Linear System

Lec 3

$$\frac{d\vec{X}}{dt} = A\vec{X} + \vec{h}(t), \quad X(0) = X_0$$

$A \equiv$ a constant matrix (autonomous)

$$\frac{dX}{dt} = A(t)X + h(t)$$

A depends on time (non-autonomous)

$$\left(X, h \in \mathbb{R}^n; \quad A \in \mathbb{R}^{n\times n} \right)$$

Constant coefficient matrix A

$$e^{At} := I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots$$

$$\left(= \sum_{i=0}^{\infty} \frac{(tA)^i}{i!} \right)$$

Properties of e^{At} (Matrix Exponential)

(1) $e^{At}\Big|_{t=0} = I$

(2) $\frac{d}{dt}\left(e^{At}\right) = Ae^{At} = e^{At}A$

(3) $e^{At}e^{As} = e^{A(t+s)}$

(4) $\left(e^{At}\right)^{-1} = e^{-At}$, i.e. $\left(e^{At}\right)\left(e^{-At}\right) = I$

i.e. e^{At} is invertible

(5) If $AB = BA$, then

$$e^{At}e^{Bt} = e^{Bt}e^{At} = e^{(A+B)t}$$

(In fact, $e^{At}e^{Bt} = e^{(A+B)t}$ for all t if and only if $AB = BA$ [M, p.64 #6])

(Use $I(t)=e^{-At}$ as integrating factor.)

$$\frac{dX}{dt} = AX + h(t)$$

$$\frac{dX}{dt} - AX = h(t)$$

$$e^{-At}\frac{d}{dt}X \underbrace{- A e^{-At}}_{= \frac{d}{dt} e^{-At}} X = e^{-At}h(t)$$

$$e^{-At}\frac{d}{dt}X + \left(\frac{d}{dt}e^{-At}\right)X = e^{-At}h(t)$$

$$\frac{d}{dt}\left(e^{-At}X\right) = e^{-At}h(t)$$

$$\int_0^t \frac{d}{ds}\left(e^{-As}X(s)\right)ds = \int_0^t e^{-As}h(s)\,ds$$

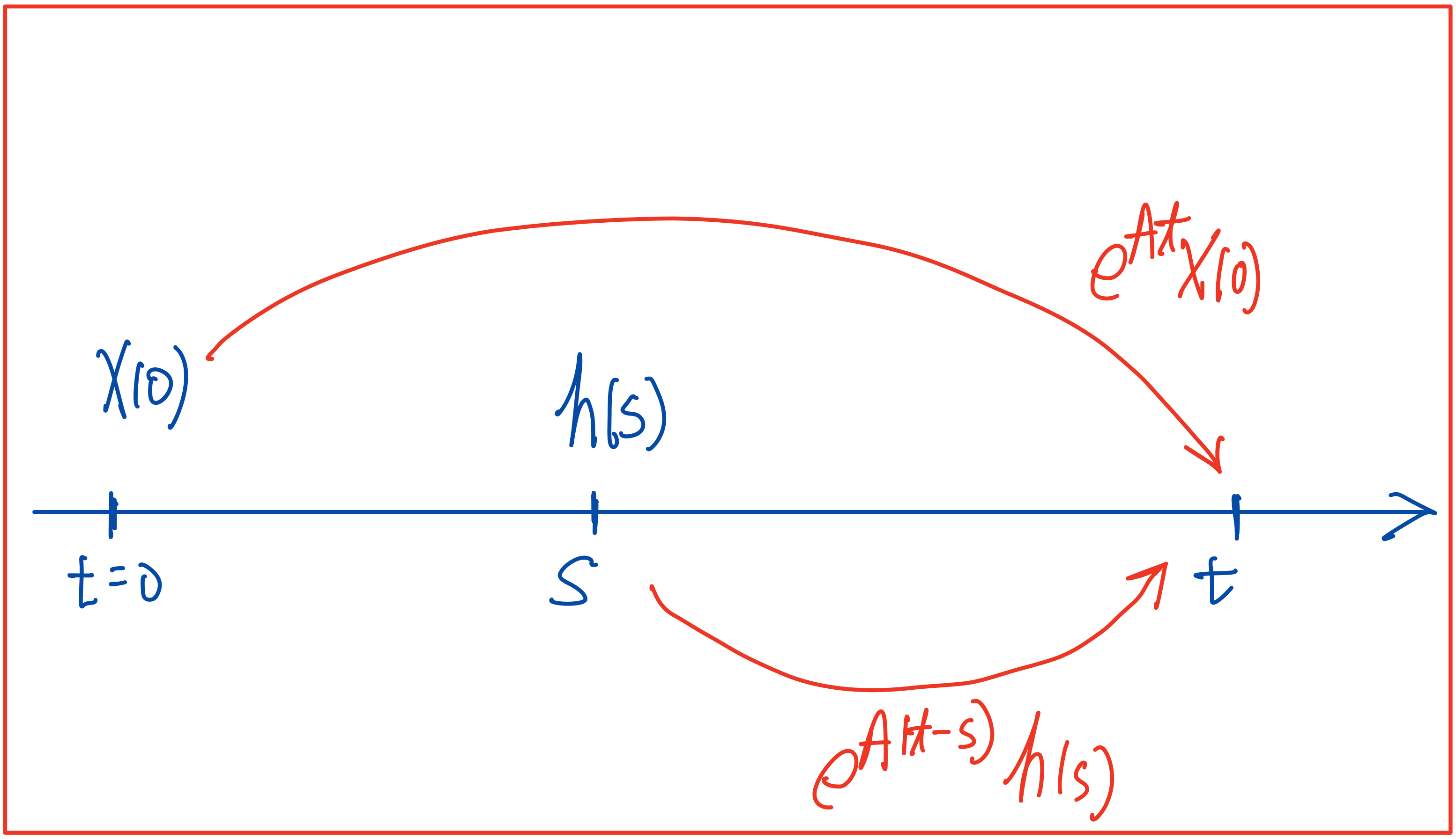
$$e^{-At}X(t) - X(0) = \int_0^t e^{-As}h(s)\,ds$$

$$e^{-At}X(t) = X(0) + \int_0^t e^{-As} h(s)\,ds$$

$$\boxed{\begin{aligned} X(t) &= e^{At}X(0) + e^{At}\int_0^t e^{-As} h(s)\,ds \\ &= e^{At}X(0) + \int_0^t e^{A(t-s)} h(s)\,ds \end{aligned}}$$

(In particular, if $h(s) \equiv 0$, i.e. $\dot{X} = AX$,

then, $X(t) = e^{At}X(0)$.)



[Meiss, p.29]

Example 2.2. Consider the 2×2 system

$$\dot{x} = \begin{pmatrix} -8 & -5 \\ 10 & 7 \end{pmatrix} x. \tag{2.5}$$

The characteristic polynomial is $p(\lambda) = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$, so there are two eigenvalues, each with algebraic multiplicity one, $\lambda_1 = 2$ and $\lambda_2 = -3$. The eigenvector equations (2.2) are

$$(A - 2I)v_1 = \begin{pmatrix} -10 & -5 \\ 10 & 5 \end{pmatrix} v_1 = 0 \;\Rightarrow\; v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

$$(A + 3I)v_2 = \begin{pmatrix} -5 & -5 \\ 10 & 10 \end{pmatrix} v_2 = 0 \;\Rightarrow\; v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This gives the two solutions

$$x_1 = c_1 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad x_2 = c_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad \blacksquare$$

M1 Let $X(t) = C_1(t) V_1 + C_2(t) V_2$

$$\dot{X} = AX$$

$$\left(C_1(t)V_1 + C_2(t)V_2 \right)' = A\left(C_1(t)V_1 + C_2(t)V_2\right)$$

$$\dot{C}_1(t) V_1 + \dot{C}_2(t) V_2 = C_1(t) A V_1 + C_2(t) A V_2$$

$$= C_1(t)\lambda_1 V_1 + C_2(t)\lambda_2 V_2$$

Compare coefficients:

$$\dot{C}_1 = \lambda_1 C_1 \; ; \quad \dot{C}_2 = \lambda_2 C_2$$

$$\Rightarrow C_1(t) = C_1(0) e^{\lambda_1 t} ; \quad C_2(t) = C_2(0) e^{\lambda_2 t}$$

Hence $$x(t) = c_1(0)e^{\lambda_1 t}V_1 + c_2(0)e^{\lambda_2 t}V_2$$

$$= \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}$$

$$= \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$t=0 \Rightarrow$ $$X_0 = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix}^{-1} X_0$$

$$X(t) = \underbrace{\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^{-1}}_{e^{At}} X_0$$

$$\boxed{e^{At} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^{-1}}$$

(M2) Diagonalize A:

$$AV_1 = \lambda_1 V_1 \quad ; \quad AV_2 = \lambda_2 V_2$$

$$\begin{bmatrix} AV_1 & AV_1 \end{bmatrix} = \begin{bmatrix} \lambda_1 V_1 & \lambda_2 V_2 \end{bmatrix}$$

$$A \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} V_1 & V_2 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} V_1 & V_2 \end{bmatrix}^{-1}}_{P^{-1}}$$

$$\boxed{A = P \Lambda P^{-1}}$$

$$e^{At} = I + \frac{At}{1!} + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots\cdots$$

$$A = P\Lambda P^{-1}; \quad A^2 = P\Lambda^2 P^{-1}; \quad A^3 = P\Lambda^3 P^{-1}$$

$$\cdots\cdots \quad A^n = P\Lambda^n P^{-1}$$

$$e^{At} = \underset{\overset{\|}{PP^{-1}}}{I} + \frac{P\Lambda P^{-1}t}{1!} + \frac{P\Lambda^2 P^{-1}t^2}{2!} + \frac{P\Lambda^3 P^{-1}t^3}{3!}$$

$$+ \cdots + \frac{P\Lambda^n P^{-1}t^n}{n!} + \cdots$$

$$= P\left(\underbrace{I + \frac{\Lambda t}{1!} + \frac{\Lambda^2 t^2}{2!} + \cdots + \frac{\Lambda^n t^n}{n!} + \cdots}_{e^{\Lambda t}}\right)P^{-1}$$

$$\Lambda^n = \begin{pmatrix} \lambda_1^n & \\ & \lambda_2^n \end{pmatrix}$$

$$= P\begin{pmatrix} 1 + \frac{\lambda_1 t}{1!} + \frac{\lambda_1^2 t^2}{2!} + \cdots & \\ & 1 + \frac{\lambda_2 t}{1!} + \frac{\lambda_2^2 t^2}{2!} + \cdots \end{pmatrix}P^{-1}$$

$$= P\underbrace{\begin{pmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{pmatrix}}_{e^{\Lambda t}}P^{-1} \qquad \boxed{= Pe^{\Lambda t}P^{-1}}$$

$$= \begin{bmatrix} v_1 & v_2 \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

$$A = P\Lambda P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 2 & \\ & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}^{-1}$$

$$e^{At} = P e^{\Lambda t} P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} & \\ & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} & \\ & e^{-3t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & e^{-3t} \\ -2e^{2t} & -e^{-3t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -e^{2t} + 2e^{-3t} & -e^{2t} + e^{-3t} \\ 2e^{2t} - 2e^{-3t} & 2e^{2t} - e^{-3t} \end{bmatrix}}$$

Non-autonomous Case

$$\boxed{\frac{dX}{dt} = A(t)X + h(t), \quad X(0) = X_0}$$

Variation of Parameters

① Let $\{\Phi(t)\}_{t \geq 0}$ be a matrix function $(n \times n)$ that solve:

$$\boxed{\frac{d\Phi(t)}{dt} = A(t)\Phi(t), \qquad \Phi(0) = I}$$

(Identity matrix)

$$\left[\Phi(t)\right]^{n \times n} = \begin{bmatrix} | & | & & | \\ \varphi_1(t) & \varphi_2(t) & \cdots & \varphi_n(t) \\ | & | & & | \end{bmatrix} \qquad \varphi_i \in \mathbb{R}^n$$

Then $\varphi_i(t)$ solves

$$\boxed{\frac{d}{dt}\varphi_i(t) = A(t)\varphi_i(t), \quad \varphi_i(0) = e_i}$$

② Let $X(t) = \Phi(t)\vec{u}(t)$ $\quad \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}$

$$\left(= u_1(t)\vec{\varphi}_1(t) + \cdots + u_n(t)\vec{\varphi}_n(t) \right)$$

Try to find $\vec{u}(t)$ $(=?)$

$$\frac{dX}{dt} = A(t)X + h(t), \quad X(0) = X_0$$

$$\frac{d}{dt}\left(\underline{\Phi(t)\,u(t)}\right) = A(t)\left(\underline{\Phi(t)\,u(t)}\right) + h(t)$$

$$\cancel{\left(\frac{d}{dt}\Phi(t)\right)u(t)} + \Phi(t)\frac{du(t)}{dt} = \cancel{A(t)\Phi(t)u(t)} + h(t)$$

$$\Phi(t)\frac{du(t)}{dt} = h(t)$$

$\Phi^{-1}(t)$ (assume exists)

$$\frac{du(t)}{dt} = \Phi^{-1}(t)\,h(t)$$

$$\int_0^t \left(\qquad \right) dt$$

$$u(t) - u(0) = \int_0^t \Phi^{-1}(s)\,h(s)\,ds$$

$$u(t) = u(0) + \int_0^t \Phi^{-1}(s)\,h(s)\,ds$$

$$X(t) = \Phi(t)\, u(t)$$

$$= \Phi(t)\, u_0 + \Phi(t) \int_0^t \Phi(s)^{-1} h(s)\, ds$$

$u_0 = ?$

$$X_0 = \Phi(0) u_0 + \cancel{\Phi(0) \int_0^0 \cdots ds}$$

$$= I\, u_0$$

$$\Longrightarrow u_0 = X_0$$

Hence

$$\boxed{\begin{aligned} X(t) &= \Phi(t) X_0 + \Phi(t) \int_0^t \Phi(s)^{-1} h(s)\, ds \\ &= \Phi(t) X_0 + \int_0^t \Phi(t)\Phi(s)^{-1} h(s)\, ds \end{aligned}}$$

Let $\Phi(t,s) = \Phi(t)\Phi(s)^{-1} \qquad (0 \leq s \leq t)$

$$\Phi(t,0) = \Phi(t)$$

Then

$$\boxed{X(t) = \Phi(t,0) X_0 + \int_0^t \Phi(t,s)\, h(s)\, ds}$$

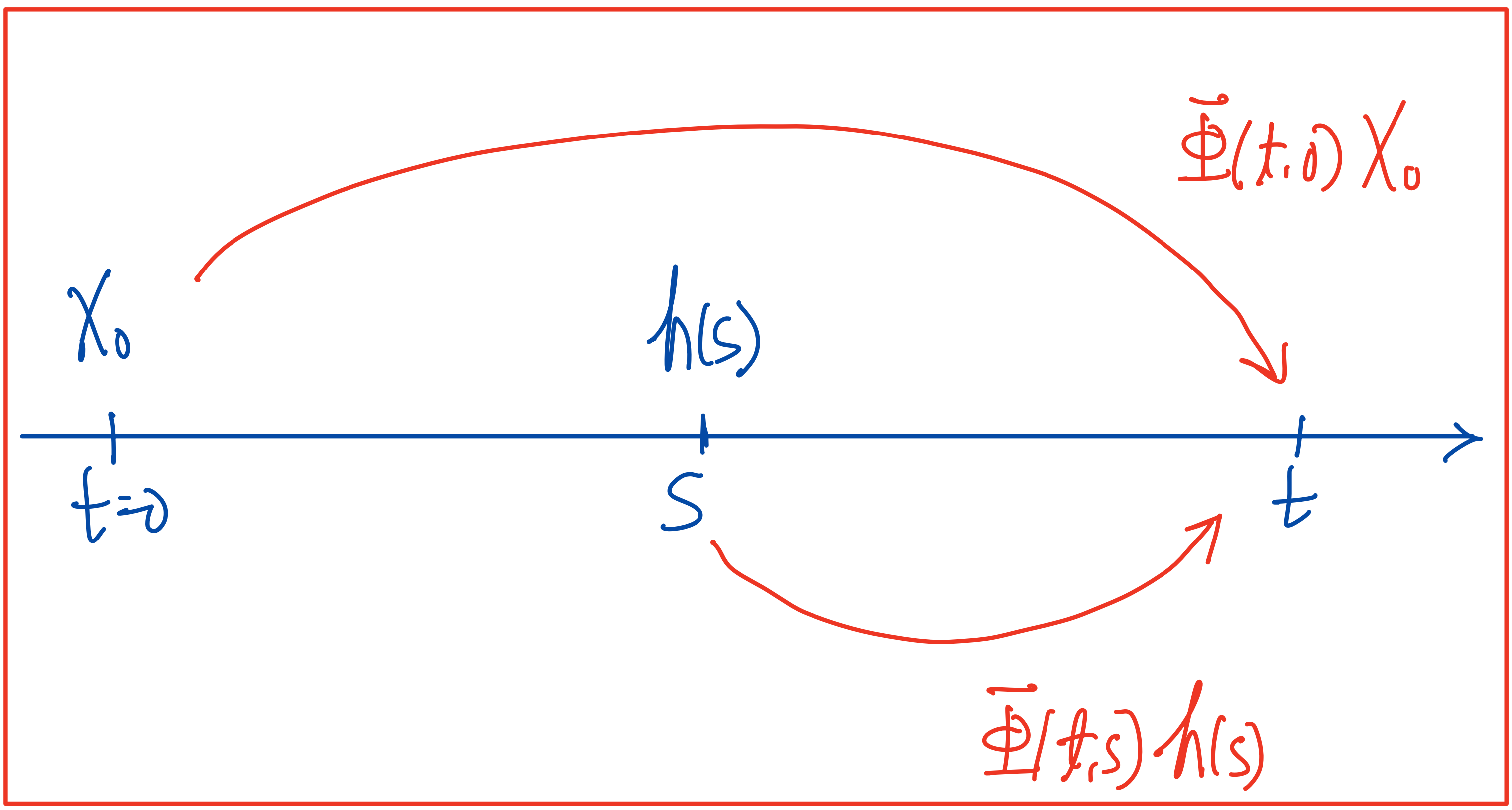
(In particular, if $h(s) \equiv 0$, then $X(t) = \Phi(t,0) X_0$)

$\Phi(t)$ is called the fundamental matrix:

$$\boxed{\frac{d\Phi(t)}{dt} = A(t)\Phi(t), \qquad \Phi(0) = I}$$

$\Phi(t,s)$ is called the propagation operator:

$$\boxed{\frac{d}{dt}\Phi(t,s) = A(t)\Phi(t,s), \qquad \Phi(s,s) = I}$$



Existence of $\Phi^{-1}(t)$

(Method 1) Let $\Psi(t)$ be the solution of

$$\dot{\Psi}(t) = -\Psi(t)A(t), \quad \Psi(0) = I$$

Consider $\frac{d}{dt}\left(\Psi(t)\Phi(t)\right) = ?$

(Method 2) Use Abel's Thm [M, Thm 2.34]

$$\frac{d\Phi(t)}{dt} = A(t)\Phi(t), \quad \Phi(0) = I$$

Then $\frac{d}{dt}\left(\det \Phi(t)\right) = \left(\operatorname{tr} A(t)\right)\left(\det \Phi(t)\right)$

$$\longrightarrow \quad \det \Phi(t) = e^{\int_0^t (\operatorname{tr} A(s))\, ds}$$

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