0 \xrightarrow{t\to+\infty} 0$ exponentially fast in t
- $x_0 \in E_u \iff$ there is $K>0$ s.t
  $$\|e^{At}x_0\| \leq Ce^{Kt}\|x_0\| \text{ for all } t<0$$
  i.e. $e^{At}x_0 \xrightarrow{t\to-\infty} 0$ exponentially fast in t

Linear Stability

[M, Sec 2.7]
[Bellman, p.25]

$$\mathbb{R}^n = E^s \oplus E^c \oplus E^u$$



Nonlinear Stability [M, Thm 4.19, p. 117]

Linear asymptotic stability implies
(nonlinear) asymptotic stability

$$\boxed{\frac{d}{dt}X = AX + g(X), \quad \|g(x)\| \ll \|X\| \text{ for } \|X\| \ll 1}$$

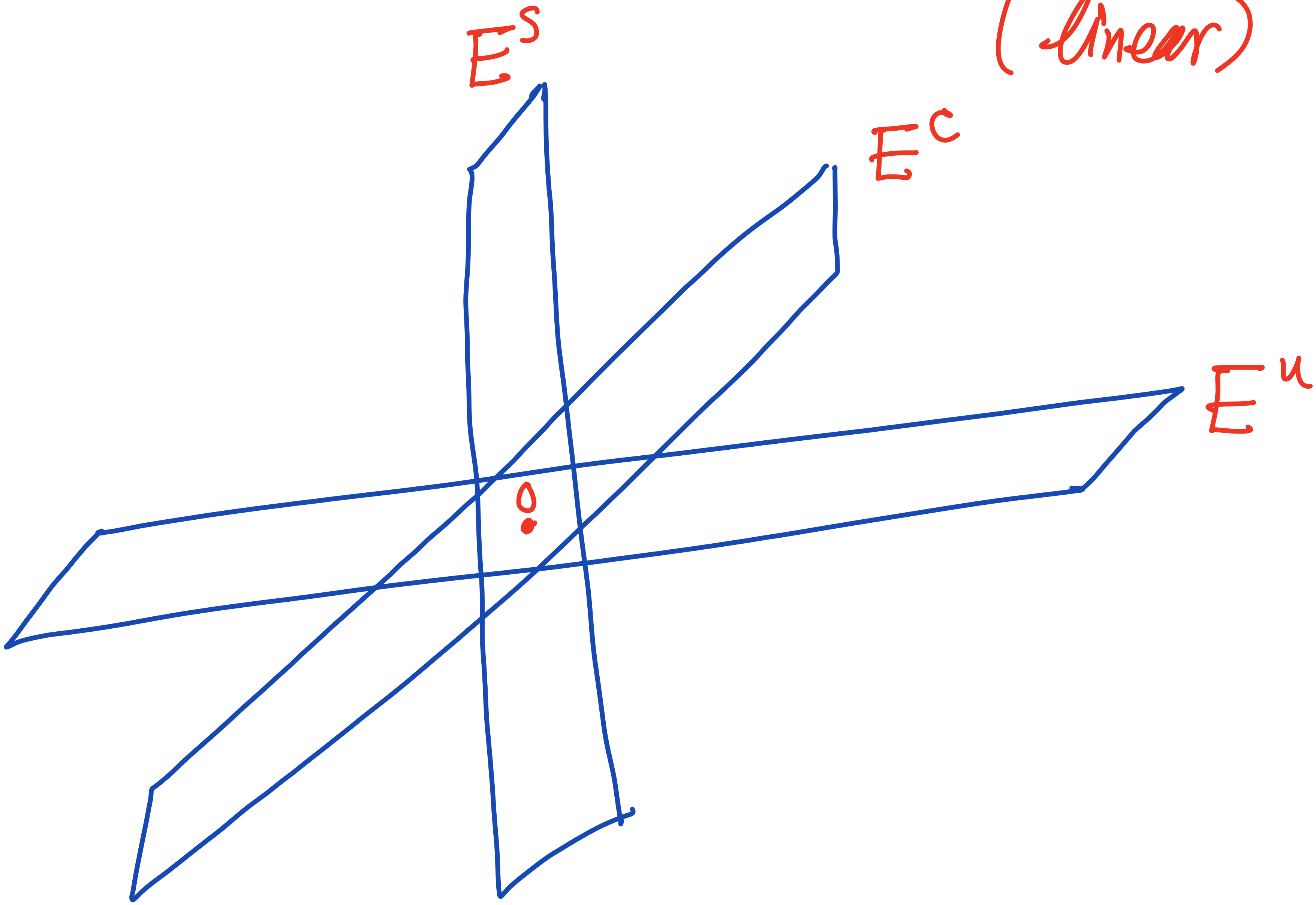
Assume $\mathrm{Re}(\lambda_i(A)) < 0$, (i.e. $\mathbb{R}^n = E^s$)

Then $X_* = 0$ is asymptotically stable.

(There is $\varepsilon > 0$, s.t. if $\|X_0\| < \varepsilon$, then $\|X(t)\| \xrightarrow{t \to +\infty} 0$ (exponentially fast))

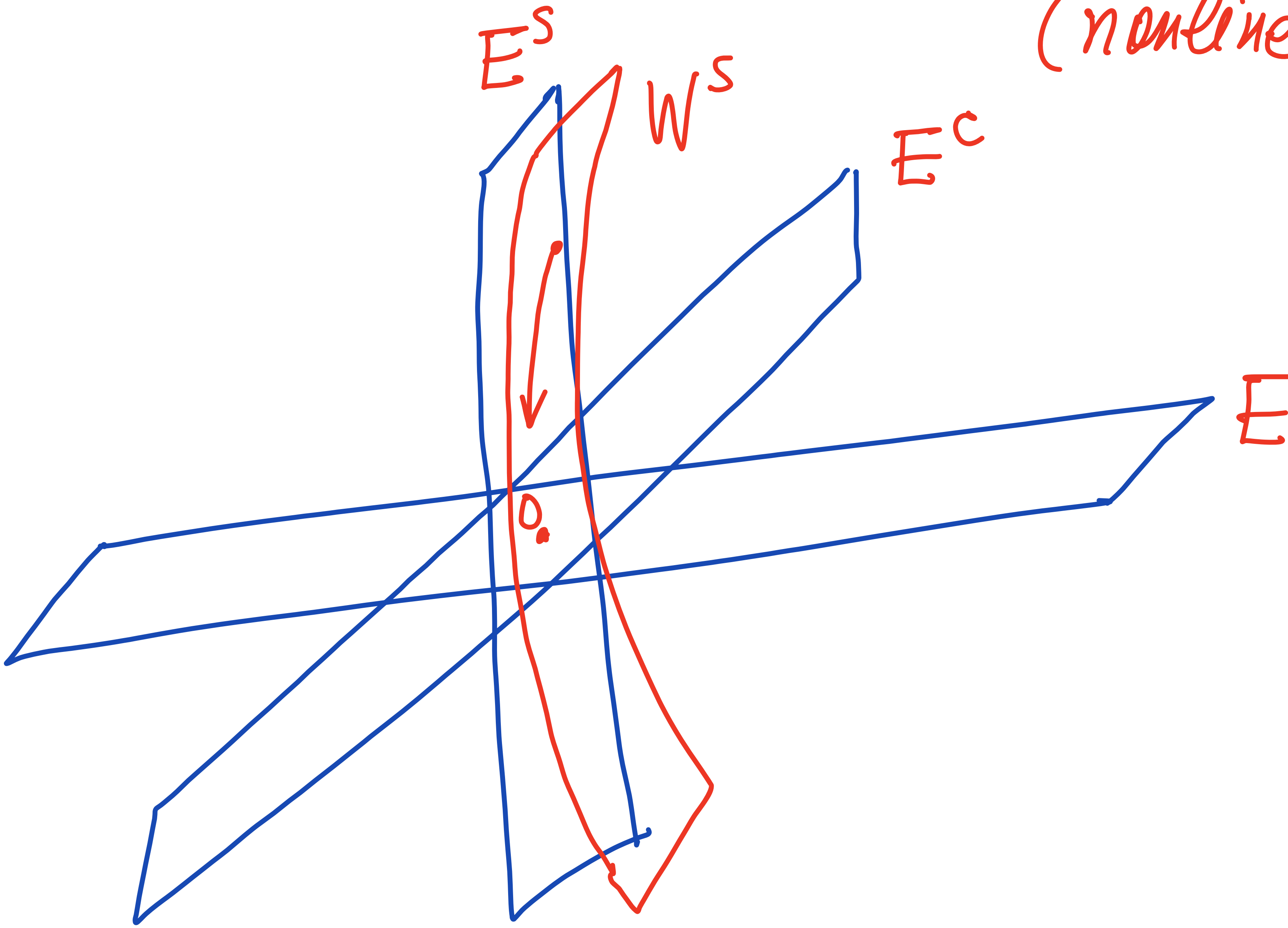
Invariant Manifolds $W^s(0)$, $W^u(0)$, $W^c(0)$

(linear)



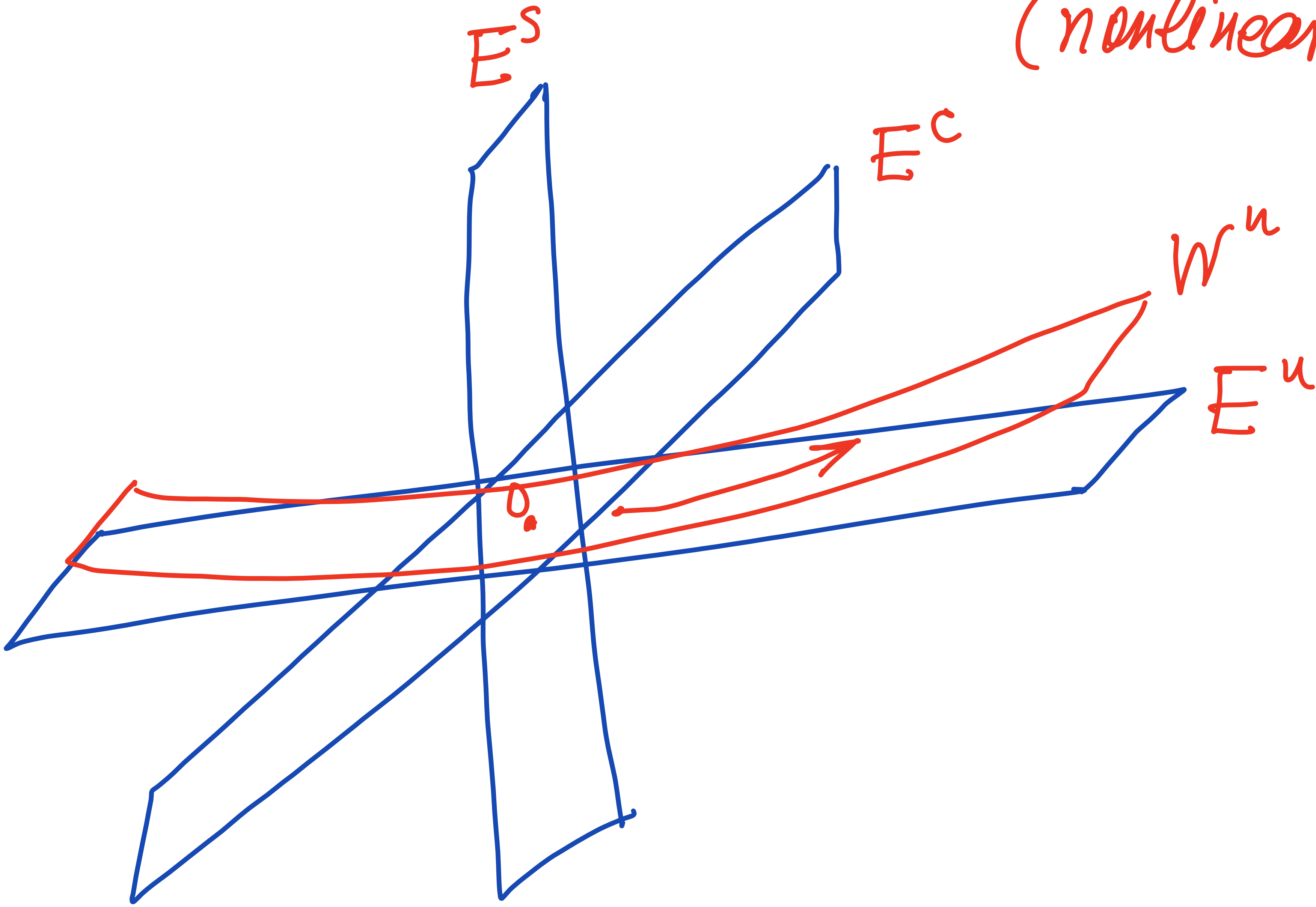
Invariant Manifolds $W^s(0)$, $W^u(0)$, $W^c(0)$

(nonlinear)



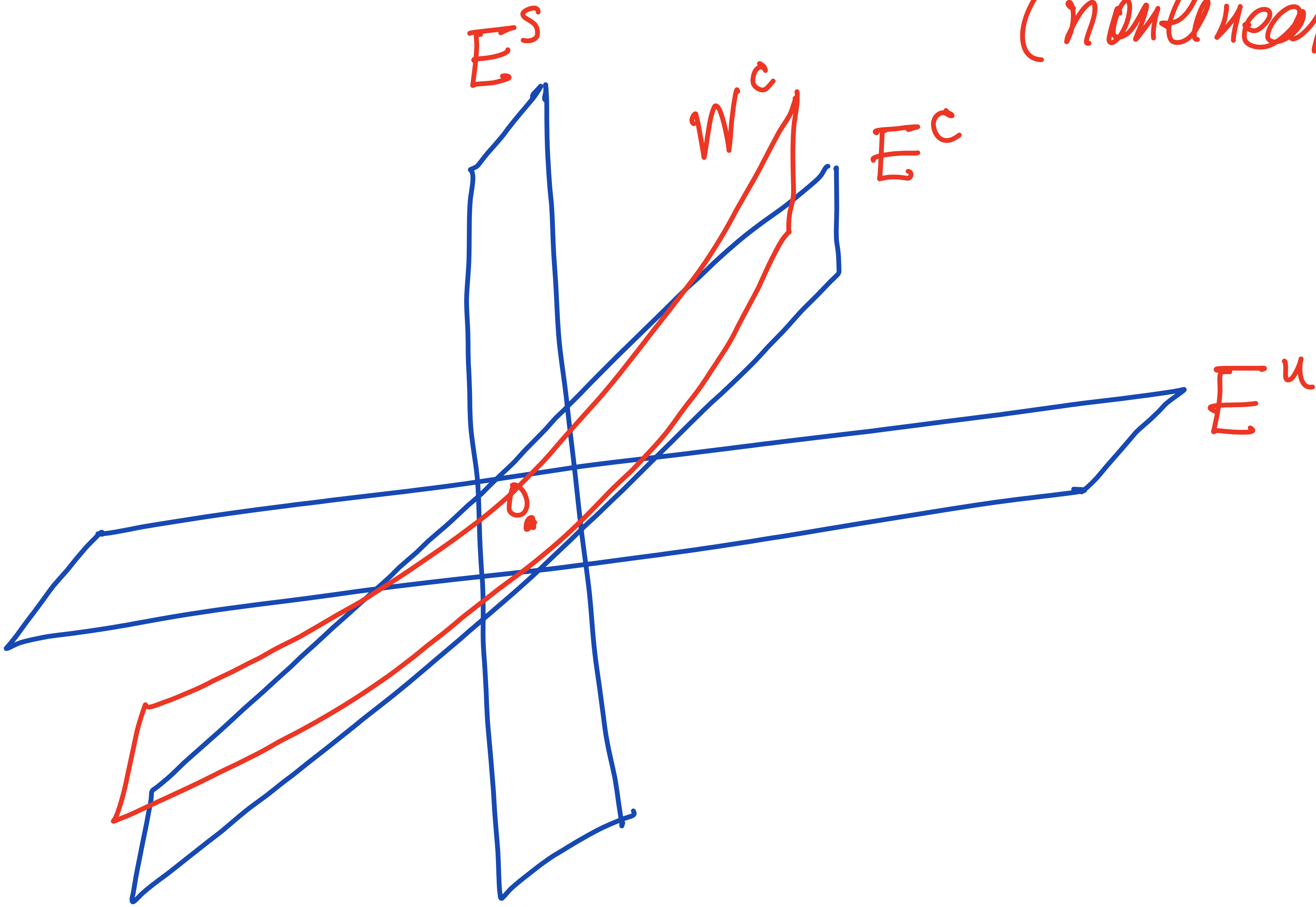
Invariant Manifolds $W^s(0)$, $W^u(0)$, $W^c(0)$

(nonlinear)



Invariant Manifolds $W^s(0)$, $W^u(0)$, $W^c(0)$

(nonlinear)



Invariant Manifolds $W^s(0)$, $W^u(0)$, $W^c(0)$

[M, Thm 5.9, Thm 5.21]

There are 3 manifolds $W^s(0)$, $W^u(0)$, $W^c(0)$ (in a neighborhood of $x_* = 0$):

① invariant under the flow:

if $x \in W^s(0)$, then $\phi_t(x) \in W^s(0)$

$$\boxed{\phi_t(W^s) \subseteq W^s}$$

(Similarly for W^u & W^c.)

Invariant Manifolds $W^s(0)$, $W^u(0)$, $W^c(0)$

[M, Thm 5.9, Thm 5.21]

There are 3 manifolds $W^s(0)$, $W^u(0)$, $W^c(0)$ (in a neighborhood of $X_* = 0$):

② W^s, W^u, W^c pass through $X_* = 0$ and tangent to E^s, E^u, E^c.

$$\dim(W^s) = \dim E^s$$

(Similarly for W^u, W^c.)

Invariant Manifolds $W^s(0)$, $W^u(0)$, $W^c(0)$

[M, Thm 5.9, Thm 5.21]

There are 3 manifolds $W^s(0)$, $W^u(0)$, $W^c(0)$ (in a neighborhood of $x_* = 0$):

③ $\forall x \in W^s$, $\underline{\phi_t(x) \in W^s \xrightarrow{t \to +\infty} 0}$

$\forall x \in W^u$, $\underline{\phi_t(x) \in W^u \xrightarrow{t \to -\infty} 0}$

(both with exponentially rates.)

Invariant Manifolds $W^s(0)$, $W^u(0)$, $W^c(0)$

[M, Thm 5.9, Thm 5.21]

There are 3 manifolds $W^s(0)$, $W^u(0)$, $W^c(0)$ (in a neighborhood of $X_*=0$):

④ W^s & W^u are unique (while W^c might not be.)

⑤ If F is C^k (has continuous k-derivatives) then so are W^s, W^u & W^c.