MA543 Lecture 1 (What are ODEs/Exs/Dyn Sys)

[M, Ch. 1]

ODEs (Ordinary Differential Eqns)

(1) Single equation, 1^{st} order

$$\begin{cases} \dfrac{dx}{dt} = f(x,t), & x = x(t) \\ x(0) = x_0 & \leftarrow \text{initial condition} \end{cases}$$

t : independent variable

$x = x(t)$: dependent variables, unknown

(2) 2x2 system, 1^{st} order

$$\begin{cases} \dfrac{dx}{dt} = f(x,y,t) \\ \dfrac{dy}{dt} = g(x,y,t) \end{cases} \qquad \begin{array}{l} x = x(t), \\ y = y(t) \end{array}$$

$x(0) = x_0, \ y(0) = y_0 \leftarrow$ initial condition

(3) $n \times n$ system, 1^{st} order (most general form)

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, x_2, \dots, x_n, t) \\ \dfrac{dx_2}{dt} = f_2(x_1, x_2, \dots, x_n, t) \\ \quad \vdots \qquad \vdots \\ \dfrac{dx_n}{dt} = f_n(x_1, x_2, \dots, x_n, t) \end{cases}$$

$$x_i(0) = x_{i0}$$

In vector form: $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$$F(X,t) = \begin{pmatrix} f_1(x_1, \dots, x_n, t) \\ \vdots \\ f_n(x_1, \dots, x_n, t) \end{pmatrix}$$

$$\boxed{\frac{d}{dt}X = F(X,t), \quad X(0) = X_0}$$

(4) Non-autonomous

$F = F(X, t)$ depends on t explicitly.

$$\frac{dX}{dt} = F(X, t)$$

vs

Autonomous $F = F(X)$,

$$\frac{dX}{dt} = F(X)$$

Non-autonomous can be converted to autonomous

$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $\in \mathbb{R}^n$ introduce $x_{n+1} = t$, $\tilde{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} X \\ x_{n+1} \end{pmatrix}$ $\in \mathbb{R}^{n+1}$

$$\begin{cases} \dfrac{dX}{dt} = F(X, x_{n+1}) \\ \dfrac{dx_{n+1}}{dt} = 1 \end{cases} \iff \frac{d\tilde{X}}{dt} = \begin{pmatrix} F(X, x_{n+1}) \\ 1 \end{pmatrix} = \hat{F}(\tilde{X})$$

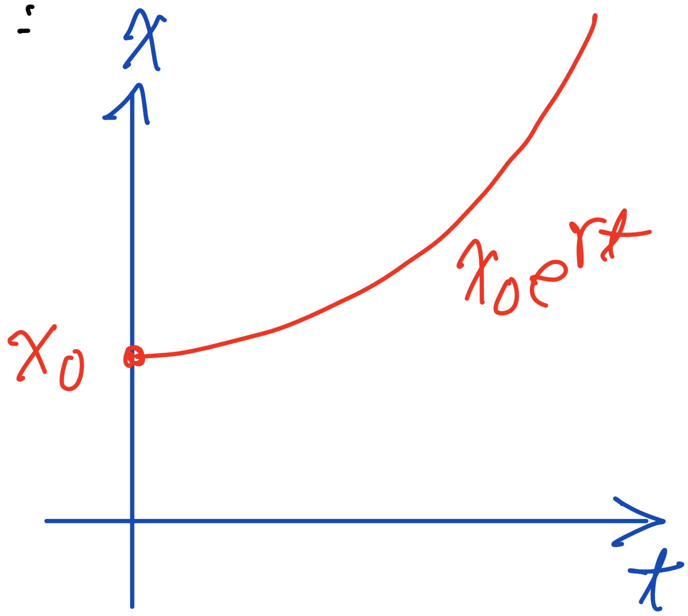
Illustrative Examples of ODEs

(1) Population growth (Single species)

$$\frac{dx}{dt} = rx, \quad x(0) = x_0$$

$$\frac{1}{x}\frac{dx}{dt} = r \leftarrow \text{rate of growth per capita}$$

Separable equation:



$$\int \frac{1}{x}\,dx = \int r\,dt$$

$$\ln|x| + C_1 = rt + C_2$$

$$\ln|x| = rt + C$$

$$|x| = e^{rt+C} = (e^C)e^{rt}$$

$$x = (\pm e^C)e^{rt}$$

$$x = Ce^{rt} \Longrightarrow x(t) = x_0 e^{rt}$$

(2) Logistic Growth

$$\frac{dx}{dt} = rx(1-bx)$$

$$\frac{1}{x}\frac{dx}{dt} = r(1-bx)$$

growth rate

depreciated by reduced/consumed resources

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right) \qquad K = \frac{1}{b}$$

K = carrying capacity

Solve, separable equation

$$\frac{dx}{x\left(1-\frac{x}{K}\right)} = r\,dt$$

$$\frac{1}{x\left(1-\frac{x}{K}\right)} = \frac{1}{x} + \frac{\frac{1}{K}}{1-\frac{x}{K}}$$

$$\int \frac{dx}{x\left(1-\frac{x}{K}\right)} = \int\left(\frac{1}{x} + \frac{\frac{1}{K}}{1-\frac{x}{K}}\right) dx$$

$$= \ln|x| - \ln\left|1-\frac{x}{K}\right|$$

$$= \ln\frac{|x|}{\left|1-\frac{x}{K}\right|}$$

Hence

$$\ln\frac{|x|}{\left|1-\frac{x}{K}\right|} = rt + C$$

$$\frac{x}{1-\frac{x}{K}} = Ce^{rt} \quad \left(C = \frac{x_0}{1-\frac{x_0}{K}}\right)$$

$$x = Ce^{rt} - \frac{Ce^{rt}x}{K}$$

$$\left(1 + \frac{Ce^{rt}}{K}\right)x = Ce^{rt}$$

$$x = \frac{Ce^{rt}}{1 + \frac{Ce^{rt}}{K}}$$

$$\boxed{x = \frac{KCe^{rt}}{K + Ce^{rt}}, \quad C = \frac{x_0}{1 - \frac{x_0}{K}}}$$

$$x(0) = x_0, \quad \lim_{t \to +\infty} x(t) = \frac{KC}{C} = K$$

carrying capacity

$$\frac{dx}{dt} = \underline{rx\left(1-\frac{x}{K}\right) = f(x)}$$

$$\frac{dx}{dt} = \text{slope of } x(t) \text{ (rate of change)}$$

$$f(x) > 0 \implies \frac{dx}{dt} > 0 \implies x \uparrow \text{ in } t$$

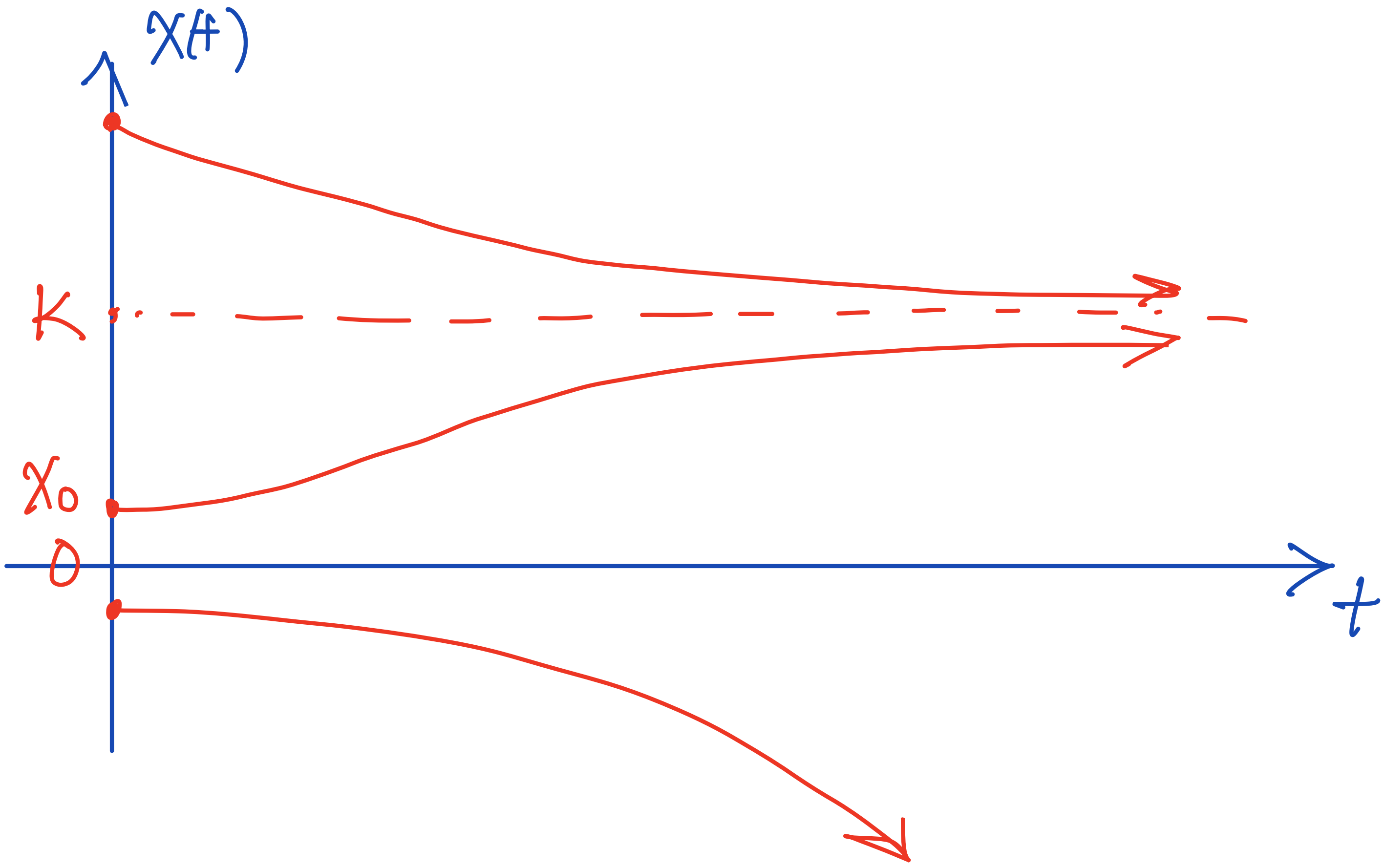
$$f(x) < 0 \implies \frac{dx}{dt} < 0 \implies x \downarrow \text{ in } t$$

$$f(x) = 0 \implies \frac{dx}{dt} = 0 \implies x \text{ - stationary (critical pt.)}$$

$$f(x) = rx\left(1-\frac{x}{K}\right) = \begin{cases} >0 & 0 < x < K \\ =0 & x = 0, K \\ <0 & x > K \end{cases}$$

$$(r, K > 0)$$

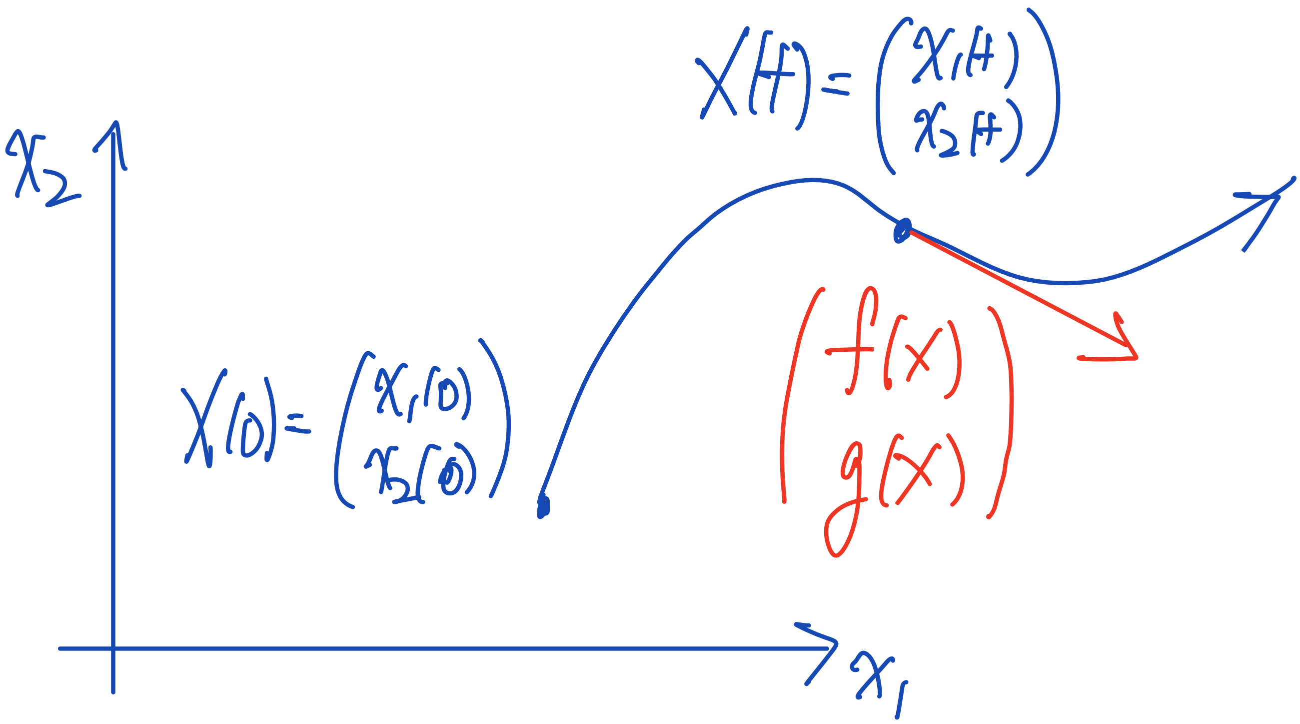
$$(<0 \quad x < 0)$$



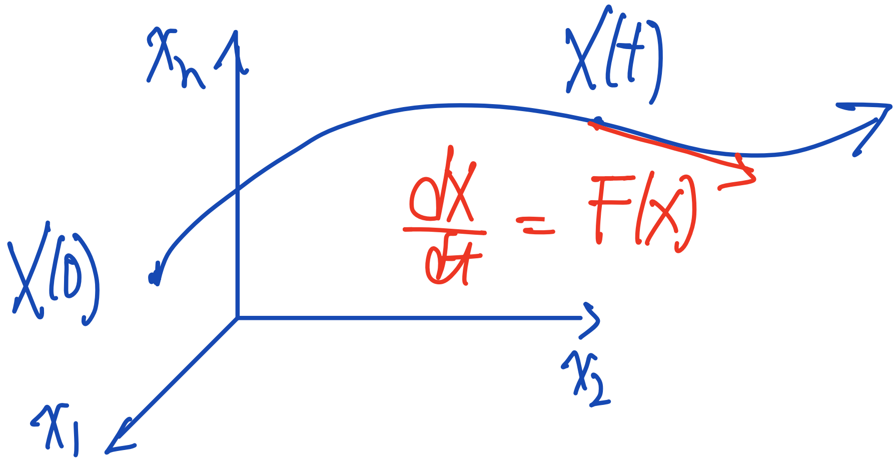
x(t)
K
x_0
0
t

Plotting two (or higher) dimensional systems

$$\begin{cases} \dfrac{dx_1}{dt} = f(x_1, x_2) \\ \dfrac{dx_2}{dt} = g(x_1, x_2) \end{cases} \qquad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



$$\frac{dX}{dt} = F(X) \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$



(3) Population Growth (Interacting species)

(i) Competition models

$$\frac{dx}{dt} = x(A - Bx - Cy)$$

$$\frac{dy}{dt} = y(D - Ex - Fy)$$

(ii) Predator-Prey

x — prey
y — predator

$$\frac{dx}{dt} = x(A - Bx - Cy)$$

$$\frac{dy}{dt} = y(Dx - Ey)$$

(4) Conversion of higher order ODE to 1st order system

$$g\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \ldots, \frac{d^nx}{dt^n}, t\right) = 0$$

(solve for $\frac{d^nx}{dt^n}$)

$$\frac{d^nx}{dt^n} = f\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right)$$

Introduce new variables

$$y_1 = x \qquad y_2 = \frac{dx}{dt} \qquad y_3 = \frac{d^2x}{dt^2} \qquad \ldots \qquad y_n = \frac{d^{n-1}x}{dt^{n-1}}$$

$$\Rightarrow \begin{cases} \dfrac{dy_1}{dt} = y_2 \\ \dfrac{dy_2}{dt} = y_3 \\ \vdots \\ \dfrac{dy_n}{dt} = f(t, y_1, y_2, \ldots, y_n) \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ f(t, y_1, y_2, \cdots, y_n) \end{pmatrix}$$

(5) Hamiltonian flow:

Hamiltonian function

$$H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \qquad H(X, Y)$$

$$(X, Y)$$

"position" "momentum"

$$\dot{X} = +\nabla_Y H(X, Y)$$

$$\dot{Y} = -\nabla_X H(X, Y)$$

(Often $H(X, Y) = V(X) + T(Y)$)

potential energy kinetic energy

$$\dot{X} = \nabla_Y T(Y)$$

$$\dot{Y} = -\nabla_X V(X)$$

H is constant along solution.

$$H(X(t), Y(t))$$

$$\frac{d}{dt} H(X(t), Y(t))$$

$$= (\nabla_x H)\dot{x} + (\nabla_y H)\dot{y}$$

$$= (\nabla_x H)(\nabla_y H) + (\nabla_y H)(-\nabla_x H) = 0$$

Hamiltonian flow with friction

$$\dot{X} = \nabla_y H(X, Y)$$

$$\dot{Y} = -\nabla_x H(X, Y) - \overbrace{\gamma \dot{X}}^{\text{(dissipation)}}$$

$$\frac{d}{dt} H(X, Y) = (\nabla_x H)\dot{x} + (\nabla_y H)\dot{y}$$

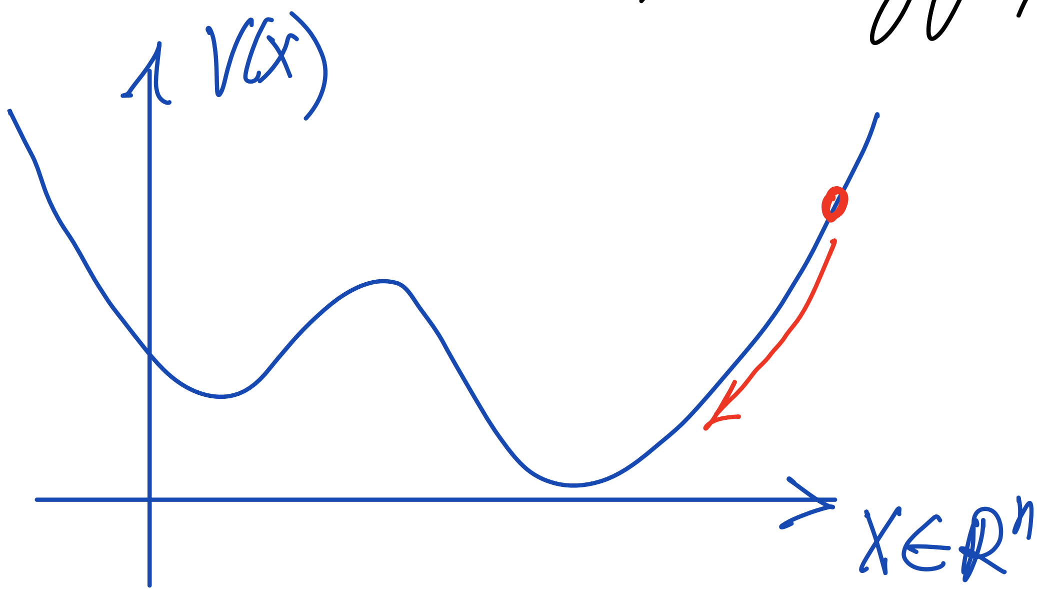
$$= \cancel{(\nabla_x H)(\nabla_y H)} + (\nabla_y H)(\cancel{-\nabla_x H} - \gamma \dot{x})$$

$$= -\langle \nabla_y H, \gamma \dot{x} \rangle$$

$$= -\gamma \langle \nabla_y H, \nabla_y H \rangle < 0 \text{ if } \gamma > 0$$

(b) Gradient flow

$V : \mathbb{R}^n \longrightarrow \mathbb{R}$, energy fct



$$\boxed{\dot{X} = -\nabla V(X)}$$

V decreases along solution

$V(X(t))$

$$\frac{d}{dt} V(X(t)) = \langle \nabla V(X(t)), \dot{X} \rangle$$

$$= \langle \nabla V(X(t)), -\nabla V(X) \rangle$$

$$= -\langle \nabla V(X(t)), \nabla V(X(t)) \rangle$$

< 0 (unless $\nabla V(X) = 0$)
(critical pt.)

Goal of Studying Dynamical Systems

$$\dot{X} = F(X), \quad X(0) = X_0$$

① Understand / Classify long time behaviors

$X(t) \longrightarrow ?$ as $t \to +\infty$

eg. - fixed/stationary pt?

- periodic orbit?

- chaotic behavior (with statistical properties?)

② Stability of description(s)?

③ Dependence of descriptions w.r.t. underlying parameters?

(bifurcation?)