Computation of e^{At}

(I) $A^{n\times n}$ is diagonalizable

has n (lin. ind) eigenvectors

$$AV_1=\lambda_1 V_1, \; \cdots, \; AV_n=\lambda_n V_n$$

(M I) Consider $\dot{X}=AX$, $X(0)=X_0$

Let $X(t)=c_1(t)V_1+\cdots+c_n(t)V_n$

express $X(t)$ as lin. comb. of the V_i's

$$\dot{X}=AX$$

$$\left(c_1(t)V_1+\cdots+c_n(t)V_n\right)^{\cdot}=A\left(c_1(t)V_1+\cdots+c_n(t)V_n\right)$$

$$\dot{c}_1V_1+\cdots+\dot{c}_nV_n=c_1AV_1+\cdots+c_nAV_n$$

$$=c_1\lambda_1V_1+\cdots+c_n\lambda_nV_n$$

Compare coefficients: $\dot{c}_1 = \lambda_1 c_1, \; \cdots\cdots, \; \dot{c}_n = \lambda_n c_n$

$$\Longrightarrow \quad c_1(t) = c_1(0)e^{\lambda_1 t}, \cdots, \; c_n(t) = c_n(0)e^{\lambda_n t}$$

$$X_n(t) = c_1(0)e^{\lambda_1 t}V_1 + \cdots + c_n(0)e^{\lambda_n t}V_n$$

$$= \begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t}c_1(0) \\ e^{\lambda_2 t}c_2(0) \\ \vdots \\ e^{\lambda_n t}c_n(0) \end{bmatrix}$$

$$= \begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} \underbrace{\begin{bmatrix} c_1(0) \\ \vdots \\ c_n(0) \end{bmatrix}}_{=?}$$

$t=0$

$$X_0 = \underbrace{\begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix}}_{P} \begin{bmatrix} c_1(0) \\ \vdots \\ c_n(0) \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} c_1(0) \\ \vdots \\ c_n(0) \end{bmatrix} = P^{-1} X_0$$

$$x(t) = P \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix} P^{-1} x_0$$

$$= e^{At} x_0$$

$$\boxed{e^{At} = P \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix} P^{-1}}$$

M2

$$AV_1 = \lambda_1 V_1, \; \cdots, \; AV_n = \lambda_n V_n$$

$$[AV_1 \; AV_2 \cdots AV_n] = [\lambda_1 V_1 \; \lambda_2 V_2 \cdots \lambda_n V_n]$$

$$A \underbrace{[V_1 \; V_2 \cdots V_n]}_{P} = \underbrace{[V_1 \; V_2 \cdots V_n]}_{P} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$AP = P\Lambda \Rightarrow \boxed{A = P\Lambda P^{-1}}$$

$$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$A = P\Lambda P^{-1}; \quad A^2 = P\Lambda^2 P^{-1}, \quad A^3 = P\Lambda^3 P^{-1}$$

$$\cdots \cdots \quad A^k = P\Lambda^k P^{-1}$$

Λ - diagonal: $$\Lambda^k = \begin{pmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{pmatrix}$$

$$e^{At} = \underset{\substack{\| \\ PP^{-1}}}{I} + \frac{P\Lambda P^{-1}t}{1!} + \frac{P\Lambda^2P^{-1}t^2}{2!} + \frac{P\Lambda^3P^{-1}t^3}{3!}$$

$$+ \cdots + \frac{P\Lambda P^{-1}t}{n!} + \cdots$$

$$
= P\underbrace{\left(I + \frac{\Lambda t}{1!} + \frac{\Lambda^2 t^2}{2!} + \cdots + \frac{\Lambda^k t^k}{k!} + \cdots\right)}_{= e^{\Lambda t}} P^{-1}
$$

$$
\begin{pmatrix} \frac{\lambda_1^k t^k}{k!} & & \\ & \ddots & \\ & & \frac{\lambda_n^k t^n}{k!} \end{pmatrix}
$$

$$
= P\begin{pmatrix} \overbrace{1 + \frac{\lambda_1 t}{1!} + \frac{\lambda_1^2 t^2}{2!} + \cdots}^{e^{\lambda_1 t}} & & \\ & \ddots & \\ & & \overbrace{1 + \frac{\lambda_n t}{1!} + \frac{\lambda_n^2 t^2}{2!} + \cdots}^{e^{\lambda_2 t}} \end{pmatrix} P^{-1}
$$

$$
\boxed{e^{At} = P e^{\Lambda t} P^{-1} = P\begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix} P^{-1}}
$$

Examples

[Meiss, p. 29]

1 **Example 2.2.** Consider the 2×2 system

$$\dot{x} = \begin{pmatrix} -8 & -5 \\ 10 & 7 \end{pmatrix} x. \tag{2.5}$$

The characteristic polynomial is $p(\lambda) = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$, so there are two eigenvalues, each with algebraic multiplicity one, $\lambda_1 = 2$ and $\lambda_2 = -3$. The eigenvector equations (2.2) are

$$(A - 2I)v_1 = \begin{pmatrix} -10 & -5 \\ 10 & 5 \end{pmatrix} v_1 = 0 \quad \Rightarrow \quad v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

$$(A + 3I)v_2 = \begin{pmatrix} -5 & -5 \\ 10 & 10 \end{pmatrix} v_2 = 0 \quad \Rightarrow \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This gives the two solutions

$$x_1 = c_1 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad x_2 = c_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad \blacksquare$$

$$A = P\Lambda P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 2 & \\ & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}^{-1}$$

$$e^{At} = P e^{\Lambda t} P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} & \\ & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} & \\ & e^{-3t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & e^{-3t} \\ -2e^{2t} & -e^{-3t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -e^{2t} + 2e^{-3t} & -e^{2t} + e^{-3t} \\ 2e^{2t} - 2e^{-3t} & 2e^{2t} - e^{-3t} \end{bmatrix}}$$

② $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, $A\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{V_1} = \underbrace{(-1)}_{\lambda_1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{V_2} = \underbrace{(-3)}_{\lambda_2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & \\ & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} e^{-t}+e^{-3t} & e^{-t}-e^{-3t} \\ e^{-t}-e^{-3t} & e^{-t}+e^{-3t} \end{bmatrix}$$

③ $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$: $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{V_1} = \underbrace{1}_{\lambda_1}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{V_2} = \underbrace{1}_{\lambda_2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}_{V_3} = \underbrace{3}_{\lambda_3}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}^{-1}$$

$$e^{At} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} e^{t} & & \\ & e^{t} & \\ & & e^{3t} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -e^{t} & 0 & e^{3t} \\ 2e^{t} & 0 & 0 \\ 0 & e^{t} & 2e^{3t} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -2 & -1 & 1 \\ 1 & \frac{1}{2} & 0 \end{bmatrix}$$

$$\boxed{e^{At} = \begin{bmatrix} e^{3t} & -\frac{1}{2}e^{t} + \frac{1}{2}e^{3t} & 0 \\ 0 & e^{t} & 0 \\ -2e^{t} + 2e^{3t} & -e^{t} + e^{3t} & e^{t} \end{bmatrix}}$$

(II) $A^{n \times n}$ is not-diagonalizable (defective)

not enough eigenvectors

There is a matrix P whose columns v_i are either

eigenvectors $A v_i = \lambda_i v_i$ i.e. $(A - \lambda_i I) v_i = 0$

or generalized eigenvectors $(A - \lambda_i I)^{k_i} v_i = 0$

Jordan Canonical Form

$$\boxed{A = P \Lambda P^{-1}}$$

where

$$P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_k} \end{bmatrix}$$

$$\begin{bmatrix} J_i \end{bmatrix} = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & 0 & & \lambda_i \end{bmatrix} = \begin{bmatrix} \lambda_i & & \\ & \ddots & \\ & & \lambda_i \end{bmatrix} + \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

Jordan block

$$= \underbrace{\lambda_i I + N_i}_{D_i}$$

Note: ① D_i is diagonal,

② N_i is nilpotent, $N_i^{\square} = 0$

③ $D_i N_i = N_i D_i$, i.e. D_i, N_i commute

$$A = P \Lambda P^{-1} \Longrightarrow e^{At} = P e^{\Lambda t} P^{-1}$$

$$e^{\Lambda t} = \begin{bmatrix} [e^{J_1 t}] & & & \\ & [e^{J_2 t}] & & \\ & & \ddots & \\ & & & [e^{J_k t}] \end{bmatrix}$$

$$e^{J_i t} = e^{(D_i + N_i)t} = e^{D_i t} e^{N_i t}$$

$$= \underbrace{e^{\lambda_i I t}}_{e^{\lambda_i t} I} \underbrace{\left(I + \frac{N_i t}{1!} + \frac{N_i^2 t^2}{2!} + \cdots \right)}_{\text{only finitely many terms survive}}$$

as $N_i^{\square} = 0$

① $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (From solution formula, $e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$)

$$e^{At} = I + At + \underbrace{\frac{A^2t^2}{2!} + \frac{A^3t^3}{2!} + \cdots}_{A^2 = A^3 = \cdots = 0,\ A \text{ is nilpotent}}$$

$$= I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

② $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ (From solution formula, $e^{At} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$)

diagonal Nilpotent

$$A = \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{N} = D + N$$

$DN = ND,\ N^2 = 0$

$$e^{At} = e^{(D+N)t} = e^{Dt}e^{Nt}$$

$$= e^{Dt}\left(I + \frac{Nt}{1!} + \cancel{\frac{N^2t^2}{2!}} + \cdots\right)$$

$$= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

③

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

A has 3 repeating eigenvalues $\lambda = 2, 2, 2$ but only 1 eigenvector $V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{N}$$

Note:

$$N^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$e^{At} = e^{(D+N)t} = e^{Dt}e^{Nt}$$

$$= e^{2It}e^{Nt}$$

$$= \left(e^{2t}I\right)\left(I + \frac{Nt}{1!} + \frac{N^2t^2}{2!} + \cancel{\frac{N^3t^3}{3!} + \cdots}\right) \quad 0$$

$$= e^{2t}\left(I + Nt + \frac{N^2t^2}{2}\right)$$

$$= e^{2t}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & t^2/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= e^{2t}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

④ [Meiss p. 51]

A simple case for this would be an upper triangular matrix with a single eigenvalue λ, such as

$$A = \begin{pmatrix} \lambda & 1 & 1 \\ 0 & \lambda & 2 \\ 0 & 0 & \lambda \end{pmatrix}, \quad S = \lambda I, \quad N = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case

$$N^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad N^3 = 0,$$

and the exponential becomes

$$\boxed{e^{tA} = e^{\lambda t} \begin{pmatrix} 1 & t & t+t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix}.}$$ ∎

Strictly speaking, this is not a Jordan form

Nevertheless, $A = \lambda I + N$

Commute : $(\lambda I) N = N (\lambda I)$

$$e^{At} = e^{(\lambda I + N)t} = e^{\lambda I t} e^{Nt}$$

$$= (e^{\lambda t} I)\left(I + Nt + \frac{N^2 t^2}{2!} + \cancel{\frac{N^3 t^3}{3!}} + \cancel{\cdots}\right)$$

$$\boxed{= e^{\lambda t}\left(I + Nt + \frac{N^2 t^2}{2!}\right)}$$

(III) Laplace Transform

$$\mathcal{L}(e^{At}) = \int_0^{\infty} e^{At} e^{-st} dt = (sI - A)^{-1}$$

$$e^{At} = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)$$

③ $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

$$\mathcal{L}(e^{At}) = (sI - A)^{-1} = \begin{pmatrix} s+2 & -1 \\ -1 & s+2 \end{pmatrix}^{-1}$$

$$= \frac{1}{(s+2)^2 - 1} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+2}{(s+3)(s+1)} & \frac{1}{(s+3)(s+1)} \\ \frac{1}{(s+3)(s+1)} & \frac{s+2}{(s+3)(s+1)} \end{bmatrix}$$

Partial fraction

$$\mathcal{L}(e^{At}) = \begin{bmatrix} \frac{1}{2}\frac{1}{(s+3)} + \frac{1}{2}\frac{1}{(s+1)} & \frac{1}{2}\frac{1}{(s+1)} - \frac{1}{2}\frac{1}{(s+3)} \\ \frac{1}{2}\frac{1}{(s+1)} - \frac{1}{2}\frac{1}{(s+3)} & \frac{1}{2}\frac{1}{(s+3)} + \frac{1}{2}\frac{1}{(s+1)} \end{bmatrix}$$

$\mathcal{L}^{-1}$

$$\boxed{e^{At} = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}}$$