Lec 08

Dependence on initial data and parameter

Continuous Dependent on initial data

$$X(t) = X_0 + \int_0^t F(X(s))\,ds$$

$$Y(t) = \underbrace{Y_0}_{\text{2 different initial data}} + \int_0^t F(Y(s))\,ds$$

$$X(t) - Y(t) = X_0 - Y_0 + \int_0^t \underbrace{\big(F(X(s)) - F(Y(s))\big)}_{\text{Lip, } L.}\,ds$$

$$\underbrace{\|X(t) - Y(t)\|}_{g(t)} \leq \|X_0 - Y_0\| + \int_0^t L\,\underbrace{\|X(s) - Y(s)\|}_{g(s)}\,ds$$

$\Downarrow$ G.I

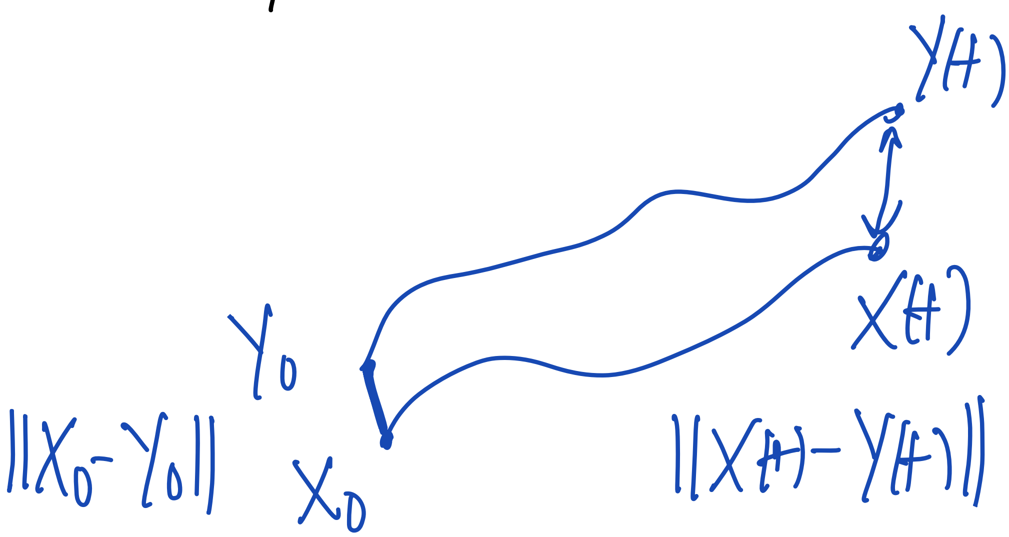
$$\boxed{\|X(t) - Y(t)\| \leq \|X_0 - Y_0\|\, e^{Lt}}$$

① Uniqueness. $\left(X_0 = Y_0 \Rightarrow X(t) = Y(t) \right)$

② The solution depends continuously on the initial data

If $X_0 \to Y_0$, then $X(t) \to Y(t)$

③ An estimate on $\|X(t) - Y(t)\|$ on the difference on initial data



$$\|X(t) - Y(t)\| \leq \|X_0 - Y_0\| e^{Lt}$$

④ Lip. dependence on initial data:

$$\|X(t) - Y(t)\| \leq \left(e^{Lt}\right) \|X_0 - Y_0\|$$

or

$$\frac{\|X(t) - Y(t)\|}{\|X_0 - Y_0\|} \leq \left(e^{Lt}\right)$$

[M, Thm 3.29]

[M, Thm 3.30]

Smooth Dependence on initial data

$$\frac{dX}{dt} = F(X), \quad X(0) = y$$

How does X depend on y?

Let $X = X(t;y)$

$$\frac{d}{dt} X(t,y) = F(X(t,y)), \quad X(0;y) = y$$

Assume F is C^1 :

$D_x F(X)$ exists and is continuous

Assume $D_y X(t;y)$ exists and try to find it.

$$D_y\left(\frac{d}{dt} X(t;y) = F(X(t,y)), \quad X(0,y) = y\right)$$

$$\frac{d}{dt}\underbrace{\left[D_y X(t;y)\right]} = D_y\left(F(X(t;y))\right)$$

$$= \underbrace{D_x F(X(t;y))}_{\substack{A(t)\ (n\times n)\\ (\text{known})}}\underbrace{\left[D_y X(t;y)\right]}$$

a time dependent $n\times n$ matrix $[\Phi(t)]$

$$\frac{d}{dt}\left[\Phi(t)\right] = A(t)\left[\Phi(t)\right], \quad \Phi(0) = ?$$

$$\Phi(0) = D_y X(0;y) = D_y y = I$$

$\Phi(t) = D_y X(t;y)$ is the fundamental matrix for

$A(t) = D_x F(X(t;y))$

Prove that $D_y X(t;y)$ indeed exists and is equal to $\Phi(t)$ $t \in [0, T]$

$$\lim_{h \to 0} \frac{\| X(t;y+h) - X(t;y) - \Phi(t)h \|}{\|h\|} = 0$$

or for any ε (eg $= 10^{-6}$), there is $\delta > 0$ s.t. for any $\|h\| \leq \delta$, we have

$$\frac{\| X(t;y+h) - X(t;y) - \Phi(t)h \|}{\|h\|} \leq \varepsilon$$

ie.

$$\| \underbrace{X(t;y+h)}_{2} - \underbrace{X(t;y)}_{1} - \underbrace{\Phi(t)h}_{3} \| \leq \varepsilon \|h\|$$

① $X(t;y) = y + \int_0^t F(X(s;y))\,ds$

② $X(t;y+h) = y+h + \int_0^t F(X(s;y+h))\,ds$

③ $\Phi(t) = I + \int_0^t D_x F(X(s,y))\,\Phi(s)\,ds$

$\Phi(t)h = h + \int_0^t D_x F(X(s;y))\,\Phi(s)\,h\,ds$

② − ① − ③:

$$X(t;y+h) - X(t;y) - \Phi(t)h$$
$$= \int_0^t \left(F(X(s,y+h)) - F(X(s,y)) - D_x F(X(s,y))\Phi(s)h \right) ds$$

$$\boxed{\begin{array}{l} \text{Let } g(t) = \| X(t;y+h) - X(t;y) - \Phi(t)h \| \\ \text{Need to show: for any } \varepsilon\text{, there is } \delta \text{ s.t.} \\ \quad \|g(t)\| \le \varepsilon \|h\| \text{ for any } \|h\| \le \delta \end{array}}$$

$$F(X(s,y+h)) - F(X(s,y)) - D_xF(X(s,y))\Phi(s)h$$

$$\begin{aligned}
= \; & F(X(s,y+h)) - F(X(s,y)) \\
& - D_xF(X(s,y))\big(X(s;y+h) - X(s;y)\big) \quad \Big] \; b \\
& + D_xF(X(s,y))\big(X(s;y+h) - X(s;y)\big) \\
& - D_xF(X(s,y))\Phi(s)h \quad \Big] \; a
\end{aligned}$$

$$a = D_xF(X(s,y))\big(X(s,y+h) - X(s,y) - \Phi(s)h\big)$$

$$\|a\| \leq \underbrace{\|D_xF(X(s,y))\|}_{\|A(s)\| = k(s)} g(s)$$

$$\begin{aligned}
b = \; & F(X(s,y+h)) - F(X(s,y)) \\
& - D_xF(X(s,y))\underbrace{\big(X(s;y+h) - X(s;y)\big)}
\end{aligned}$$

we already have:

$$\|X(s;y+h) - X(s;y)\| \leq \underbrace{\|h\|e^{Ls}}_{\leq \delta}$$

$$\|(b)\| =$$

$$\|F(X(s,y+h)) - F(X(s,y)) - D_xF(X(s,y))(X(s,y+h) - X(s,y))\|$$

$$\leq \varepsilon \|X(s,y+h) - X(s,y)\|$$

$$\leq \varepsilon \|h\| e^{Ls}$$

$$\leq C \varepsilon \|h\| \qquad \underline{s \in [0,T]}$$

$$g(t) \leq \int_0^t k(s) g(s)\, ds + \int_0^t \varepsilon \|h\| e^{Ls} ds$$

$$\leq \underbrace{\frac{\varepsilon (e^{Lt} - 1)}{L}}_{\leq C \quad t \in [0,T]} \|h\| + \int_0^t k(s) g(s)\, ds$$

$$g(t) \leq C \varepsilon \|h\| + \int_0^t k(s) g(s)\, ds$$

$$\boxed{g(t) \leq C \varepsilon \|h\| e^{\int_0^t k(s) ds} \leq \tilde{C} \varepsilon \|h\|}$$