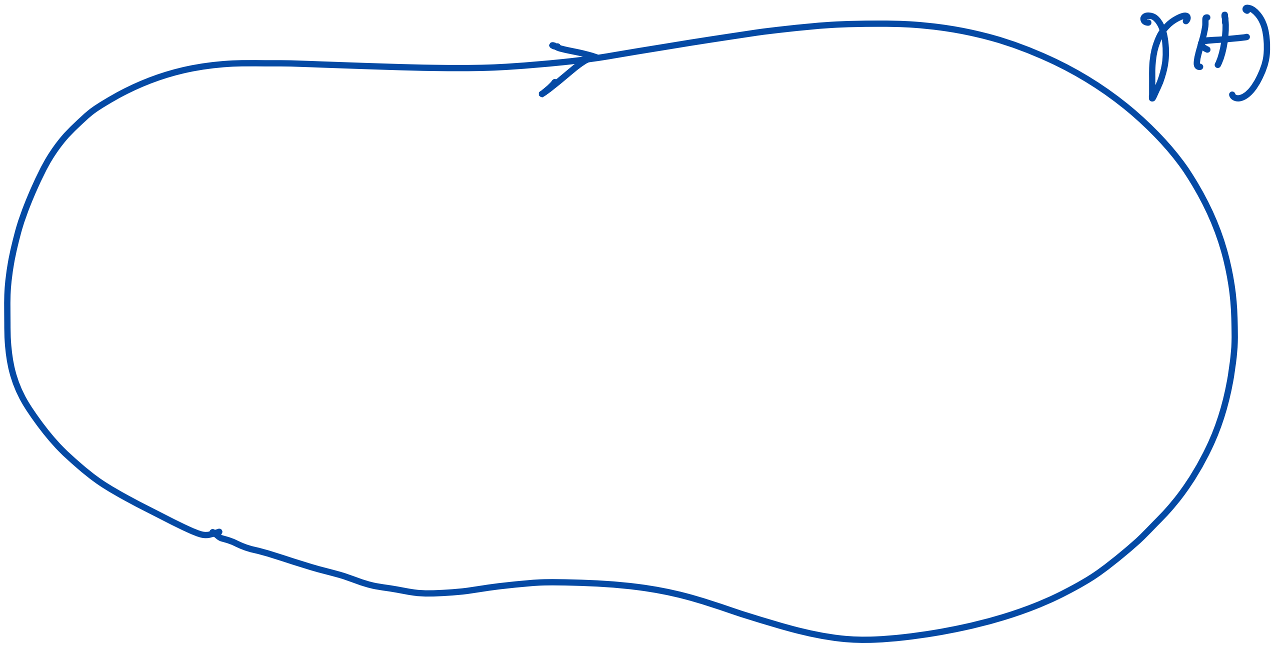
Analysis of Periodic Solutions

Autonomous system: $\dfrac{dX}{dt} = F(X)$

Suppose $\gamma(t)$ is a T-periodic solution

$$\boxed{\frac{d\gamma(t)}{dt} = F(\gamma(t)), \quad \gamma(t+T) = \gamma(t)}$$

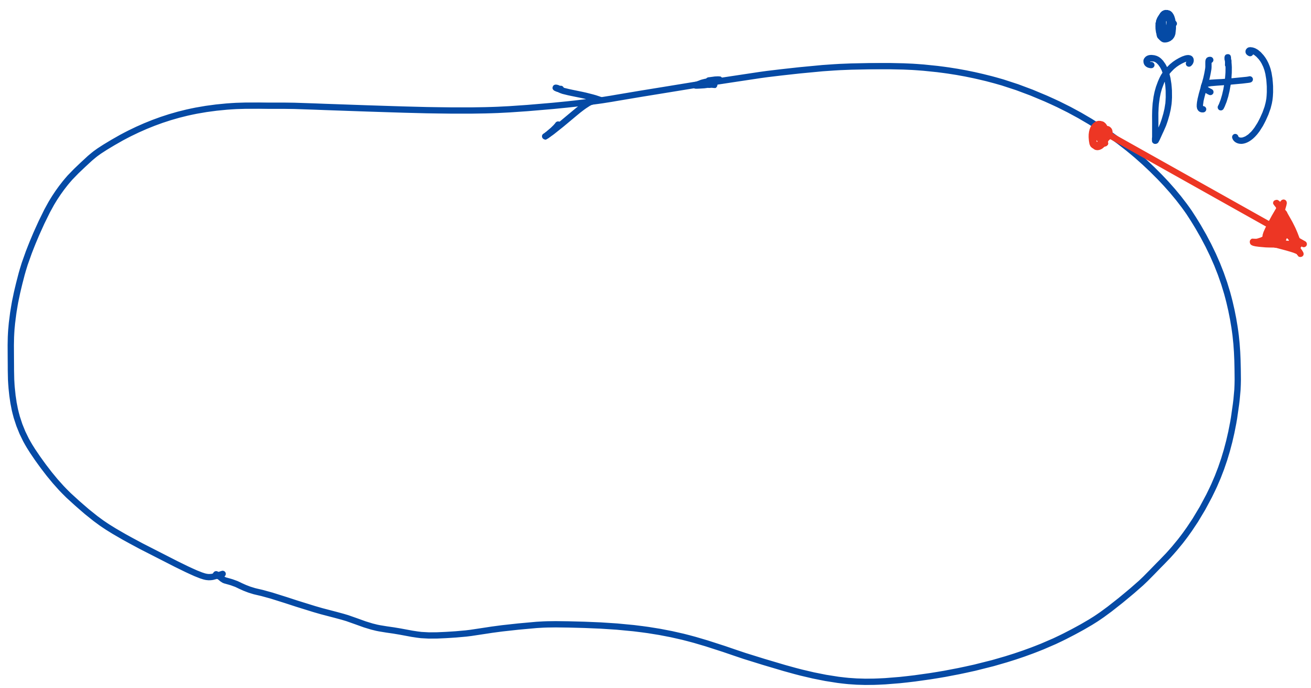


Analysis of Periodic Solutions

Autonomous system: $\frac{dX}{dt} = F(X)$

Suppose $\gamma(t)$ is a **T-periodic solution**

$$\boxed{\frac{d}{dt}\left(\dot{\gamma}(t)\right) = \left[D_X F(\gamma(t))\right]\dot{\gamma}(t), \quad \dot{\gamma}(t+T) = \dot{\gamma}(t)}$$



The velocity vector $\dot{\gamma}(t)$ solves a linear equation with T-per. coefficient:

$$\frac{d}{dt}\dot{\gamma}(t) = [A(t)]\,\dot{\gamma}(t) \qquad [D_x F(\gamma(t))]$$

corresponds to time shift

(1) $\dot{\gamma}(t) = \Phi(t)\,\dot{\gamma}(0)$ Fundamental matrix

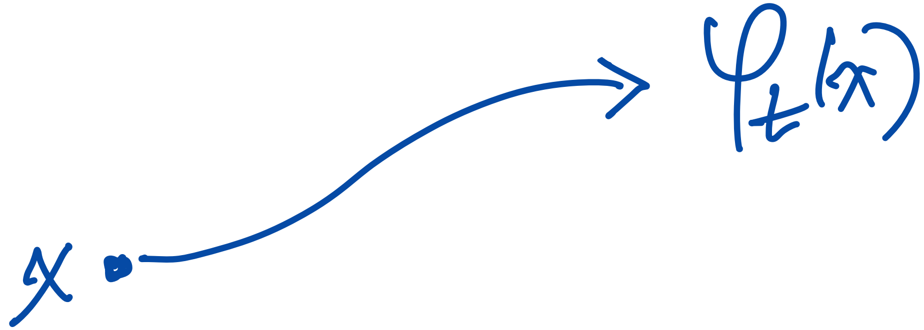
(2) $\dot{\gamma}(0) = \dot{\gamma}(T) = \Phi(T)\,\dot{\gamma}(0)$

(3) $\dot{\gamma}(0)$ is an eigenvector of $\Phi(T)$ with $\lambda = 1$

$(I - \Phi(T))^{-1}$ does not exist.

The Fundamental Matrix

$$\boxed{\Phi(t) = D_x \underbrace{\varphi_t}_{\text{flow map}}(x)}$$



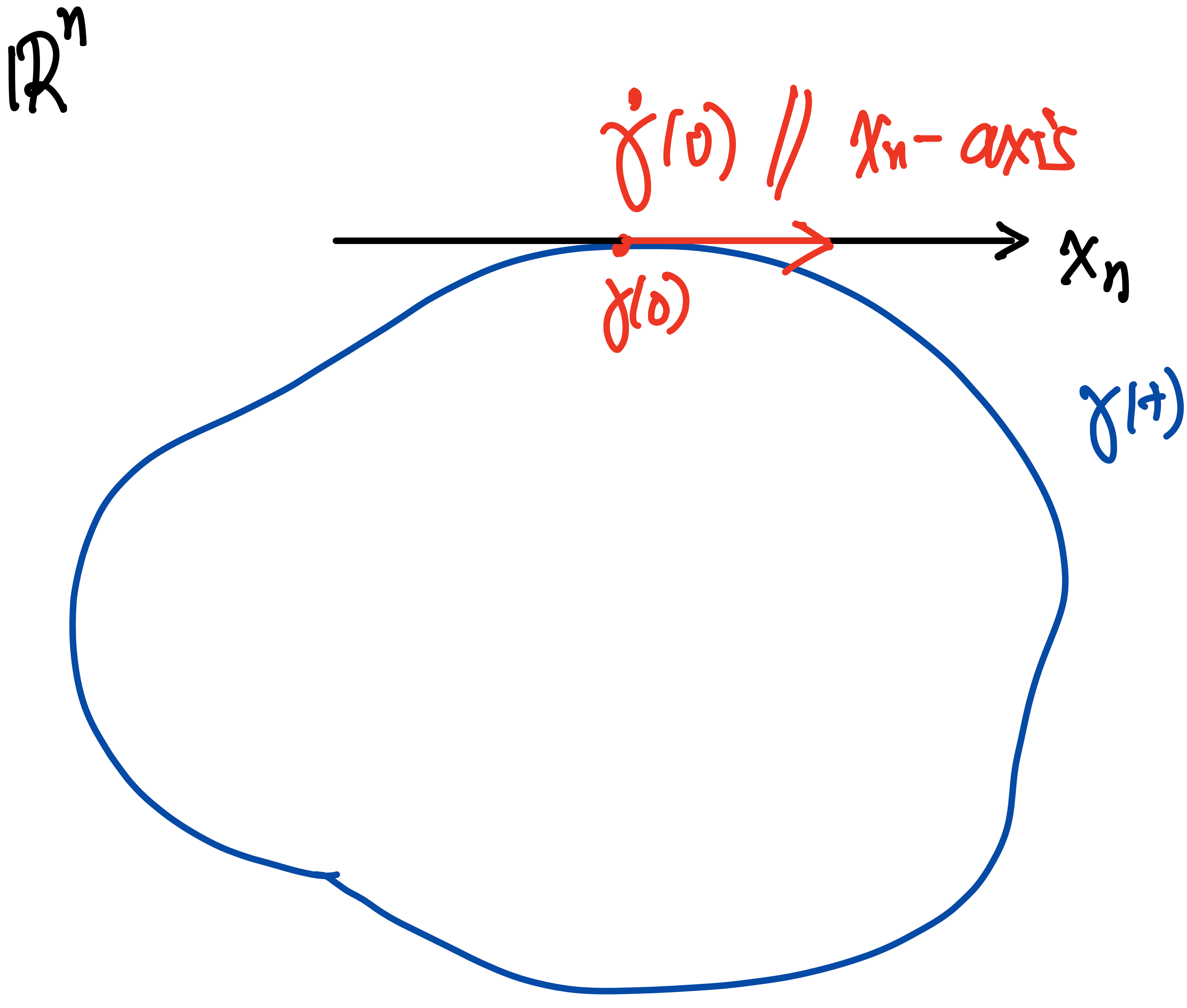
D_x

$$\frac{d}{dt}\varphi_t(x) = F(\underbrace{\varphi_t(x)}_{x}), \quad \varphi_0(x) = \underbrace{x}_{\text{init. data}}$$

$$\frac{d}{dt}\left[D_x \varphi_t(x)\right] = \underbrace{\left[D_x F(\varphi_t(x))\right]}_{A(t)}\left[D_x \varphi_t(x)\right]$$

$$\left[D_x \varphi_0(x)\right] = \left[D_x x\right] = I$$

Poincare Map P (near $\gamma(t)$)

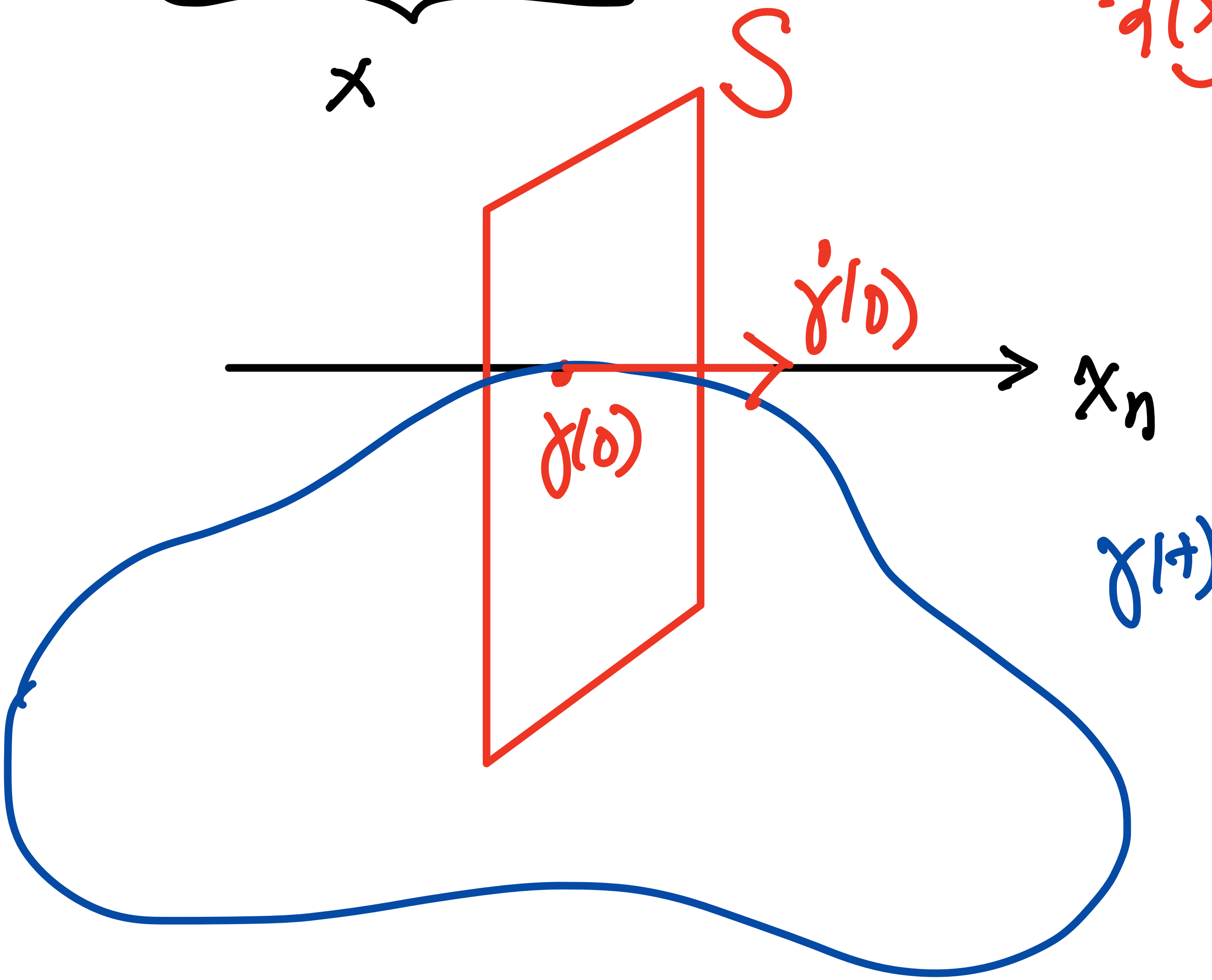


Poincaré Map P (near $\gamma(t)$)

$$\mathbb{R}^n = \{\underbrace{(x_1, x_2, \cdots x_{n-1}, x_n)}_{X}\}$$

$S \cong \mathbb{R}^{n-1}$, cross section

$$= \{\underbrace{(x_1, x_2, \cdots x_{n-1})}_{\tilde{X}}\}$$



$$X = (\tilde{X}, x_n)$$

Poincare Map P (near $\gamma(t)$)

$$\mathbb{R}^n = \{ \underbrace{(x_1, x_2, \cdots x_{n-1}, x_n)}_{x} \}$$

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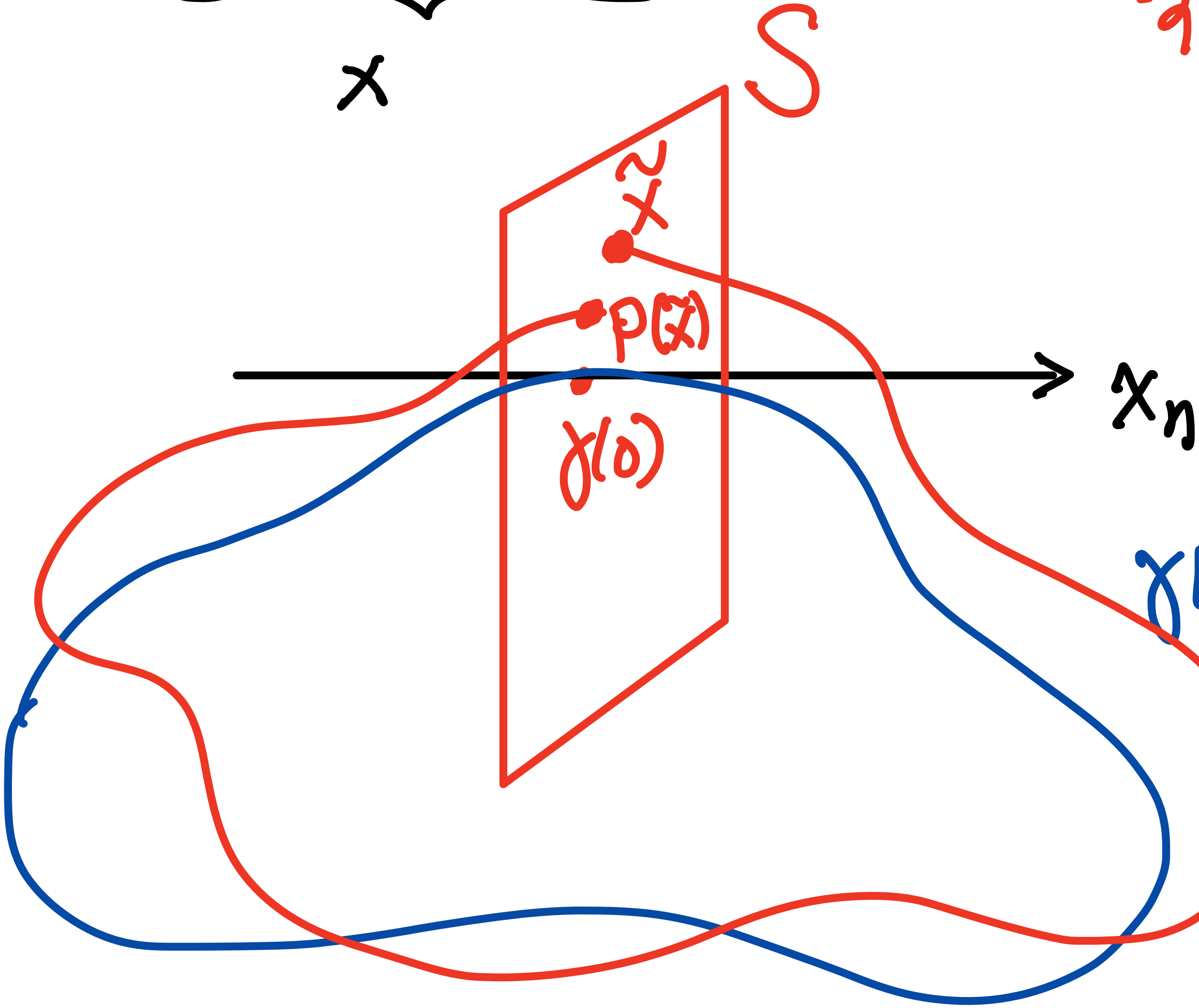

$$\underline{X = (\tilde{x}, x_n)}$$

$X(0) = \tilde{x} \in S$

$P(\tilde{x}) = \varphi_{\tau(\tilde{x})}(\tilde{x})$

$\in S$

Poincaré Map P (near $\gamma(t)$)

$$P : S(\cong \mathbb{R}^{n-1}) \longrightarrow S(\cong \mathbb{R}^{n-1})$$

A solution starting from $X(0) = \tilde{x} \in S$ is periodic if and only if

$$\boxed{P(\tilde{x}) = \tilde{x}},$$

i.e. $\tilde{x}$ is a fixed point of P

Floquet Theory vs Poincare Map

$$A(t) = [D_x F(y(t))] \Longrightarrow \Phi(t)$$

$$\Phi(t) = Q(t)e^{tB}, \qquad Q(t+T) = Q(t)$$

$$\Phi(T) = Q(T)e^{TB} = \underbrace{Q(0)}_{=I} e^{TB} = \underbrace{e^{TB}}_{M}$$

$$\boxed{M = \Phi(T) = [D_x \varphi_T(x)] \quad \text{Monodromy Matrix}}$$

Floquet Theory vs Poincare Map

[M, Thm 4.55]

$$\boxed{\operatorname{Spec}(M) = \operatorname{Spec}(D_{\tilde{x}} P(0)) \cup \{1\}}$$

Floquet Theory vs Poincare Map

[M, Thm 4.55]

collection of eigenvalues

$$\boxed{\mathrm{Spec}(M) = \mathrm{Spec}(D_{\tilde{x}}P(0)) \cup \{1\}}$$

$M^{n\times n}$, n eigenvalues

$(DP)^{(n-1)\times(n-1)}$, $n-1$ eigenvalues

Comes from time shift.

Floquet Theory vs Poincare Map

[M, Thm 4.55]

$$\begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix}^{n\times n} = \left[\begin{array}{c|c} & 0 \\ DP(0)^{(n-1)\times(n-1)} & \vdots \\ & 0 \\ \hline \times\ \times\ \times\ \times\ \times\ \times\times\times\ \times\times & 1 \end{array}\right]$$

Floquet Theory vs Poincaré Map

[M, Thm 4.55]