Stability of Periodic Orbits

\[ \frac{d}{dt} X = F(X) \]

minimal

\( X(t) \) is periodic with period \( T > 0 \) if \( X(t+T) = X(t) \) for all \( t \).

\( \gamma(t) \) - periodic solution.

\( \gamma(t) \) is stable if \( X(0) \approx \gamma(0) \) close, then \( X(t) \approx \gamma(t) \) close.

\( \gamma(t) \) is asymptotically stable if \( X(0) \approx X(0) \)

then \( X(t) \to \gamma(t) \) as \( t \to +\infty \)

Unfortunately, the above_defs is "wrong" or not useful.
\[ \text{eg } \quad f(t) \text{ is periodic } T, \]

\[ X(\omega) = \delta(\omega) \quad \text{time shift} \]

Then \[ X(t) = f(\omega + t) \] is a solution!!!

But \[ f(\omega + t) \xrightarrow{t \to +\infty} f(\omega) \]

\[ T = 1.01 \text{ yr} \]

\[ X(\omega + T) \]

\[ X(T) \]

\[ X(\omega) \]

\[ T = 1 \text{ yr} \]

We introduce the following def:

**Orbital Stability**

instead of comparing \[ X(\omega) \sqrt{f(\omega)}, \]
we compare \( \{ X(t) | t \geq 0 \} \) and \( \{ Y(t) | t \geq 0 \} \)

\[ x(0) \]

\[ x(t) \]

(1) \( \Gamma'' = \{ f(y') | y' \geq 0 \} \) is stable

if \( X(0) \) is close to \( f(0) \), then

\( X(t) \) is close \( \Gamma \) for all \( t \).

(2) \( \Gamma \) is asymptotically stable

if \( X(0) \) is close to \( f(0) \), then

\( X(t) \xrightarrow{t \to +\infty} \Gamma \).
$X(t)$ will converge to $\Gamma$ as $t \to \infty$

(3) **Orbital Stability**

If $X(t_0)$ is close to $\Gamma$, then there is a $\delta$ (a time shift) such that:

$$|X(t) - \Gamma(t + \delta)| \leq \delta$$

(Small forces for stability)

$$\lim_{t \to \infty} \delta = 0$$ as $t \to \infty$
Linearization around \( y(t) \)

Write \( X(t) \) as \( y(t) + \delta(t) \)

\[
\frac{d}{dt} (y(t) + \delta(t)) = F(y(t) + \delta(t))
\]

The eqn. satisfied by \( \delta(t) \) is:

\[
\frac{d\delta}{dt} = \left[ D F(y(t)) \right] \delta + O(\delta^2)
\]

Since \( y(t+\delta) = y(t) \)
then \( A(t+\delta) = A(t) \)

\[
\frac{d\delta}{dt} = A(t) \delta + O(\delta^2)
\]
A is $T$ periodic

$x(t) \rightarrow 0 \ ? \ |x(t)| \leq 1 \ ?$

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**Floquet Theorem**

Consider

$$\frac{dX}{dt} = A(t)X$$

Then the fundamental solution of (1)

$$\Phi(t) : \quad \frac{d\Phi}{dt} = A(t)\Phi, \quad \Phi(0) = I$$

can be written as:

$$\Phi(t) = Q(t) e^{tB}$$

where $Q(t)$ is $T$-periodic, $Q(t+T) = Q(t)$

$B$ is a constant matrix.

(Note: all of $\Phi(t), Q(t), e^{tB}$ are invertible.)
For stability, what is important is $e^{TB} = M$ (monodromy matrix).

The solution

$$X(t) = \Phi(t)X_0$$

$$= Q(t)e^{TB}X_0$$

$T$-periodic

$$\frac{dy}{dt} = By$$

$$y(t) = e^{tB}y_0$$

Hence

$$X(t) = Q(t)Y(t)$$
time periodic, invertible

\[ X = Q(t) Y \implies Y = Q^T(t) X \]

\[ X(t) = \Phi(t) X_0 \]
\[ = Q(t) e^{tB} X_0 \]
\[ X(T) = \Phi(T) e^{TB} X_0 \]
\[ X(T) = e^{TB} X_0 = M X_0 \]
\[ X(2T) = \Phi(2T) X_0 \]
\[ = Q(2T) e^{2TB} X_0 \]
\[ = (e^{TB})^2 X_0 \]
\[ = e^{TB} X_0 \]
\[ X(3T) = M^3 X_0 \]

\[ X(nT) = M^n X_0 \]

\[ X(nT) \xrightarrow{n \to \infty} 0 \iff M^n \xrightarrow{n \to \infty} 0 \]
Poincaré Map $P: \Sigma \rightarrow \Sigma$

$\forall x \in \Sigma_{\text{(initial data)}} \Rightarrow \phi_t(x) \in \Sigma_{\text{(solution set)}}$
1) Fixed points of $\mathcal{P}$: $P(x) = x$

periodic orbits
(of course $P(0) = 0$)

2) Stability of periodic orbits (e.g., $P(0) = 0$)

Stability of Poincaré map
Stability of Poincaré Map $P$ at $0$

1. $0$ is **stable** and $P$ if for any $\varepsilon$, there is $\delta$ s.t. $|x(0)| \leq \varepsilon$, then $|P^n(x(0))| \leq \delta$
(b) $0$ is **asymptotically stable** if there is $\varepsilon > 0$, s.t. if $|x(0)| \leq \varepsilon$, then $|P^n(x(0))| \xrightarrow{n \to \infty} 0$.

(c) $0$ is **unstable** if there is $\varepsilon > 0$, s.t. for any $\delta > 0$, there is $x = x(0)$ s.t. $|x| \leq \delta$ but $|P^n(x)| \geq \varepsilon$ for some $n$. 

(a)

(b)
\( (c) \)

\[ \frac{dX}{dt} = A(t)X \quad A(t) = A(t + T) \]

\[ \Phi(t) = \Phi(t + T) = e^{TB} \]

\[ M = e^{TB} \quad \text{monodromy matrix} \]

\[ X(t) = e^{TB}X_0 \]

\[ X(kT) = (e^{TB})^kX_0 = M^kX_0 \]

\[ M^k \rightarrow 0? \]
Consider the map $P: \mathbb{R}^n \rightarrow \mathbb{R}^m \rightarrow \mathbb{R}^m$ with the diagram:

\[ P^{(n)}(x) = P(P(P(\cdots P(x)))) \]

This represents the \emph{n-time composition} of the map $P$.

**Stability of 0**

If $P(0) = 0$,
\[ P^{(n)}(x) \rightarrow 0 \]
Relationship between $M + \rho$

Thm 4.20

$\lambda(M) = \lambda(DP(\theta)) \cup \{1\}$

$\lambda = \text{Spec} \Rightarrow \text{Spectrum (eigenvalues)}$

$M^{n \times n} - n \lambda$'s

$[DP]^{(n-1) \times (n-1)} - (n-1) \lambda$'s

Thm 4.21

If $\lambda(DP) \subseteq \{\lambda : |\lambda| \leq 1\}$

then $f$ is asymptotically stable.

$\lambda$ can be complex.
Lemma 1 is an eigenvalue of $M = e^{TB}$.

$M$ comes from $e^{TB}$, which in turn comes from

$$\frac{dX}{dt} = A(t)X$$
\[ A = 1 \iff \text{time shift} \]

\[ \frac{d\phi(t)}{dt} = F(\phi(t)) \]

\[ \phi(t) = 0 \quad \phi(0) = 0 \]

\[ \frac{1}{\lambda} \left( \frac{\partial}{\partial t} \right) \phi(t) = \frac{1}{\lambda} F(\phi(t)) \]

\[ = (DF) \phi(t) \frac{\partial}{\partial t} \]

\[ \Rightarrow \]

\[ \frac{1}{\lambda} \left( \frac{\partial}{\partial t} \right) \phi(t) = A(t) \frac{\partial}{\partial t} \phi(t) \]

\[ j(t) = Q(t) e^{\mathbf{B}^t} j(0) \]

\[ x(t + T) = x(t) \]
\[ j(t) = \lambda j(0) \]
\[ j(t) = e^{\lambda t} j(0) \]
\[ j(0) = M j(0) \]

\[ \lambda \] is an eigenvalue with eigenvector \( j(0) \).

\[ j(t) = e^{\lambda t} j(0) \]

\[ j(t) \] is still a solution!!
In fact, \( M = \begin{bmatrix} \text{top} \end{bmatrix} \) is such that

\[
(M \dot{\gamma}(0)) = 1 \cdot \dot{\gamma}(0)
\]
\[ M : \Sigma \rightarrow \Sigma \]

\[ x \in \Sigma \iff x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 0 \end{pmatrix} \rightarrow \Sigma \]

\[ M : \text{xn-axis} = \text{span} \{ f(0) \} \rightarrow \text{xn-axis} \]

\[ \sum \rightarrow \sum \]

Thm 4.2: \[ |f(x_{BP})| < 1 \]

\[ \implies \text{stable.} \]

\[ \frac{dx}{dt} = A(T)x \]

\[ X(T) = \overline{A}(T)X_0 = Q(T)e^{AB}X_0 \]

\[ X(kT) = M^kX_0 \]
\[ X(kT) \rightarrow 0 \iff [M^k x_0] \rightarrow 0 \]

\[ \iff \left| MX_0 \right| < \left| X_0 \right| \]

**Contraction**

\[ \lambda(M) < 1 \]

But we are only interested in the action of \( M \) on \( \Sigma \) (not on \( \Sigma_{\text{axis}} \))

\[ \lambda(M) \Big|_{\Sigma} = X_0 \Delta p \]

Hence

\[ \left| x(M) \right|_{\Sigma} < 1 \iff \left| \lambda(DP) \right| < 1 \]