Existence of (local) Stable Manifold $W^s_{loc}(0)$ (for hyperbolic system)

$$\boxed{\frac{d}{dt}X = AX + g(X)}$$

$(X \in \mathbb{R}^n, \quad A^{n\times n})$

$|g(X)| \sim O(|X|^2) \ll |X|$ for $|X| \ll 1$

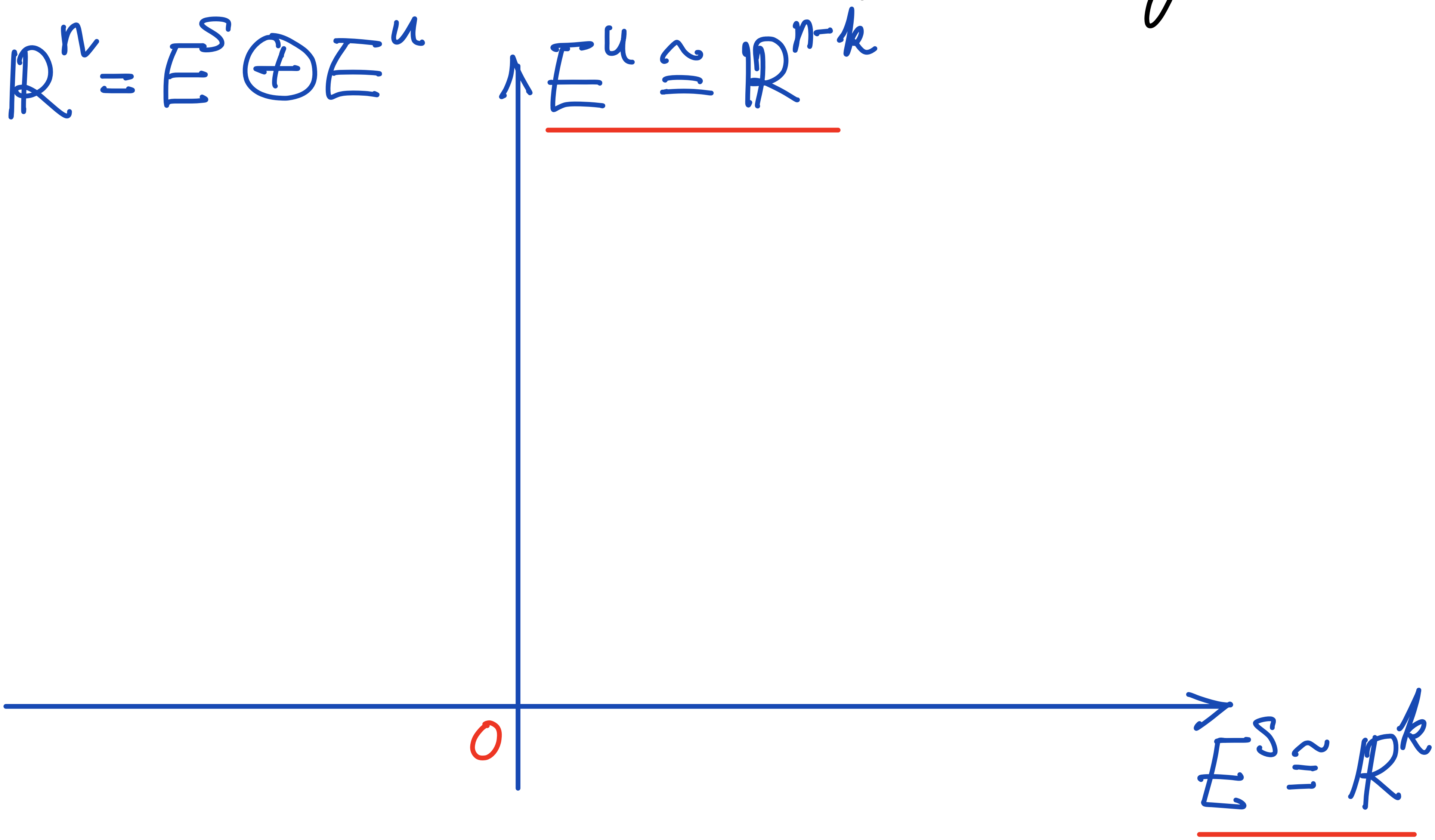
(1) A is hyperbolic: $\mathrm{Re}(\lambda_i(A)) \neq 0$

(2) E^s = stable subspace: $\dim(E^s) = k$

(3) E^u = unstable subspace: $\dim(E^u) = n-k$

Existence of (local) Stable Manifold $W^s_{loc}(0)$ (for hyperbolic system)

$\mathbb{R}^n = E^s \oplus E^u$



Existence of (local) Stable Manifold $W^s_{loc}(0)$

(for hyperbolic system)

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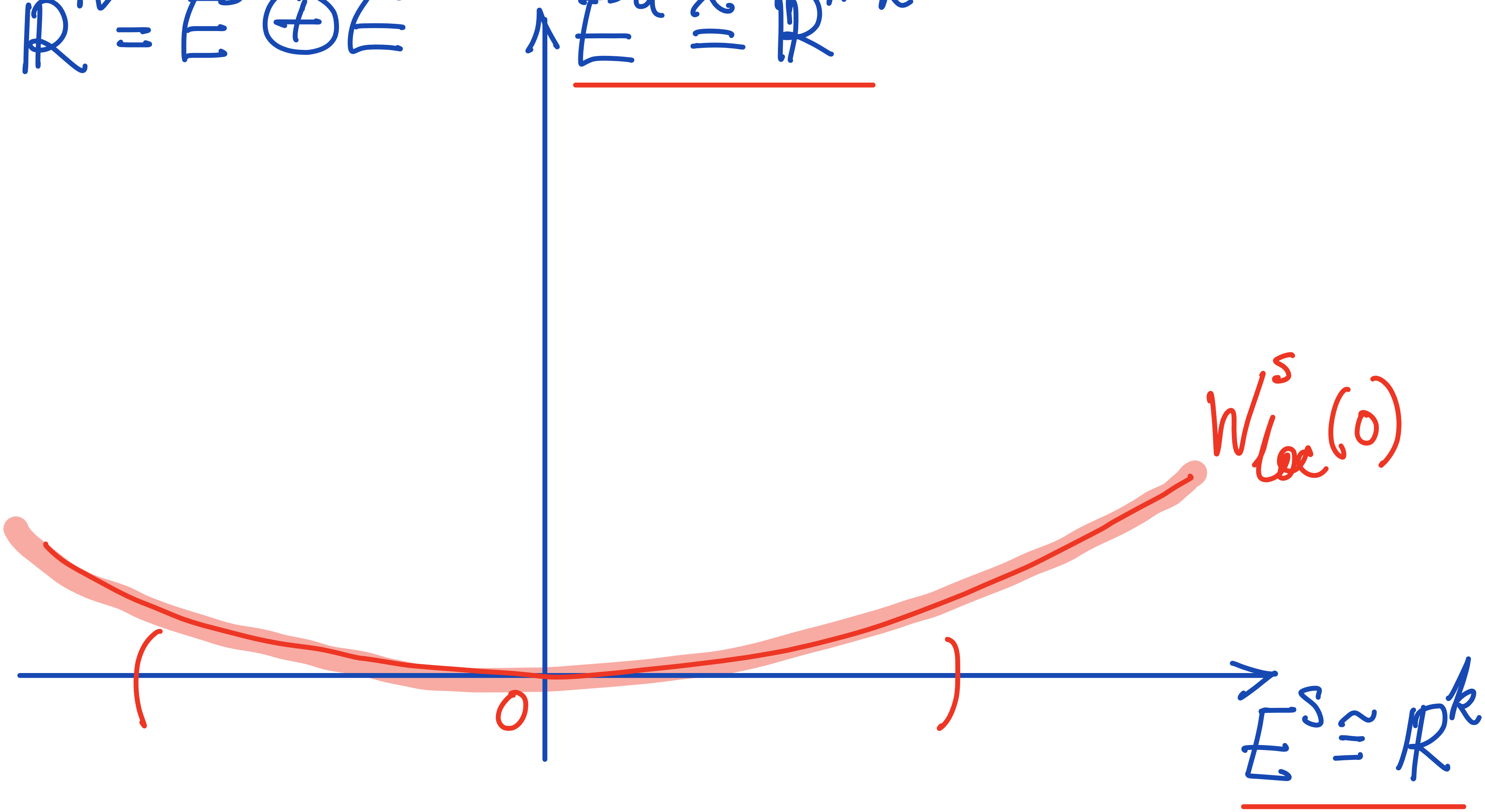

Existence of (local) Stable Manifold $W^s_{loc}(0)$
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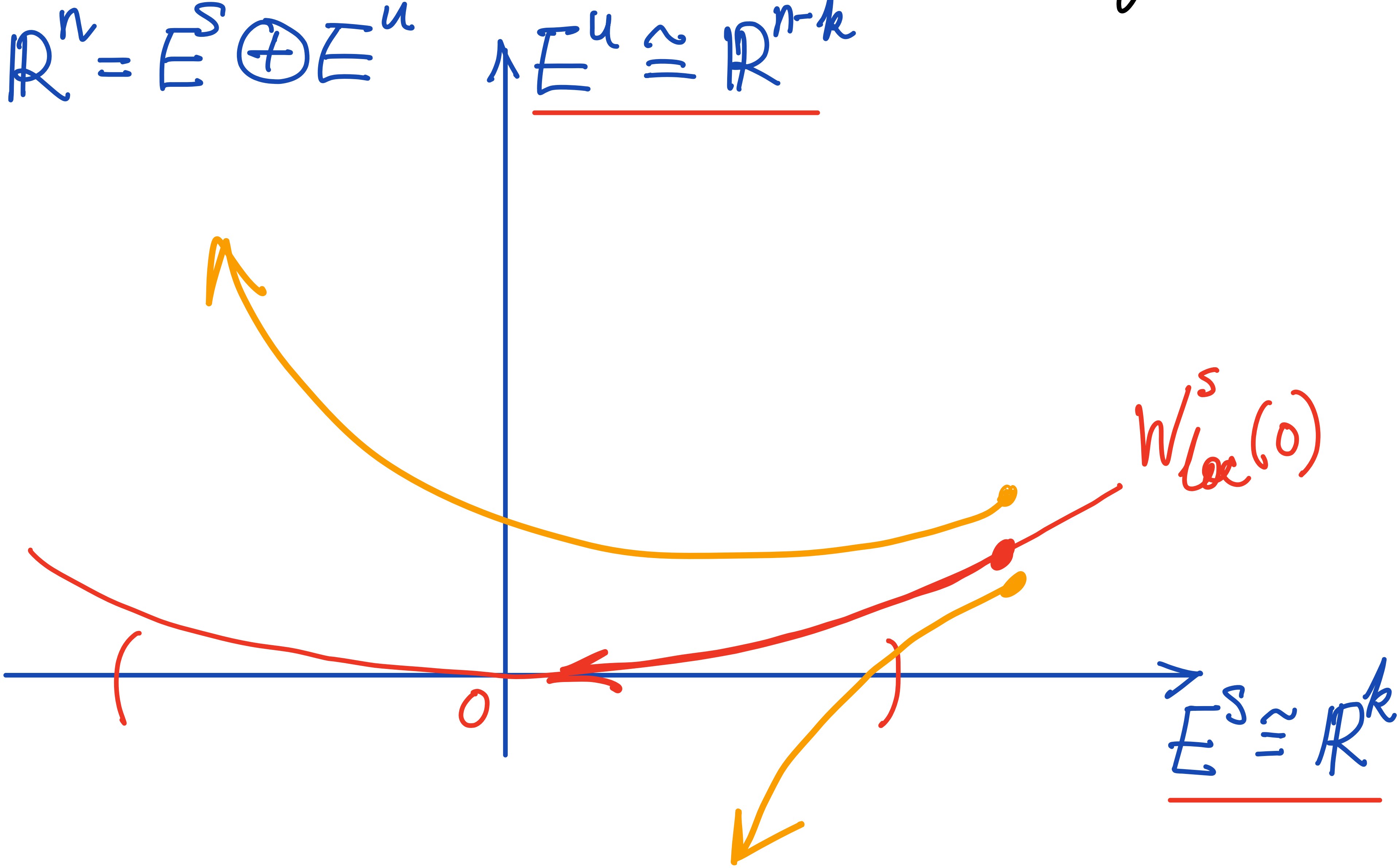


Existence of (local) Stable Manifold $W^s_{loc}(0)$

[M, Thm 5.9, p. 169] (for hyperbolic system)

There is a unique manifold $W^s_{loc}(0)$ in a neighborhood of 0 satisfying:

(1) $W^s(0)$ is invariant under the flow;
($x \in W^s(0) \Rightarrow \phi_t(x) \in W^s(0)$ for all $t > 0$)

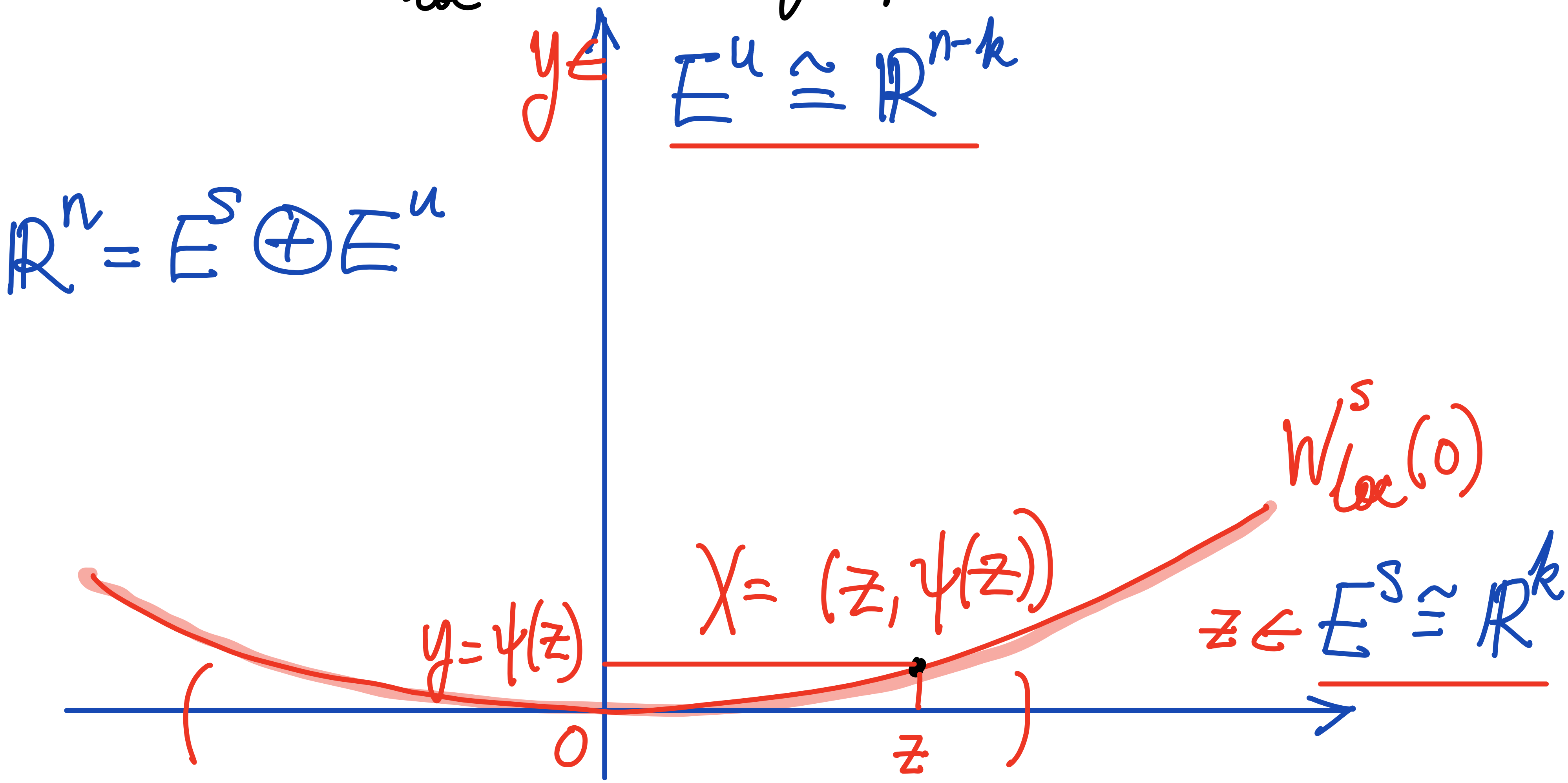
(2) $W^s(0)$ passes through 0 and is tangent to E^s at 0

(3) $\dim(W^s(0)) = k = \dim(E^s)$

(4) if $x \in W^s(0)$, then $\phi_t(x) \xrightarrow{t \to +\infty} 0$

Existence of (Local) Stable Manifold $W^s_{loc}(0)$
(for hyperbolic system)

① Write $W^s_{loc}(0)$ as a graph over $E^s \cong \mathbb{R}^k$:



Existence of (local) Stable Manifold $W^s_{loc}(0)$

(for hyperbolic system)

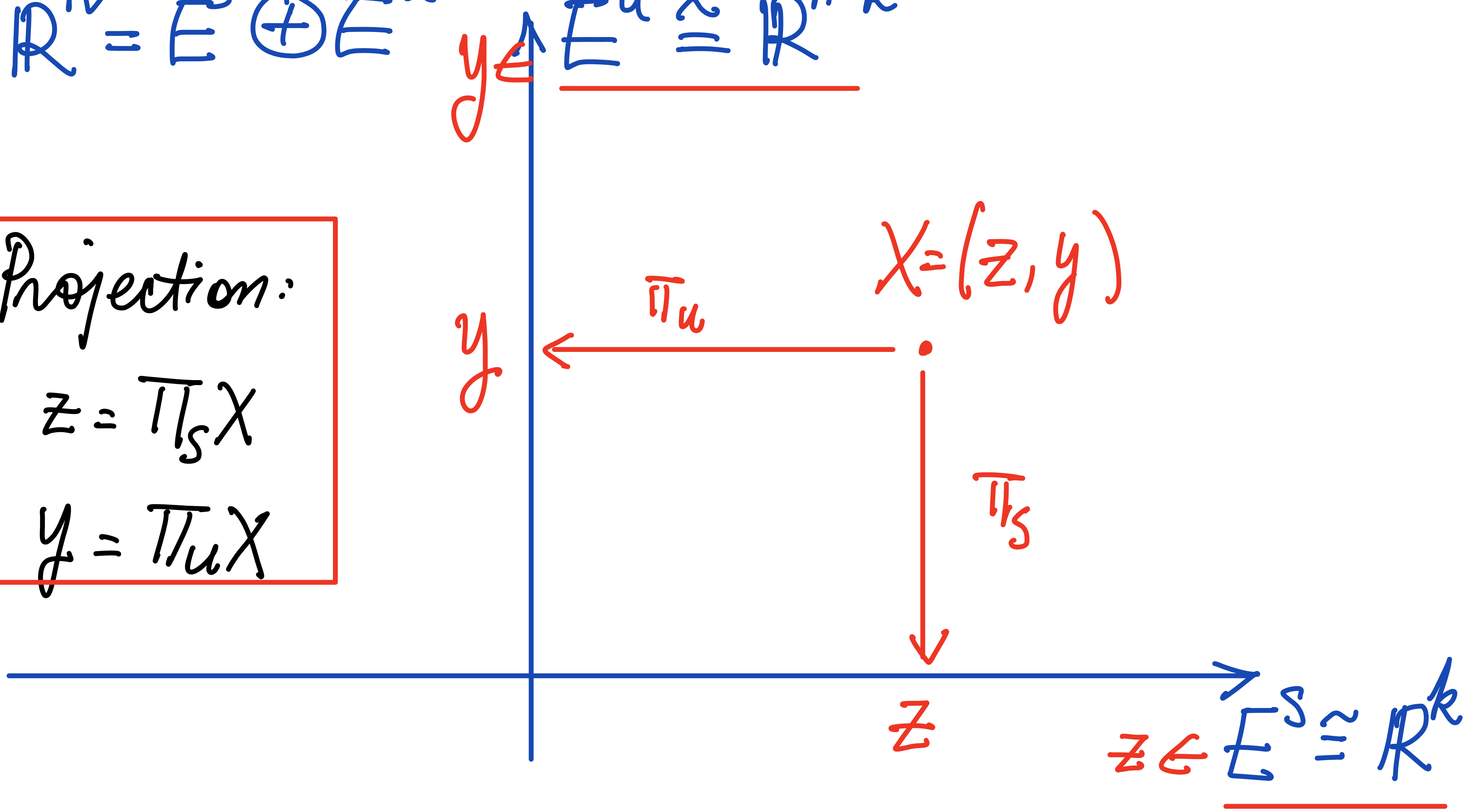
① Write $W^s_{loc}(0)$ as a graph over $E^s \cong \mathbb{R}^k$:

$$\begin{cases} W^s_{loc}(0) = \{ (z, \psi(z)) : z \in E^s \cong \mathbb{R}^k \} \\ \psi : \mathbb{R}^k (E^s) \longrightarrow \mathbb{R}^{n-k} (E^u) \\ \psi(0) = 0, \\ D\psi(0) = 0, \text{ ie. } |\psi(z)| \leq C|z|^2 \end{cases}$$

Existence of (local) Stable Manifold $W^s_{loc}(0)$

(for hyperbolic system)

② $\mathbb{R}^n = E^s \oplus E^u$



Projection:

$z = \pi_s x$

$y = \pi_u x$

Existence of (local) Stable Manifold $W^s_{loc}(0)$

(for hyperbolic system)

There is $K, \lambda > 0$ such that

for any $x_0 \in \mathbb{R}^n \triangleq E^s \oplus E^u$,

$$\| e^{At} \Pi_s x_0 \| < K e^{-\lambda t} \| x_0 \| \quad \text{for } t > 0$$

$$\| e^{At} \Pi_u x_0 \| < K e^{\lambda t} \| x_0 \| \quad \text{for } t < 0$$

Existence of (local) Stable Manifold $W^s_{loc}(0)$
(for hyperbolic system)

③

(a) Consider nonlinear equation as a linear equation with inhomogeneous term:

$$\frac{dX(t)}{dt} = \underbrace{AX(t)}_{\text{linear equation}} + \underbrace{g(X(t))}_{\text{inhomogeneous term}}$$

(b) and then use fixed point theorem.

Linear Inhomogeneous Equation (M, Lemma 5.8)

(I) Consider $\boxed{\dot{X} = AX + \gamma(t), \quad \pi_s X(0) = \sigma}$

(1) A is hyperbolic ($\mathrm{Re}(\lambda_i(A)) \neq 0$)

for all $t \geq 0$

(2) $\gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ is bounded ($\|\gamma(t)\| \leq M < \infty$)

Then $\exists!$ bounded solution $X(t)$, $t \geq 0$, given by

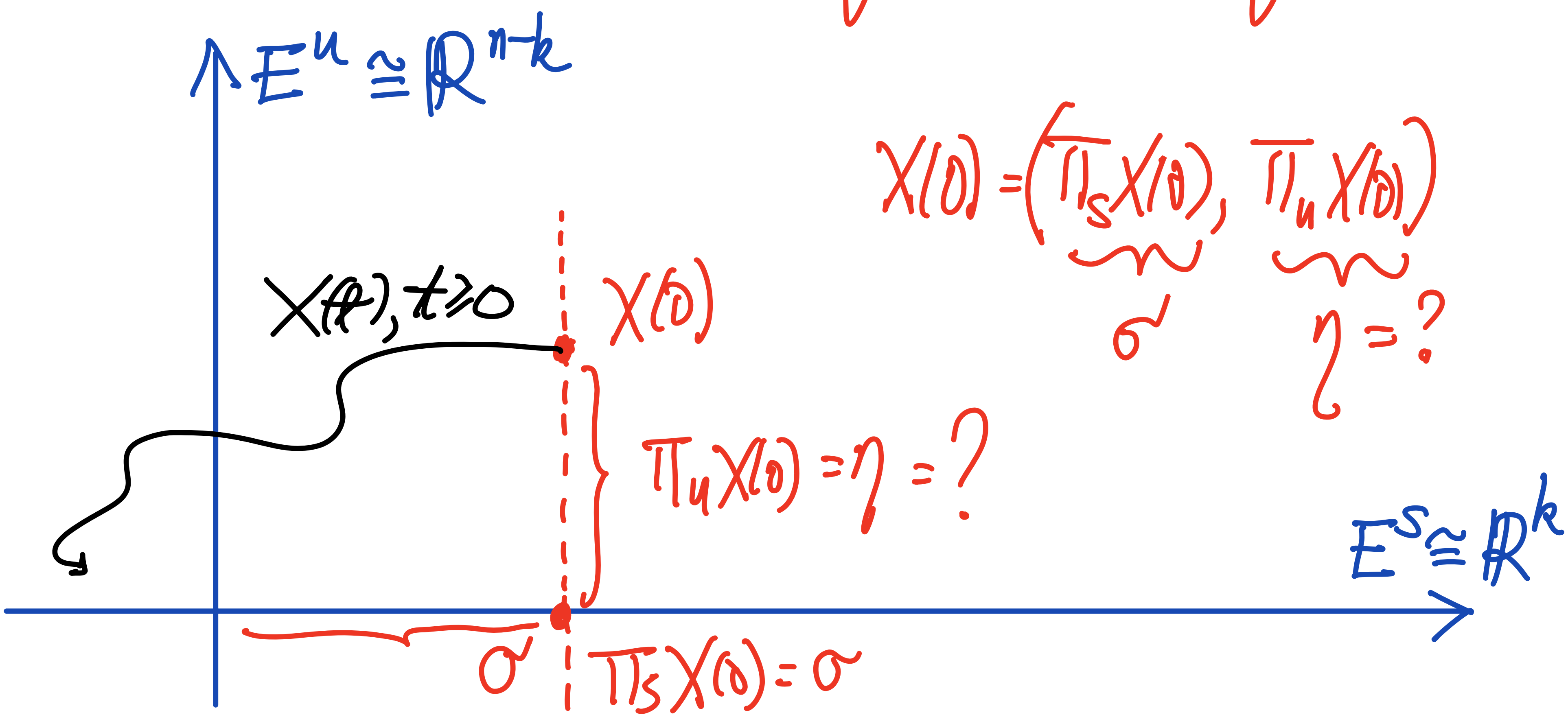
$$\boxed{X(t) = e^{tA}\sigma + \int_0^t e^{(t-s)A} \pi_s \gamma(s)\, ds - \int_t^\infty e^{(t-s)A} \pi_u \gamma(s)\, ds}$$

Linear Inhomogeneous Equation (M, Lemma 5.8)

(I) Consider $\boxed{\dot{X} = AX + \gamma(t), \quad \pi_s X(0) = \sigma}$

given given

$$X(0) = (\underbrace{\pi_s X(0)}_{\sigma}, \underbrace{\pi_u X(0)}_{\eta = ?})$$



Linear Inhomogeneous Equation (M, Lemma 5.8)

(I) Consider $\boxed{\dot{X} = AX + \gamma(t), \quad \pi_s X(0) = \sigma}$

$$\boxed{X(t) = e^{tA}\sigma + \int_0^t e^{(t-s)A}\pi_s \gamma(s)\,ds - \int_t^\infty e^{(t-s)A}\pi_u \gamma(s)\,ds}$$

$t=0$

$$X(0) = \underbrace{\sigma}_{\sigma \in E^s} - \underbrace{\int_0^\infty e^{-As}\pi_u \gamma(s)\,ds}_{\eta \in E^u}$$

Nonlinear Version (II)

$\eta = \eta(\sigma)$

$$\frac{dX}{dt} = AX + \underbrace{g(X)}_{\gamma(t)}, \quad X(0) = (\sigma; \eta)$$

given ?

$$\boxed{\text{given } \sigma, \text{ choose } \eta = \eta(\sigma) \text{ s.t. } X(t) \xrightarrow{t \to +\infty} 0}$$

① $X(t)$

$$= e^{At}\sigma + \int_0^t e^{A(t-s)} \pi_s g(X(s))\,ds - \int_t^\infty e^{A(t-s)} \pi_u g(X(s))\,ds$$

② $$\eta = \eta(\sigma) = -\int_0^\infty e^{-As} \pi_u g(X(s))\,ds$$

Banach Fixed Point Theorem (II)

Define Let $X: t \in \mathbb{R}_+ \longrightarrow X(t) \in \mathbb{R}^n$

$$\boxed{(TX)(t) = e^{tA}\sigma + \int_0^\infty e^{A(t-s)}\Pi_s g(X(s))\,ds - \int_t^\infty e^{A(t-s)}\Pi_u g(X(s))\,ds}$$

$$X(t) = e^{At}\sigma + \int_0^t e^{A(t-s)}\Pi_s g(X(s))\,ds - \int_t^\infty e^{A(t-s)}\Pi_u g(X(s))\,ds$$

$$\Updownarrow$$

X is a fixed pt. of T:

$$\boxed{\begin{array}{ll} \text{i.e.} & X = TX \\ \text{or,} & X(t) = (TX)(t) \end{array}}$$

Banach Fixed Point Theorem (II)

Define Let $X: t \in \mathbb{R}_+ \longrightarrow X(t) \in \mathbb{R}^n$

$$\boxed{(TX)(t) = e^{tA}\sigma + \int_0^\infty e^{A(t-s)} \Pi_s g(X(s))\,ds - \int_t^\infty e^{A(t-s)} \Pi_u g(X(s))\,ds}$$

Let $\delta > 0$. $C_\delta = \{ X : X(t) \in \mathbb{R}^n,\ t \geq 0,$ continuous in t, & $\|X(t)\| \leq \delta \}$

Then there is a unique fixed pt. $X \in C_\delta$, i.e. $X = TX$,

$$X(t) = e^{tA}\sigma + \int_0^t e^{A(t-s)} \Pi_s g(X(s))\,ds - \int_t^\infty e^{A(t-s)} \Pi_u g(X(s))\,ds$$

Banach Fixed Point Theorem (II)

Define Let $X: t \in \mathbb{R}_+ \longrightarrow X(t) \in \mathbb{R}^n$

$$\boxed{(TX)(t) = e^{tA}\sigma + \int_0^{\infty} e^{A(t-s)}\Pi_s g(X(s))\,ds - \int_t^{\infty} e^{A(t-s)}\Pi_u g(X(s))\,ds}$$

Let $\delta > 0$. $C_\delta = \left\{ X : X(t) \in \mathbb{R}^n,\ t \geq 0,\ \text{continuous int, } \&\ \|X(t)\| \leq \delta \right\}$

(1) $T : C_\delta \longrightarrow C_\delta$ contraction map

(2) $\|TX - TY\| \leq C\|X - Y\|$, for some $0 < C < 1$

Banach Fixed Point Theorem (II)

Two Properties of $g(x)$

$$\boxed{\|g(x)\| \leq O(|x|^2) \ll |x| \quad \text{for } |x| \ll 1}$$

(1) $\forall \varepsilon > 0, \exists \delta > 0$ such that

if $\|x\| \leq \delta$, then $\boxed{\|g(x)\| \leq \varepsilon \|x\|}$

(2) $\forall \varepsilon > 0, \quad \exists \delta > 0$ such that

if $\|x\|, \|y\| \leq \delta$, then

$$\boxed{\|g(x) - g(y)\| \leq \varepsilon \|x - y\|}$$

Banach Fixed Point Theorem (II)

Recap on the notion of norms

① $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n,$

$$\boxed{\|X\| = \sqrt{x_1^2 + \cdots + x_n^2}}$$

② $X = \{X(t)\} \in C^0(\mathbb{R}_+, \mathbb{R}^n)$

$$\boxed{\|X\|_{C^0(\mathbb{R}_+, \mathbb{R}^n)} = \sup_{t \in \mathbb{R}_+} \|X(t)\|}$$

Properties of the fixed point X (III)

$$\boxed{X(t) = e^{tA}\sigma + \int_0^t e^{A(t-s)}\Pi_s g(X(s))ds - \int_t^\infty e^{A(t-s)}\Pi_u g(X(s))ds}$$

Given $\sigma \in E^s$,

$\Rightarrow$ $X = TX$, fixed pt

$\Rightarrow$ $X(0) = \sigma \underbrace{- \int_0^\infty e^{-As}\Pi_u g(X(s))ds}_{\eta}$

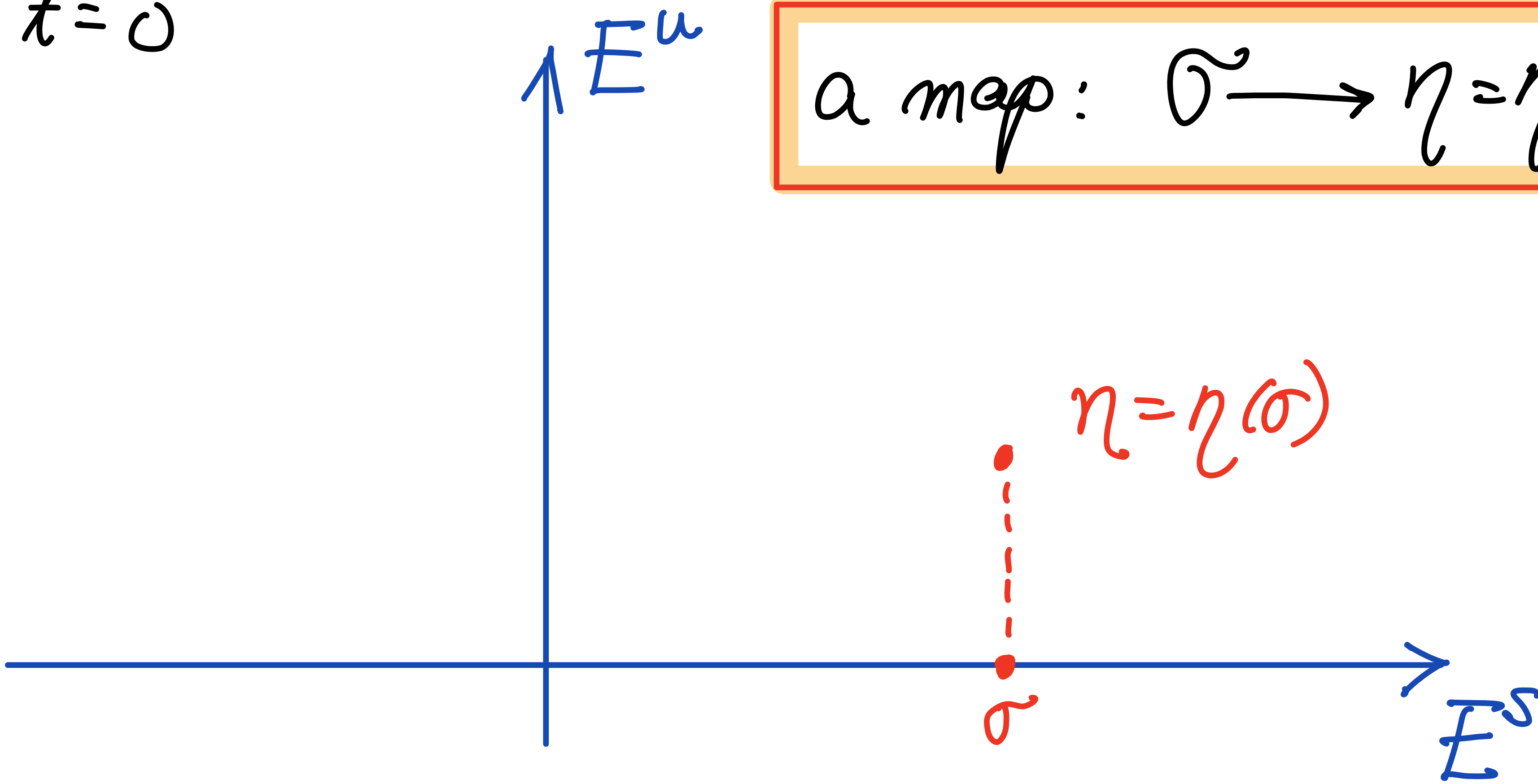
$\Rightarrow$ a map: $\sigma \longrightarrow \eta = \eta(\sigma)$

Properties of the fixed point X (III)

$$X(t) = e^{tA}\sigma + \int_0^t e^{A(t-s)}\pi_s g(X(s))ds - \int_t^\infty e^{A(t-s)}\pi_u g(X(s))ds$$

$t = 0$

a map: $\sigma \longrightarrow \eta = \eta(\sigma)$



Properties of the fixed point X (III)

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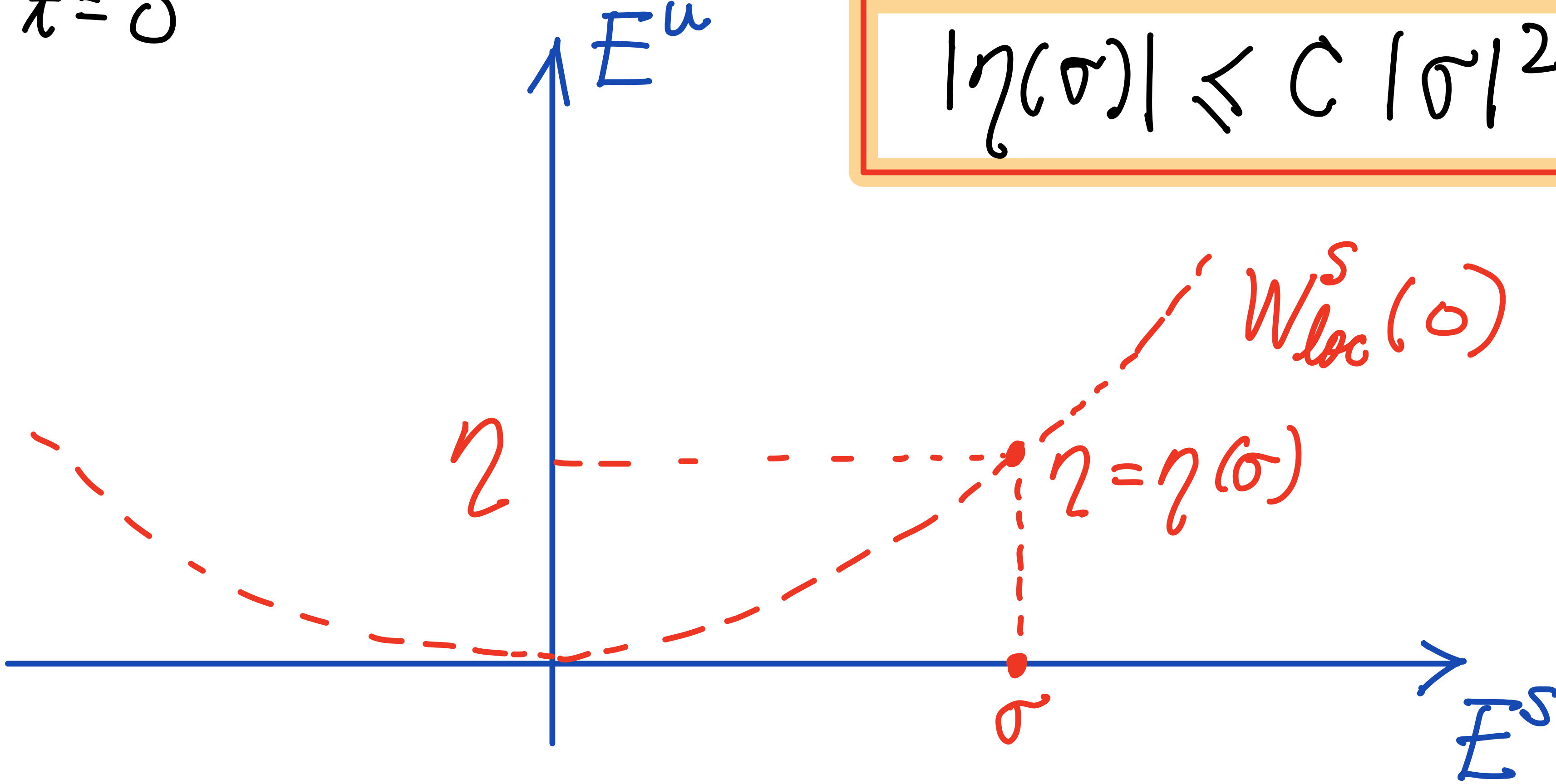
$t = 0$

$$\boxed{|\eta(\sigma)| \leqslant C\,|\sigma|^2} \qquad (1)$$



Properties of the fixed point X (III)

$$\boxed{X(t) = e^{tA}\sigma + \int_0^t e^{A(t-s)}\pi_s g(X(s))ds - \int_t^\infty e^{A(t-s)}\pi_u g(X(s))ds}$$

$$\boxed{|X(t)| \leq a e^{-bt} \text{ for } t \geq 0} \quad (2)$$

for some $a, b > 0$

Properties of the fixed point X (III)

Generalized Gronwall (M, Lemma 5.10, P. 170)

Let $u : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a bounded function s.t.

$$\boxed{u(t) \leq e^{-\alpha t} M + L \int_0^t e^{-\alpha(t-s)} u(s)\,ds + L \int_t^\infty e^{\alpha(t-s)} u(s)\,ds}$$

$(\alpha, M, L > 0,\ L < \alpha/3)$

Then

$$\boxed{u(t) \leq \frac{M}{\beta} e^{-\left(\alpha - \frac{L}{\beta}\right)t}, \quad \beta = 1 - \frac{2L}{\alpha}}$$