

Existence and Uniqueness of Solutions

- (1) Does the solution exist for all $t > 0$?
 - (2) Is the solution unique?
-

Explicit Solution formula

(I) Linear Equation

$$\dot{x} = ax \quad - a \equiv \text{constant, lin. homog.}$$

$$\dot{x} = ax + h(t) \quad - \text{linear inhomog.}$$

$$\dot{x} = a(t)x + h(t), \quad x(0) = x_0$$

integrating factor $I(t)$ (=?)

$$I \times \dot{x} - I(t)a(t)x = I(t)h(t)$$

$$\text{Let } I(t) = e^{-\int_0^t a(s)ds}$$

$$\text{Then } \dot{I}(t) = e^{-\int_0^t a(s)ds} (-a(t)) = -a(t)I$$

$$I(t) \dot{x}(t) + \dot{I}(t) x(t) = I(t) h(t)$$

$$\frac{d}{dt} (I(t) x(t)) = I(t) h(t)$$

$$\int_0^t \frac{d}{ds} (I(s) x(s)) ds = \int_0^t (I(s) h(s)) ds$$

$$I(t) x(t) - \cancel{I(0)} \cancel{x(0)} = \int_0^t I(s) h(s) ds$$

1 x_0

$$I(t) x(t) = x_0 + \int_0^t I(s) h(s) ds$$

$$x(t) = I(t)^{-1} x_0 + I(t)^{-1} \int_0^t I(s) h(s) ds$$

$$= e^{\int_0^t a(r) dr} x_0$$

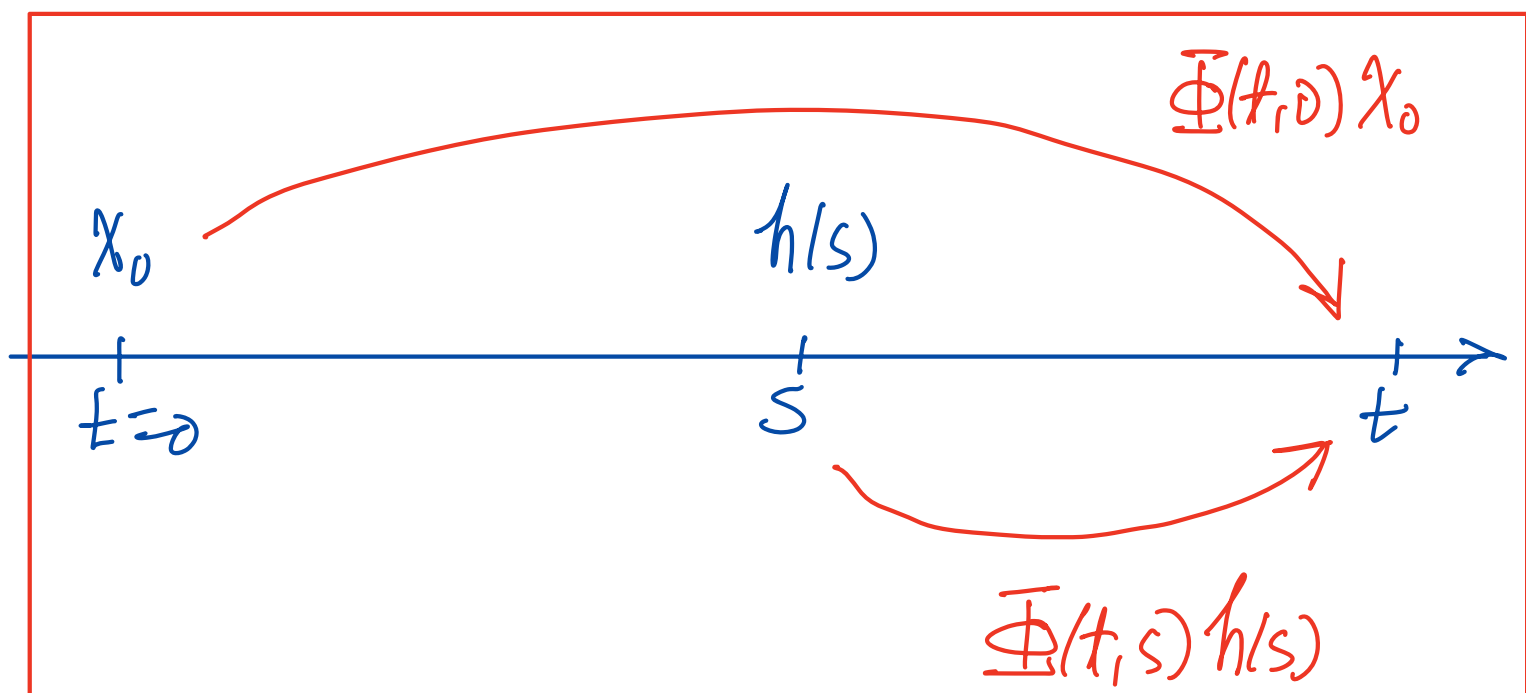
$$+ e^{\int_0^t a(r) dr} \int_0^t e^{\int_0^s -a(r) dr} h(s) ds$$

$$= e^{\int_0^t a(r) dr} x_0 + \int_0^t e^{\int_0^t a(r) dr} e^{\int_0^s -a(r) dr} h(s) ds$$

$$x(t) = e^{\int_0^t a(r) dr} x_0 + \int_0^t e^{\int_s^t a(r) dr} h(s) ds$$

(solution exists for as long as $\int_s^t a(r) dr$ is well-defined.)

$$\Phi(t,s) = e^{\int_s^t a(r) dr} : \text{propagator from } s \text{ to } t$$



(II) Nonlinear Example

$$\begin{cases} \dot{x} = x^p \\ x(0) = x_0 (>0) \end{cases} \quad \underline{p > 1 \text{ (eg. } p=2\text{)}} \quad \uparrow \text{super-linear}$$

$$\int \frac{dx}{x^p} = \int dt$$

$$\int_{x_0}^{x(t)} \frac{dx}{x^p} = \int_0^t dt = t$$

$$\frac{x^{-p+1}}{-p+1} \Big|_{x_0}^{x(t)} = t$$

$$\frac{x(t)^{-p+1} - x_0^{-p+1}}{-p+1} = t$$

$$x(t)^{-p+1} = x_0^{-p+1} - (p-1)t$$

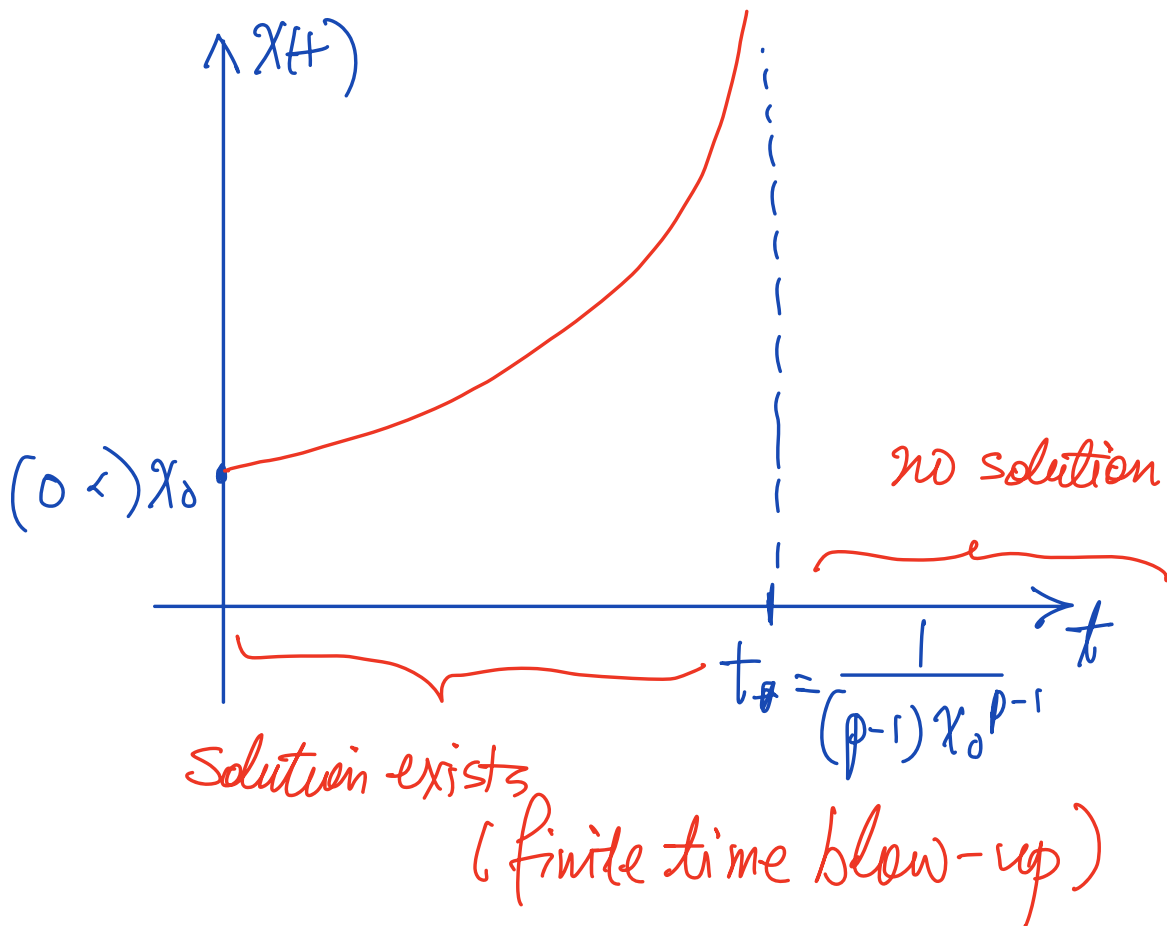
$$X(t) = \left(\frac{1}{x_0^{-p+1} - (p-1)t} \right)^{\frac{1}{p-1}}$$



Solution exists until

$$x_0^{-p+1} = (p-1)t$$

$$t = t_* = \left(\frac{1}{p-1} \right) \frac{1}{x_0^{p-1}}$$



(Note: The larger the x_0 is, the smaller is the existence interval.)

$$\begin{cases} \dot{x} = x^p & 0 < p < 1, \text{ (eg. } p = \frac{1}{2} \text{)} \\ x(0) = x_0 = 0 \end{cases}$$

$$\begin{cases} \dot{x} = x^{\frac{1}{2}} \\ x(0) = 0 \end{cases}$$

(1) Trivial Solution: $x(t) \equiv 0$

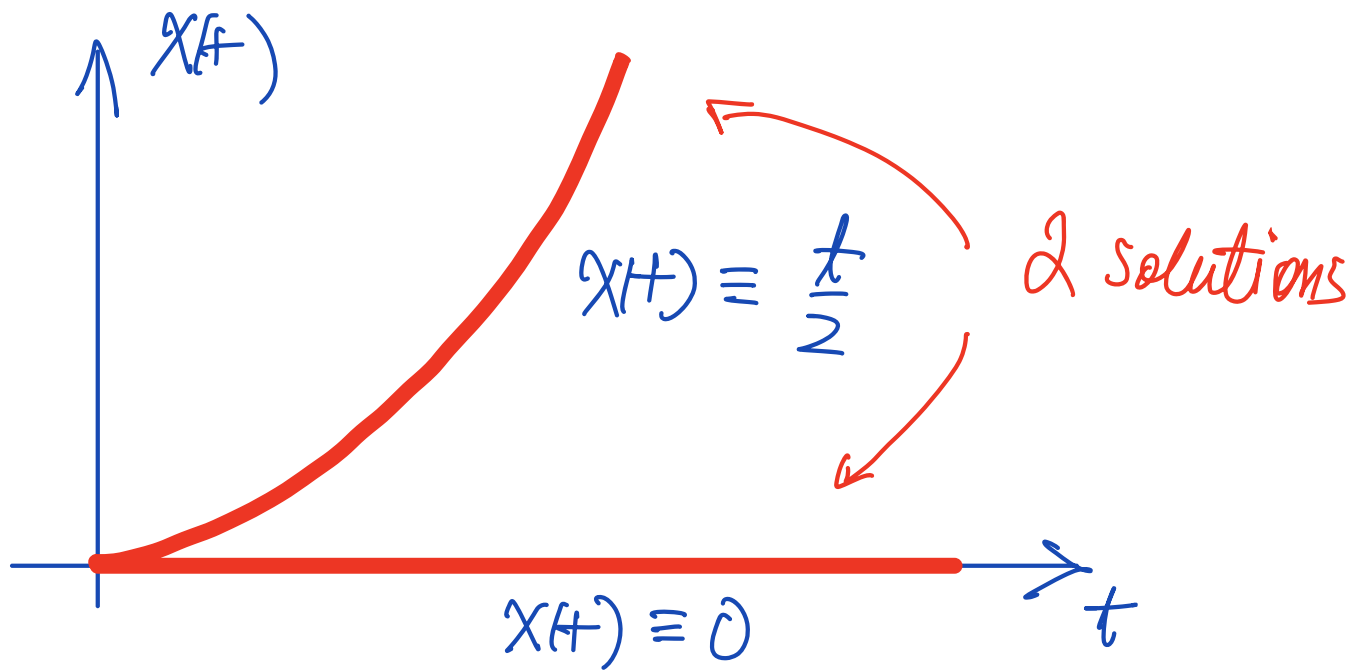
(2) Non-trivial Solution:

$$\int_0^{x(t)} \frac{dx}{x^{\frac{1}{2}}} = \int_0^t dt$$

$$2(x^{\frac{1}{2}}(t) - \cancel{x_0^{\frac{1}{2}}}) = t$$

$$x(t) = \left(\frac{t}{2}\right)^2 \quad (x(0) = 0)$$

$$\left(\dot{x} = 2\left(\frac{t}{2}\right)^{\frac{1}{2}} = \frac{t}{2} = x^{\frac{1}{2}} = \frac{t}{2} \right)$$



In fact, there are infinitely many solns

$$X(t) = \begin{cases} 0 & 0 \leq t \leq t_* \\ \left(\frac{t - t_*}{2}\right)^2 & t \geq t_* \end{cases}, \quad \boxed{t_* = \text{Waiting period}}$$

