[M] Sec. 2.9 #2 Hw #1

$$H(x,y) = \frac{1}{2}(y^2 - x^2) + xy$$

$$\dot{x} = H_y = x + y$$
$$\dot{y} = -H_x = x - y$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

(Note: A is symmetric $A^T = A$)

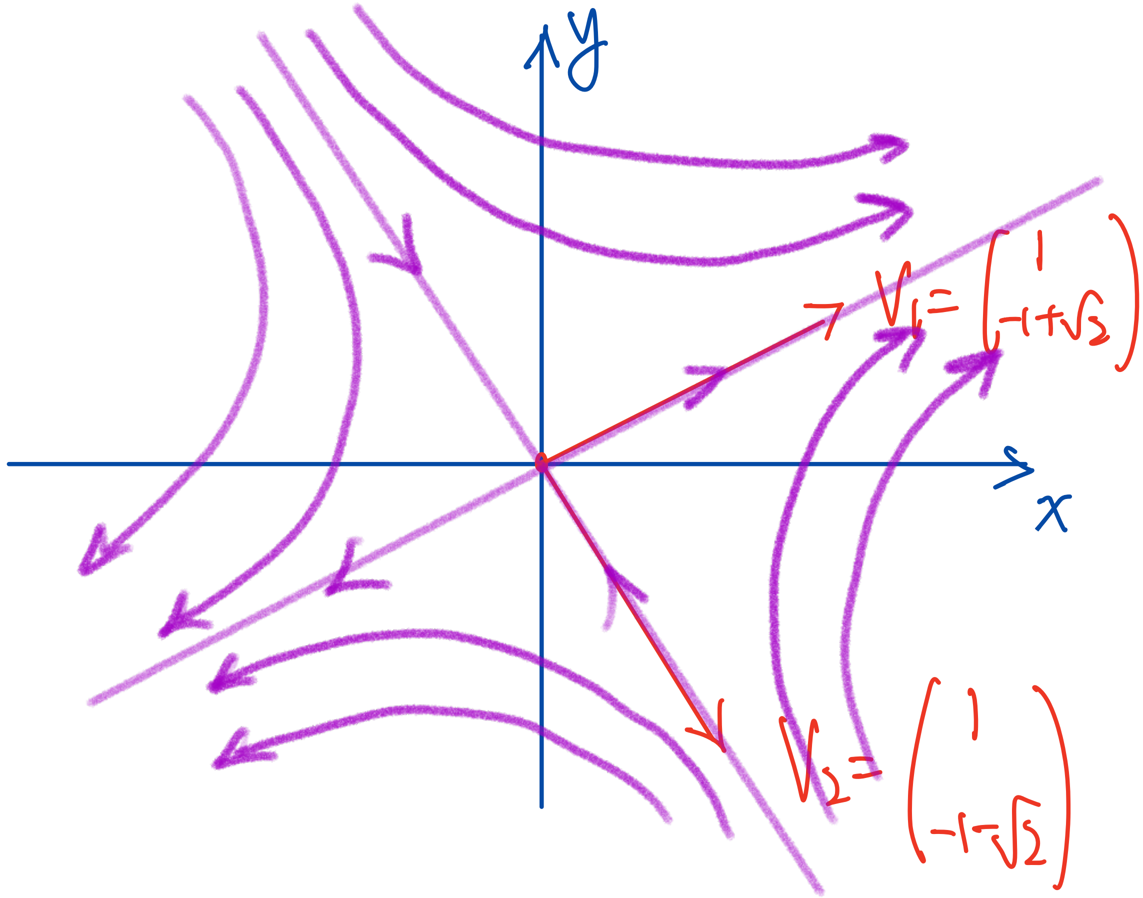
$$\det(A - \lambda I) = 0 \Rightarrow (\lambda+1)(\lambda-1) - 1 = 0$$
$$\lambda^2 - 2 = 0$$
$$\lambda = \pm\sqrt{2}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix} = \sqrt{2} \begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix} \leftarrow V_1$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1-\sqrt{2} \end{bmatrix} = -\sqrt{2} \begin{bmatrix} 1 \\ -1-\sqrt{2} \end{bmatrix} \leftarrow V_2$$

(Note: $V_1 \perp V_2$: $(1)^2 + (-1+\sqrt{2})(-1-\sqrt{2}) = 0$)

General Solution: $X(t) = c_1 e^{\sqrt{2}t} V_1 + c_2 e^{-\sqrt{2}t} V_2$



$H(x,y) = C$ $\leftarrow$ c-level curve of H

$$\frac{1}{2}(y^2 - x^2) + xy = C$$

$$-x^2 + 2xy + y^2 = 2C \qquad C$$

$$\underbrace{(x \quad y)\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}}_{M \ (M = M^T)}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det(M - \lambda I) = \lambda^2 - 2 = 0, \quad \lambda = \pm\sqrt{2}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} = \sqrt{2}\begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} \leftarrow u_1$$

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} = -\sqrt{2}\begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} \leftarrow u_2$$

Normalize u_1, u_2 : $u_1 \to \hat{u}_1 = \dfrac{u_1}{\| \ \|}$

$u_2 \longrightarrow \hat{u}_2 = \dfrac{u_2}{\|u_2\|}$

Diagonalize M :

$$M = \underbrace{\begin{bmatrix} \hat{u}_1 & \hat{u}_2 \end{bmatrix}}_{O} \begin{bmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{bmatrix} \underbrace{\begin{bmatrix} \hat{u}_1 & \hat{u}_2 \end{bmatrix}^T}_{O^T}$$

O // orthogonal matrix

$X = \begin{pmatrix} x \\ y \end{pmatrix}$

$$H(x,y) = (x \ \ y)\, M \begin{pmatrix} x \\ y \end{pmatrix} = X^T M X$$

$$= X^T O D O^T X$$

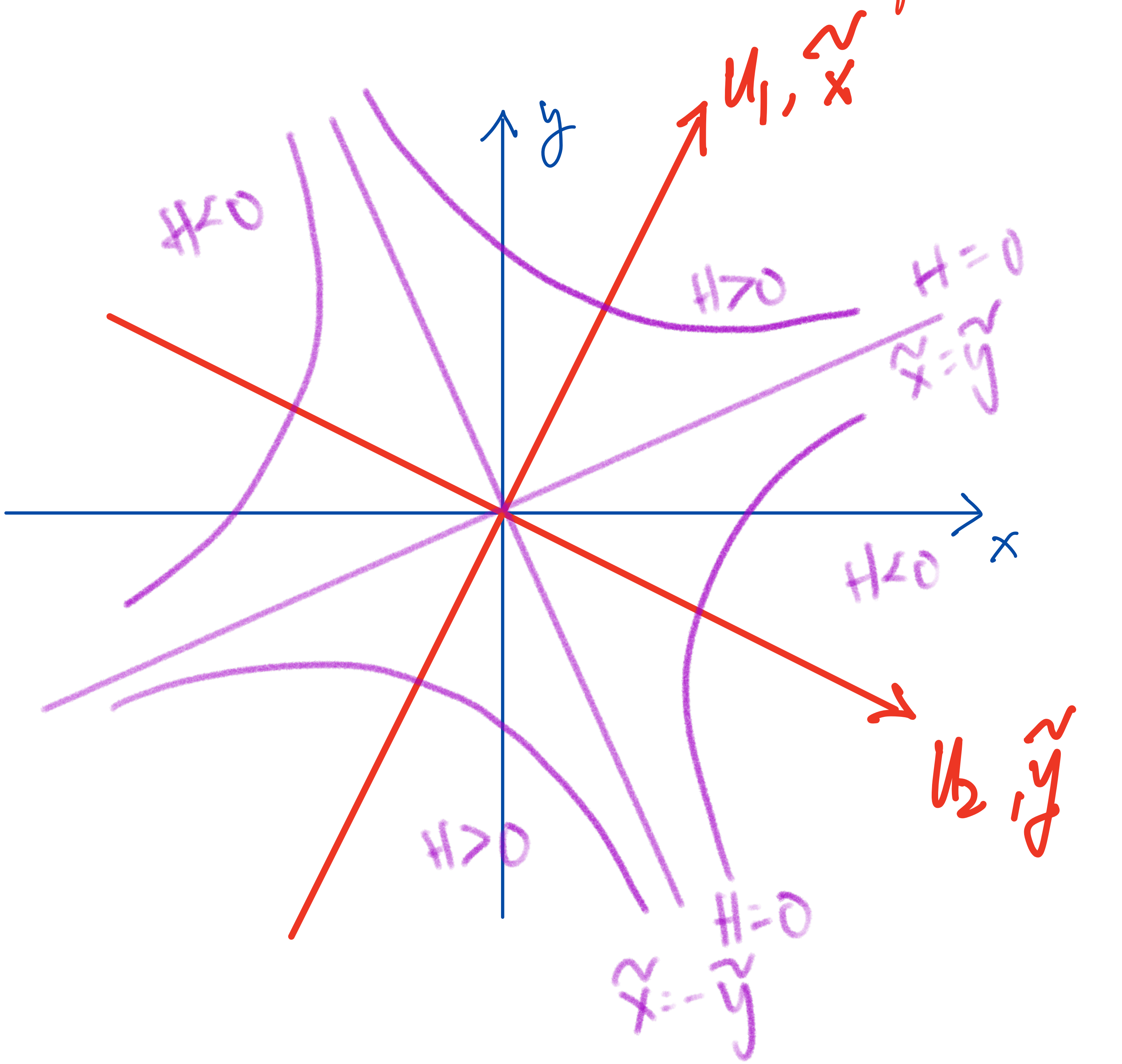
$$= (O^T X)^T D (O^T X) \leftarrow \tilde{X} = O^T X$$

$$= \tilde{X}^T D \tilde{X}$$

Let $\tilde{X} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = O^T X = O^T \begin{pmatrix} x \\ y \end{pmatrix}$

Then $H(x,y) = \frac{1}{2}(y^2 - x^2) + xy = \sqrt{2}\,\tilde{x}^2 - \sqrt{2}\,\tilde{y}^2$

Hence $H = c \iff \boxed{\tilde{x}^2 - \tilde{y}^2 = \frac{c}{2}}$ Hyperbola in the $\tilde{x}\tilde{y}$-plane



[M] Sec. 2.9 #19(c) $\quad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & t \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$y' = -y \Rightarrow y(t) = e^{-t} y_0$$

$$\underline{x' =} \ x + ty = \underline{x + te^{-t} y_0}$$

$$\Rightarrow \quad x(t) = e^t x_0 + \int_0^t e^{(t-s)} s e^{-s} y_0 \, ds$$

$$= e^t x_0 + e^t y_0 \int_0^t s e^{-2s} ds$$

$$= e^t x_0 + e^t y_0 \int s \, d\frac{e^{-2s}}{-2} \Big|_0^t$$

$$= e^t x_0 + e^t y_0 \left[\frac{se^{-2s}}{-2} \Big|_0^t + \frac{1}{2} \int_0^t e^{-2s} ds \right]$$

$$= e^t x_0 + e^t y_0 \left[\frac{te^{-2t}}{-2} - \frac{1}{4} (e^{-2t} - 1) \right]$$

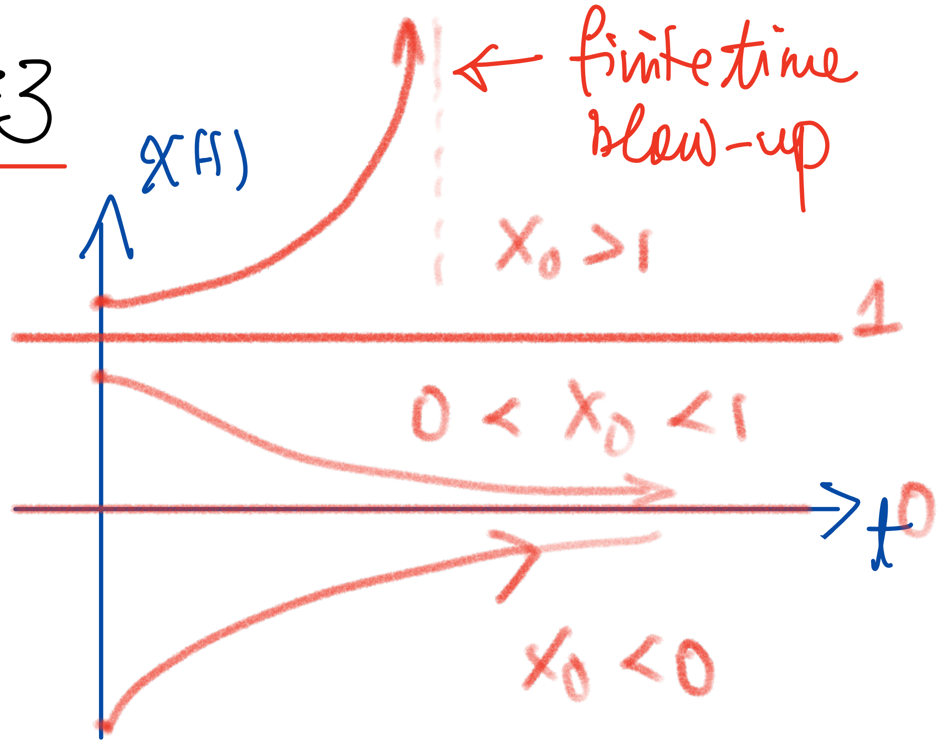
$[\Phi(t)]$

$$= e^t x_0 + \left[-\frac{1}{2} te^{-t} - \frac{1}{4} e^{-t} + \frac{1}{4} e^t \right] y_0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} e^t & -\frac{1}{2} te^{-t} - \frac{1}{4} e^{-t} + \frac{1}{4} e^t \\ 0 & e^{-t} \end{pmatrix}} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Hw #1 Prob #3

$$\dot{x} = -x + x^2$$



$$\frac{dx}{x^2 - x} = dt$$

$$\frac{dx}{x(x-1)} = dt$$

$$\int \frac{1}{x-1} - \frac{1}{x} \, dx = \int dt$$

$$\ln \left| \frac{x-1}{x} \right| \Bigg|_{x_0}^{x} = t$$

$$\ln \left| \frac{x-1}{x} \right| - \ln \left| \frac{x_0 - 1}{x_0} \right| = t$$

1 $X_0 > 1 \Rightarrow X(t) > 1$

$$\ln\left(\frac{x-1}{x}\right) - \ln\left(\frac{x_0-1}{x_0}\right) = t$$

$$\frac{x-1}{x} = \frac{x_0-1}{x_0} e^t$$

$$x-1 = \left(\frac{x_0-1}{x_0}\right) e^t x$$

$$x = \frac{1}{1-\left(\frac{x_0-1}{x_0}\right)e^t} = +\infty \text{ as}$$

$$t = \ln\left(\frac{x_0}{x_0-1}\right)$$

finite time blow up

2 $0 < x_0 < 1 \Rightarrow 0 < X(t) < 1$

$$\ln\left(\frac{1-x}{x}\right) - \ln\left(\frac{1-x_0}{x_0}\right) = t$$

$$\frac{1-x}{x} = \frac{1-x_0}{x_0} e^t$$

$$1-x = \frac{1-x_0}{x_0} e^t x$$

$$x(t) = \frac{1}{1+\left(\frac{1-x_0}{x_0}\right)e^t} \longrightarrow 0$$

as $t \to +\infty$

3

$$x_0 < 0 \Rightarrow x(t) < 0$$

$$\ln\left(\frac{x-1}{x}\right) - \ln\left(\frac{x_0-1}{x_0}\right) = t$$

$$x = \frac{1}{1-\left(\frac{x_0-1}{x_0}\right)e^t} \longrightarrow 0$$

as $t \to +\infty$

(Note: $\frac{x_0-1}{x_0} > 1$

Hence $1-\left(\frac{x_0-1}{x_0}\right)e^t \neq 0$ for all t)

Hw #1 Prob #4

(a) Let $\frac{d\Phi}{dt} = A(t)\Phi, \quad \Phi(0) = I$

$$\frac{d\Psi}{dt} = -\Psi A(t), \quad \Psi(0) = I$$

Consider

$$\frac{d}{dt}\left(\Psi\Phi\right) = \dot{\Psi}\Phi + \Psi\dot{\Phi}$$

$$= -\Psi A\Phi + \Psi A\Phi$$

$$= 0$$

Hence $\Psi(t)\Phi(t) = \Psi(0)\Phi(0) = I$

$$\Rightarrow \Psi(t) = \Phi(t)^{-1}$$

(b) $\det(\Phi(t)) = e^{\int_0^t \operatorname{tr} A(s)\,ds} \neq 0$

$\Rightarrow \Phi(t)^{-1}$ exists.

#2

HW #2

(b) $$e^{At}e^{Bt} = e^{(A+B)t} \quad \forall t$$

$\frac{d}{dt}$

$$Ae^{At}e^{Bt} + e^{At}Be^{Bt} = (A+B)e^{(A+B)t}$$

$\frac{d}{dt}$

$$A^2e^{At}e^{Bt} + 2Ae^{At}Be^{Bt} + e^{At}B^2e^{Bt} = (A+B)^2e^{(A+B)t}$$

$t=0$

$$\cancel{A^2} + 2AB + \cancel{B^2} = (A+B)^2 = (A+B)(A+B)$$

$$= \cancel{A^2} + AB + BA + \cancel{B^2}$$

$$\Longrightarrow \quad \underline{AB = BA}$$

(c) $$e^{At}e^{Bt} = e^{(A+B)t}$$

$$\left(I + At + \frac{A^2t^2}{2} + \cdots\right)\left(I + Bt + \frac{B^2t^2}{2} + \cdots\right)$$

$$= I + (A+B)t + \underline{(A+B)^2 t^2/2} + \cdots$$

$$LHS = I + (A+B)t + \underline{\left(\frac{B^2}{2} + AB + \frac{A^2}{2}\right)} t^2 + \cdots$$

Hence $(A+B)^2 = B^2 + 2AB + A^2$

$$\begin{array}{c} (A+B)^2 \\ \| \\ (A+B)(A+B) \\ \| \\ A^2 + AB + BA + B^2 \end{array} \quad \| \quad \Rightarrow \quad \underline{AB = BA}$$

#4

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -3 & a & b \\ 0 & -2 & c \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\dot{z} = -2z \Longrightarrow z(t) = z_0 e^{-2t}$$

$$\dot{y} = -2y + cz = -2y + cz_0 e^{-2t}$$

$$y(t) = e^{-2t} y_0 + \int_0^t e^{-2(t\cancel{-s})} c z_0 \cancel{e^{-2s}} \, ds$$

$$= e^{-2t} y_0 + c z_0 e^{-2t} \int_0^t ds$$

$$= e^{-2t} y_0 + c z_0 e^{-2t} t$$

$$\dot{x} = -3x + ay + bz$$

$$= -3x + a y_0 e^{-2t} + a c z_0 e^{-2t} t + b z_0 e^{-2t}$$

$$x(t) = e^{-3t} x_0$$

$$+ \int_0^t e^{-3(t-s)} \left[a y_0 e^{-2s} + a c z_0 e^{-2s} s + b z_0 e^{-2s} \right] ds$$

$$= e^{-3t} x_0 + e^{-3t} \int_0^t \left[a y_0 e^s + ac z_0 s e^s + b z_0 e^s \right] ds$$

$$= e^{-3t} x_0 + e^{-3t} \int_0^t (a y_0 + ac z_0 s + b z_0) e^s \, ds$$

$$= e^{-3t} x_0 + e^{-3t} \int (a y_0 + ac z_0 s + b z_0) \, d e^s \Big|_0^t$$

$$= e^{-3t} x_0 + e^{-3t} \Big[(a y_0 + ac z_0 t + b z_0) e^t - (a y_0 + b z_0) - \int_0^t e^s \, ac z_0 \, ds \Big]$$

$$= \cancel{e^{-3t}} \underline{x_0} + e^{-3t} \Big[(\cancel{a y_0} + \cancel{ac z_0 t} + \cancel{b z_0}) e^t - \cancel{(a y_0 + b z_0)} - \cancel{ac z_0} (e^t \cancel{-1}) \Big]$$

$$= (x_0 - a y_0 - b z_0 + ac z_0) e^{-3t} + (a y_0 + b z_0 - ac z_0 + \underline{ac z_0 t}) e^{-2t}$$

Solution:

$$\begin{cases} x(t) = (x_0 - ay_0 - bz_0 + ac\,z_0)e^{-3t} \\ \qquad + (ay_0 + bz_0 - acz_0 + acz_0 t)e^{-2t} \\ y(t) = e^{-2t}y_0 + cz_0 e^{-2t} t \\ z(t) = e^{-2t} z_0 \end{cases}$$

Because of the te^{-2t} terms

$$\left\| \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \right\| \leq C_k \left\| \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \right\| e^{-kt}$$

for some $0 < \underline{k < 2}$

Depends on k, $C_k \to \infty$ as $k \to 2$

[M Sec 4.13 # 9] Hw#3

$$(A^TS+SA=-I)$$

e^{tA^T} e^{tA}

$$\underbrace{e^{tA^T}A^TSe^{tA}+e^{tA^T}SAe^{tA}}=-e^{tA^T}e^{tA}$$

$$\frac{d}{dt}\left(e^{tA^T}Se^{tA}\right)=-e^{tA^T}e^{tA}$$

$$\underbrace{e^{tA^T}Se^{tA}}_{\downarrow\, t\to+\infty \atop 0}-S=-\int_0^t e^{\tau A^T}e^{\tau A}\,d\tau$$

$$S=\int_0^t e^{\tau A^T}e^{\tau A}\,d\tau$$

(Check

$$A^TS+SA=\int_0^\infty\left(A^Te^{\tau A^T}e^{\tau A}+e^{\tau A^T}e^{\tau A}A\right)d\tau$$

$$=\int_0^\infty\frac{d}{d\tau}\left(e^{\tau A^T}e^{\tau A}\right)d\tau=-I$$

$$=e^{\tau A^T}e^{\tau A}\Big|_0^\infty=-I\ )$$

[M Sec 4.13 #10]

$$L = x^T S x$$

$$\begin{aligned}
\frac{d}{dt} L &= \dot{x}^T S x + x^T S \dot{x} \\
&= (x^T A^T + g^T) S x + x^T S (Ax + g) \\
&= x^T (A^T S + SA) x + g^T S x + x^T S g \\
&= -\|x\|^2 + g^T(x) S x + x^T S g(x) \\
&\leq -\|x\|^2 + \delta \|x\|^2 \\
&\leq -(1-\delta)\|x\|^2 < 0
\end{aligned}$$

$g(x) = o(x)$

i.e. $\|g(x)\| \leq \delta \|x\|$ if $\|x\| \ll 1$

$0 < \delta < 1$

8

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\dot{y} = -2y \Rightarrow y(t) = y_0 e^{-2t}$$

$$\dot{x} = -x + 10y$$

$$x(t) = e^{-t} x_0 + 10\int_0^t e^{-(t-s)} y_0 e^{-2s}\, ds$$

$$= e^{-t} x_0 + 10 y_0 e^{-t} \int_0^t e^{-s} ds$$

$$= e^{-t} x_0 + 10 y_0 e^{-t}(1 - e^{-t})$$

$$= (x_0 + 10y_0) e^{-t} - 10 y_0 e^{-2t}$$

$$x^2(t) + y^2(t)$$

$$= \left((x_0 + 10y_0)e^{-t} - 10 y_0 e^{-2t}\right)^2 + y_0^2 e^{-4t}$$

$$= (x_0 + 10y_0)^2 e^{-2t} - 20(x_0 + 10y_0) y_0 e^{-3t} + 100 y_0^2 e^{-4t}$$

$$+ y_0^2 e^{-4t}$$

$$= \underbrace{(x_0+10y_0)^2 e^{-2t} - 20(x_0+10y_0)y_0 e^{-3t} + 101 y_0^2 e^{-4t}}_{\leq (0.1)^2 \quad \forall t>0}$$

Do some simple estimate

$$e^{-2t},\ e^{-3t},\ e^{-4t} \leq 1$$

$$x^2(t) + y^2(t)$$

$$\leq (|x_0| + 10|y_0|)^2 + 20(|x_0| + 10|y_0|)|y_0| + 101|y_0|^2$$

$$= |x_0|^2 + 20|x_0||y_0| + 10^2|y_0|^2$$

$$+ 20|x_0||y_0| + 200|y_0|^2 + 101|y_0|^2$$

$$= |x_0|^2 + 40|x_0||y_0| + 401|y_0|^2 \qquad (2ab \leq a^2+b^2)$$

$$\leq |x_0|^2 + 20|x_0|^2 + 20|y_0|^2 + 401|y_0|^2$$

$$= 21|x_0|^2 + 421|y_0|^2$$

$$\leq 421(|x_0|^2 + |y_0|^2) \leq 0.1^2$$

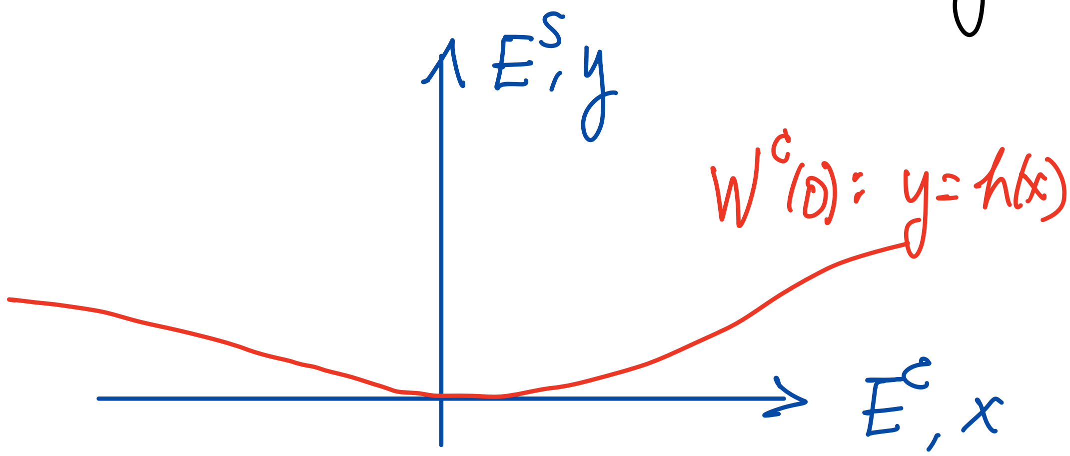
$$\underline{|x_0|^2 + |y_0|^2 \leq \sqrt{\frac{0.01}{421}}}$$

#2 $\quad \dot{x} = xy + ax^3 + by^2x$

$\dot{y} = -y + cx^2 + dx^2y$

HW 4

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} ax^3 + by^2x \\ cx^2 + dx^2y \end{pmatrix}$$



$$h'(x)\,\dot{x} = -h + cx^2 + dx^2h$$

$$h'\left[xh + ax^3 + bh^2x\right] = -h + cx^2 + dx^2h$$

Let $h(x) = \alpha x^2 + \beta x^3 + \gamma x^4 + \delta x^5 + \varepsilon x^6 + \cdots$

$$(2\alpha x + \cancel{3\beta x^2} + 4\gamma x^3 + 5\delta x^4 + \cdots)\times$$
$$\times\left[\alpha x^3 + \cancel{\beta x^4} + \gamma x^5 + \delta x^6 + \cdots + ax^3 + bx(\alpha x^2 + \beta x^3 + \gamma x^4 + \delta x^5 + \cdots)^2\right]$$
$$= -\cancel{\alpha x^2} - \cancel{\beta x^3} - \cancel{\gamma x^4} - \delta x^5 - \cdots$$
$$+ \cancel{cx^2} + dx^2(\cancel{\alpha x^2} + \beta x^3 + \gamma x^4 + \delta x^5 + \cdots)$$

$O(x^2) \Rightarrow \quad -\alpha + c = 0 \Rightarrow \alpha = c$

$O(x^3) \Rightarrow \quad \beta = 0$

$$\underline{h(x) = cx^2 + O(x^4)}$$

<u>Dynamics on $W^c(0)$:</u>

$$\begin{aligned}
\dot{x} &= xy + ax^3 + by^2x \\
&= xh + ax^3 + bh^2x \\
&= x\left(cx^2 + \cdots\right) + ax^3 + bh^2x \\
&= (a+c)x^3 + \cdots
\end{aligned}$$

$$\boxed{a+c \begin{cases} > 0 \Rightarrow \text{unstable} \\ < 0 \Rightarrow \text{stable} \end{cases}}$$

If $\underline{a+c=0}$, Compute $O(x^4)$ in h

$$2\alpha(\alpha + a) = -\gamma + d\alpha$$

$$\cancel{2c(c+a)} = -\gamma + cd \Rightarrow \gamma = cd$$

$$\underline{h(x) = cx^2 + cdx^4 + \cdots}$$

$$\dot{x} = xy + ax^3 + by^2x$$

$$= xh + ax^3 + bh^2x$$

$$= x(\cancel{cx^2} + cdx^4 + \cdots) + \cancel{ax^3}$$

$$+ b(cx^2 + cdx^4 + \cdots)^2 x$$

$$= (cd + bc^2)x^5 + \cdots$$

$$\boxed{cd + bc^2 \begin{cases} > 0 & \text{unstable} \\ < 0 & \text{stable} \end{cases}}$$

<u>If $cd + bc^2 = 0$</u> $\Longrightarrow$ Compute $O(x^5)$ in h

$O(x^5) \Rightarrow 0 = -\delta + d\beta \Rightarrow \delta = 0$

$O(x^6) \Rightarrow 2\alpha(\gamma + b\alpha^2) + 4\gamma(\alpha + a)$

$$= -\varepsilon + d\gamma$$

$$2c(\cancel{cd + bc^2}) + 4\cancel{cd(c+a)} = -\varepsilon + dcd$$

$$\varepsilon = cd^2$$

$$h(x) = cx^2 + cdx^4 + cd^2x^6 + \cdots$$

$$\dot{x} = xy + ax^3 + by^2x$$

$$= xh + ax^3 + b\,h^2 x$$

$$= x\left(cx^2 + cdx^4 + cd^2x^6 + \cdots\right)$$

$$+ ax^3 + b\left(cx^2 + cdx^4 + cd^2x^6 + \cdots\right)^2 x$$

$$= \cancel{cx^3} + \cancel{cdx^5} + cd^2x^7 + \cdots$$

$$+ \cancel{ax^3}$$

$$+ \cancel{bc^2x^5} + b\,2\,c^2 d\, x^7 + \cdots$$

$$= cd\,(d + 2bc)\,x^7 + \cdots$$

$$cd + bc^2 = 0 \implies c(d + bc) = 0$$

assume $c \neq 0$ $\quad$ $bc = -d$

$$\dot{x} = -cd^2x^7 + \cdots$$

$$\boxed{cd^2 \begin{cases} > 0 & \text{stable} \\ < 0 & \text{unstable} \end{cases}}$$

(what if $c = 0$ or even $a, b, \& d = 0$?)

(The above question is from
Carr, Appl. of Centre Manifold Theory)

where $\gamma > 0$ is a constant.

If we substitute $y(t) = h(x(t))$ into the second equation in (1.3.2) we obtain

$$h'(x)[Ax + f(x,h(x))] = Bh(x) + g(x,h(x)). \tag{1.3.6}$$

Equation (1.3.6) together with the conditions $h(0) = 0$, $h'(0) = 0$ is the system to be solved for the centre manifold. This is impossible, in general, since it is equivalent to solving (1.3.2). The next result, however, shows that, in principle, the centre manifold can be approximated to any degree of accuracy.

For functions $\phi\colon \mathbb{R}^n \to \mathbb{R}^m$ which are C^1 in a neighborhood of the origin define

$$(M\phi)(x) = \phi'(x)[Ax + f(x,\phi(x))] - B\phi(x) - g(x,\phi(x)).$$

Note that by (1.3.6), $(Mh)(x) = 0$.

<u>Theorem 3</u>. Let ϕ be a C^1 mapping of a neighborhood of the origin in $\mathbb{R}^n$ into $\mathbb{R}^m$ with $\phi(0) = 0$ and $\phi'(0) = 0$. Suppose that as $x \to 0$, $(M\phi)(x) = O(|x|^q)$ where $q > 1$. Then as $x \to 0$, $|h(x) - \phi(x)| = O(|x|^q)$.

1.4. <u>Examples</u>

We now consider a few simple examples to illustrate the use of the above results.

<u>Example 1</u>. Consider the system

$$\begin{aligned} \dot{x} &= xy + ax^3 + by^2x \\ \dot{y} &= -y + cx^2 + dx^2y. \end{aligned} \tag{1.4.1}$$

By Theorem 1, equation (1.4.1) has a centre manifold $y = h(x)$. To approximate h we set

$$(M\phi)(x) = \phi'(x)[x\phi(x) + ax^3 + bx\phi^2(x)] + \phi(x) - cx^2 - dx^2\phi(x).$$

If $\phi(x) = O(x^2)$ then $(M\phi)(x) = \phi(x) - cx^2 + O(x^4)$. Hence, if $\phi(x) = cx^2$, $(M\phi)(x) = O(x^4)$, so by Theorem 3, $h(x) = cx^2 + O(x^4)$. By Theorem 2, the equation which determines the stability of the zero solution of (1.4.1) is

$$\dot{u} = uh(u) + au^3 + buh^2(u) = (a+c)u^3 + O(u^5).$$

Thus the zero solution of (1.4.1) is asymptotically stable if $a + c < 0$ and unstable if $a + c > 0$. If $a + c = 0$ then we have to obtain a better approximation to h.

Suppose that $a + c = 0$. Let $\phi(x) = cx^2 + \psi(x)$ where $\psi(x) = O(x^4)$. Then $(M\phi)(x) = \psi(x) - cdx^4 + O(x^6)$. Thus, if $\phi(x) = cx^2 + cdx^4$ then $(M\phi)(x) = O(x^6)$ so by Theorem 3, $h(x) = cx^2 + cdx^4 + O(x^6)$. The equation that governs the stability of the zero solution of (1.4.1) is

$$\dot{u} = uh(u) + au^3 + buh^2(u) = (cd+bc^2)u^5 + O(u^7).$$

Hence, if $a + c = 0$, then the zero solution of (1.4.1) is asymptotically stable if $cd + bc^2 < 0$ and unstable if $cd + bc^2 > 0$. If $cd + bc^2 = 0$ then we have to obtain a better approximation to h (see Exercise 1).

<u>Exercise 1</u>. Suppose that $a + c = cd + bc^2 = 0$ in Example 1. Show that the equation which governs the stability of the zero solution of (1.4.1) is $\dot{u} = -cd^2u^7 + O(u^9)$.

<u>Exercise 2</u>. Show that the zero solution of (1.2.2) is asymptotically stable if $a \le 0$ and unstable if $a > 0$.

what if $c = 0$
or even $a, b, c, d = 0$?

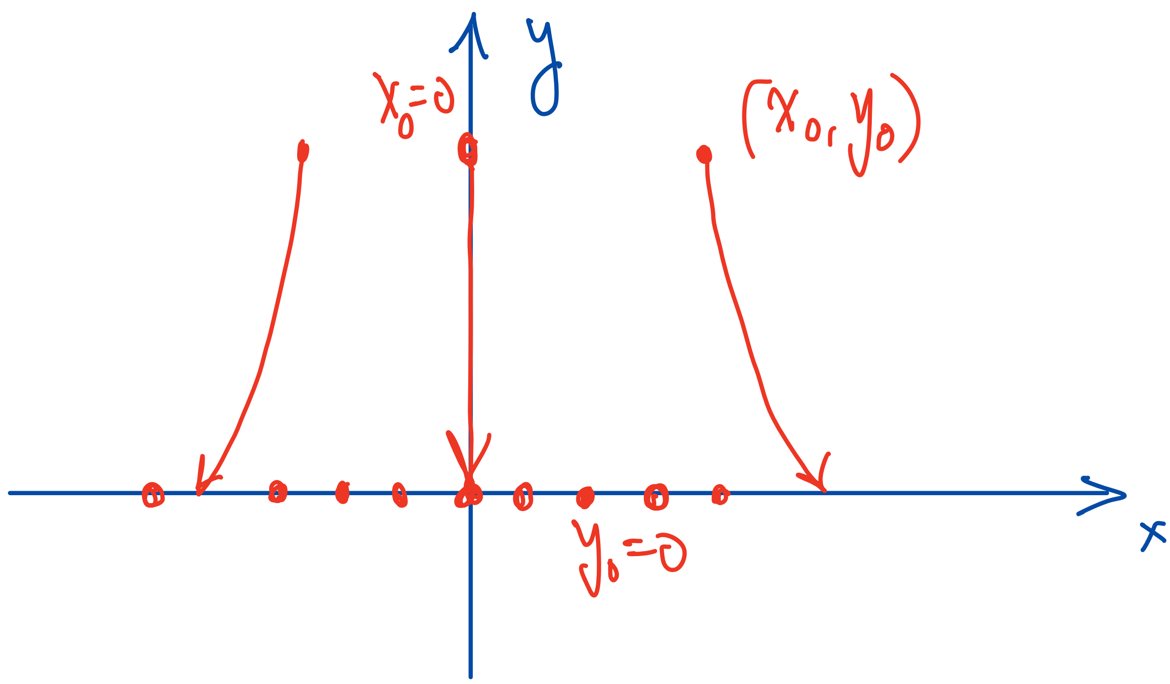
Compute W^s, W^c for $\dot{x} = xy$, $\dot{y} = -y$

$$\dot{x} = xy$$
$$\dot{y} = -y \implies y(t) = y_0 e^{-t}$$

$$\dot{x} = y_0 e^{-t} x$$

$$x(t) = x_0 e^{\int_0^t y_0 e^{-s} ds} = x_0 e^{y_0 (1 - e^{-t})}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{y_0(1-e^{-t})} \\ y_0 e^{-t} \end{pmatrix} \xrightarrow{t \to +\infty} \begin{matrix} x_0 e^{y_0} \\ 0 \end{matrix}$$



W^c = x-axis, W^s = y-axis

#3 $\dot{x} = -\nabla f(x)$ vs $\ddot{x} = -a\dot{x} - \nabla f(x)$

$f(x) = \frac{1}{2}\lambda x^2$

① $\boxed{\dot{x} = -\lambda x \Longrightarrow x(t) = x_0 e^{-\lambda t}, \ \text{rate} = \lambda}$

$\ddot{x} = -a\dot{x} - \lambda x$

$\ddot{x} + a\dot{x} + \lambda x = 0$

$r^2 + ar + \lambda = 0$

$r = \frac{-a \pm \sqrt{a^2 - 4\lambda}}{2}$ $\qquad a = 2\sqrt{\lambda}$

$= -\sqrt{\lambda}$

② $\boxed{x(t) = Ae^{-\sqrt{\lambda}t} + Bte^{-\sqrt{\lambda}t}, \ \text{rate} \sim \sqrt{\lambda}}$

$\underline{\sqrt{\lambda} > \lambda \quad \text{if} \quad \lambda < 1}$