Stability using Lyapunov Function

$$\frac{dX}{dt} = F(X), \quad X(0) = X_0$$

Let X_* - equilibrium pt. of F : $F(X_*) = 0$

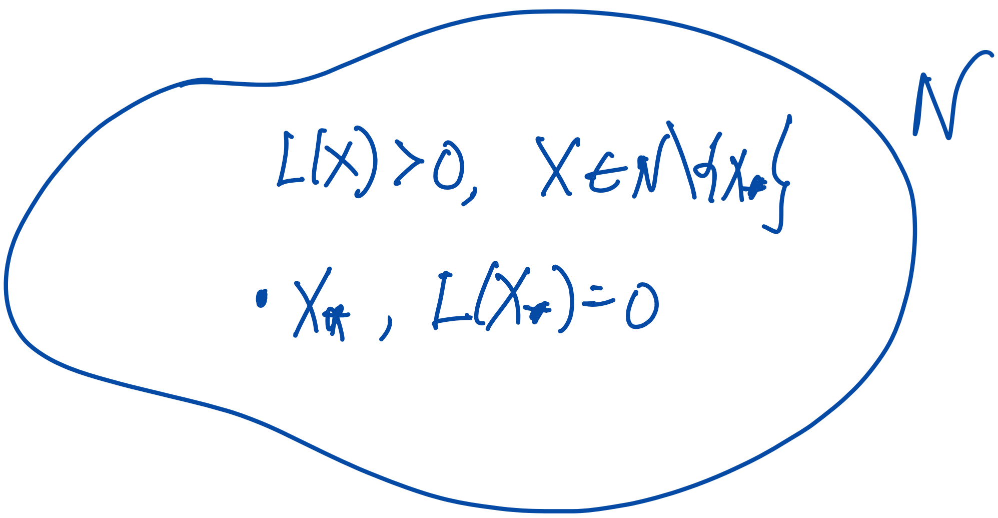
$L : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a (strong) Lyapunov fct for X_*

if (i) $L(X_*) = 0$,

(ii) $L(X) > 0$ for $X \neq X_*$ in a neighbourhood N of X_*

(i.e. X_* is a unique min. of L in N)

(iii) $L(\varphi_t(x)) < L(x)$ for $t > 0$, $x \in N \setminus \{x_*\}$



(Weak) Lyapunov fct if

$L(\varphi_t(x)) \leq L(x)$ for $t > 0$, $x \in N \setminus \{X_*\}$

A sufficient condition for L to be a Lyapunov fct:

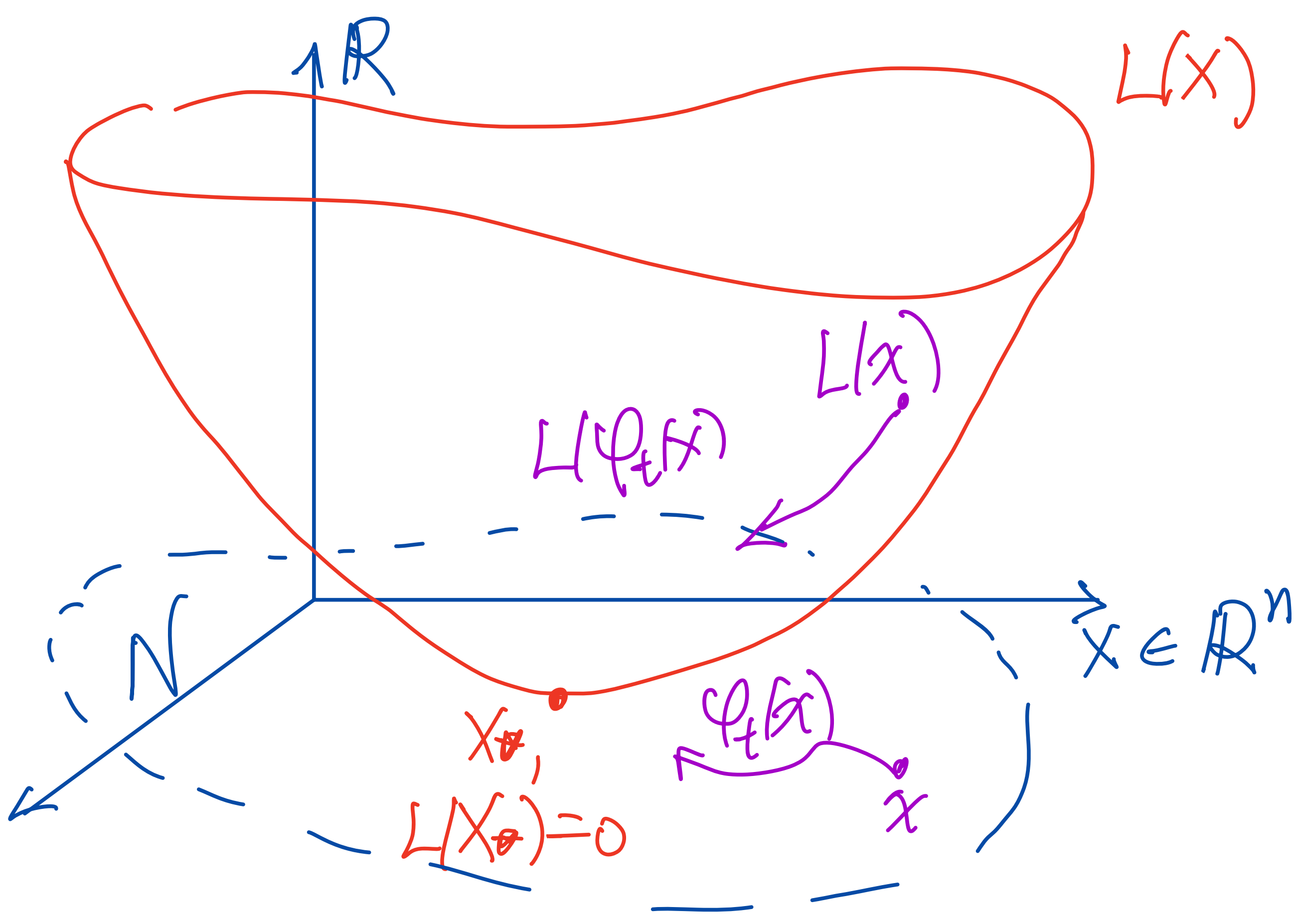
$$\boxed{\langle \nabla L(x), F(x) \rangle < 0}$$

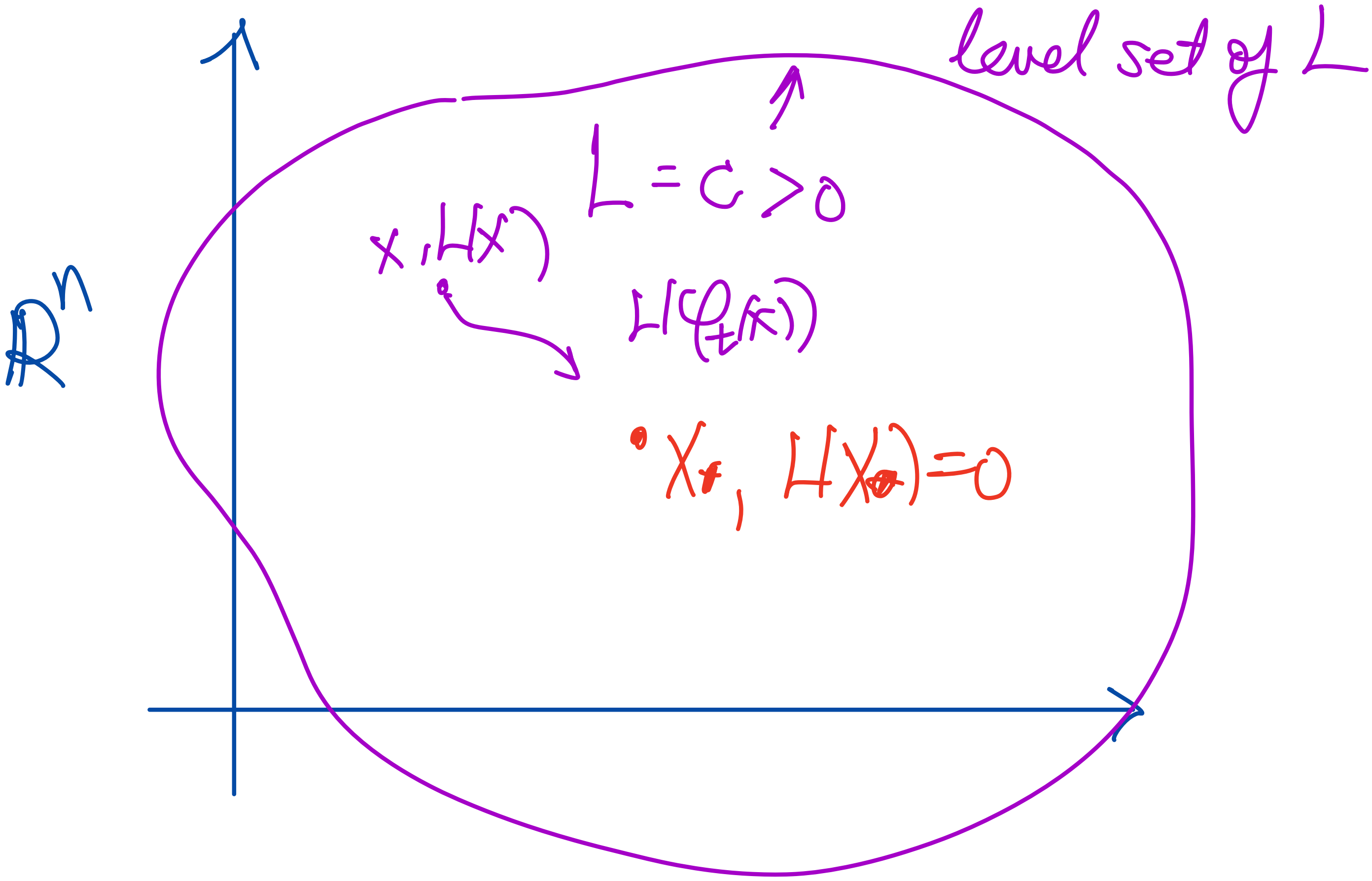
$$\left(\frac{d}{dt} L(\varphi_t(x)) = \left\langle \nabla L(\varphi_t(x)), \frac{d\varphi_t(x)}{dt} \right\rangle \right.$$

$$= \langle \nabla L(\varphi_t(x)), F(\varphi_t(x)) \rangle$$

$$\left. < 0 \right)$$

Theorem 4.21 (Lyapunov Functions). *Let x^* be an equilibrium point of a flow $\varphi_t(x)$. If L is a weak Lyapunov function for x^*, then x^* is stable. If L is a strong Lyapunov function, then x^* is asymptotically stable.*





Example 4.20. The origin is an equilibrium of the system

$$\dot{x} = -x - y - r^2, \qquad (r^2 = x^2 + y^2)$$
$$\dot{y} = x - y + r^2,$$

where r is the polar radius. The origin is a stable focus since

nonlinear part

$$Df(0,0) = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

linear part

$$\lambda_1, \lambda_2 = -1 \pm i$$

$$\mathrm{Re}(\lambda_i) < 0$$

"Let" $L(x,y) = x^2+y^2$ ← distance to the origin

$L(x,y) > 0$, $= 0$ if and only if $(x,y) = (0,0)$
↑
an equilibrium pt.

$$\frac{d}{dt}L(x(t),y(t)) = 2x\dot{x}+2y\dot{y}$$

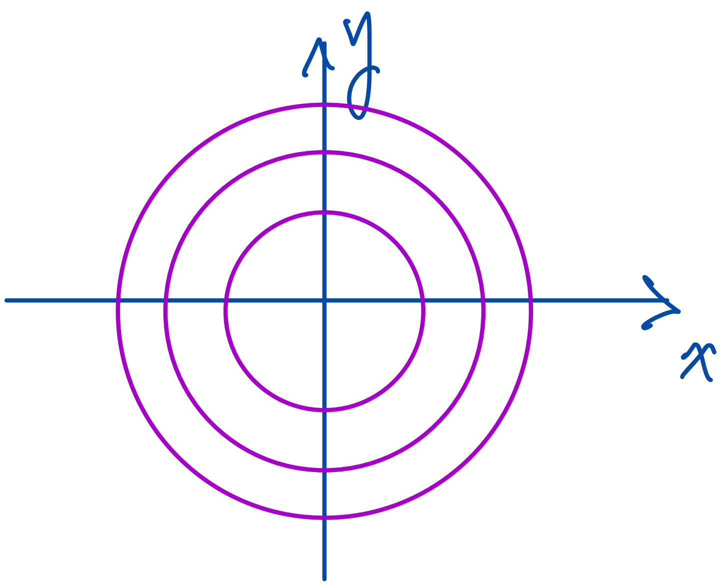
$$= 2x(-x-y-r^2)+2y(x-y+r^2)$$

$$= -2x^2-\cancel{2xy}-2xr^2+\cancel{2xy}-2y^2+2yr^2$$

$$= -2r^2-r^2(2x-2y)$$

$$= \underbrace{-2r^2}_{<0}(\underbrace{1+x-y}) > 0 \quad \text{if } \sqrt{x^2+y^2} < \frac{1}{2}$$

$$\left(\begin{aligned} |x| &\le \sqrt{x^2+y^2} < \tfrac{1}{2} \\ |y| &\le \sqrt{x^2+y^2} < \tfrac{1}{2} \\ 1+x-y &\ge 1-|x|-|y| \\ &> 1-\tfrac{1}{2}-\tfrac{1}{2} \\ &> 0. \end{aligned} \right)$$



Hence $\underline{\frac{d}{dt}L(x(t),y(t)) < 0}$ on $\{x^2+y^2 < \frac{1}{4}\}$

Example 1. Consider the system

[P, p. 132]

$$\begin{aligned}\dot{x}_1 &= -x_2^3\\ \dot{x}_2 &= x_1^3.\end{aligned}$$

The origin is a nonhyperbolic equilibrium point of this system and

$$V(\mathbf{x}) = x_1^4 + x_2^4$$

is a Liapunov function for this system. In fact

$$\dot{V}(\mathbf{x}) = 4x_1^3\dot{x}_1 + 4x_2^3\dot{x}_2 = 0.$$

Hence the solution curves lie on the closed curves

$$x_1^4 + x_2^4 = c^2$$

which encircle the origin. The origin is thus a stable equilibrium point of this system which is not asymptotically stable. Note that $D\mathbf{f}(\mathbf{0}) = 0$ for this example; i.e., $D\mathbf{f}(\mathbf{0})$ has two zero eigenvalues.

Example 2. Consider the system

[P, p.133]

$$
\begin{aligned}
\dot{x}_1 &= -2x_2 + x_2x_3 \\
\dot{x}_2 &= x_1 - x_1x_3 \\
\dot{x}_3 &= x_1x_2.
\end{aligned}
$$

The origin is an equilibrium point for this system and

$$
D\mathbf{f}(\mathbf{0}) = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

Thus $D\mathbf{f}(\mathbf{0})$ has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm 2i$; i.e., $\mathbf{x} = \mathbf{0}$ is a nonhyperbolic equilibrium point. So we use Liapunov's method. But how do we find a suitable Liapunov function? A function of the form

$$
V(\mathbf{x}) = c_1x_1^2 + c_2x_2^2 + c_3x_3^2
$$

with positive constants c_1, c_2 and c_3 is usually worth a try, at least when the system contains some linear terms. Computing $\dot{V}(\mathbf{x}) = DV(\mathbf{x})\mathbf{f}(\mathbf{x})$, we find

$$
\frac{1}{2}\dot{V}(\mathbf{x}) = (c_1 - c_2 + c_3)x_1x_2x_3 + (-2c_1 + c_2)x_1x_2.
$$

Hence if $c_2 = 2c_1$ and $c_3 = c_1 > 0$ we have $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$ and $\dot{V}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbf{R}^3$ and therefore by Theorem 3, $\mathbf{x} = \mathbf{0}$ is stable. Furthermore, choosing $c_1 = c_3 = 1$ and $c_2 = 2$, we see that the trajectories of this system lie on the ellipsoids $x_1^2 + 2x_2^2 + x_3^2 = c^2$.

We commented earlier that all sinks are asymptotically stable. However, as the next example shows, not all asymptotically stable equilibrium points are sinks. (Of course, a *hyperbolic* equilibrium point is asymptotically stable iff it is a sink.)

[P, p. 133]

Example 3. Consider the following modification of the system in Example 2:

$$\begin{aligned}\dot{x}_1 &= -2x_2 + x_2x_3 - x_1^3\\ \dot{x}_2 &= x_1 - x_1x_3 - x_2^3\\ \dot{x}_3 &= x_1x_2 - x_3^3.\end{aligned}$$

The Liapunov function of Example 2,

$$V(\mathbf{x}) = x_1^2 + 2x_2^2 + x_3^2,$$

satisfies $V(\mathbf{x}) > 0$ and

$$\dot{V}(\mathbf{x}) = -2(x_1^4 + 2x_2^4 + x_3^4) < 0$$

for $\mathbf{x} \neq \mathbf{0}$. Therefore, by Theorem 3, the origin is asymptotically stable, but it is not a sink since the eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm 2i$ do not have negative real part.

Gradient Flow

$$V : \mathbb{R}^n \longrightarrow \mathbb{R}$$

Some kind of energy functional

Gradient flow w.r.t. V is given by:

$$\boxed{\frac{dx}{dt} = -\nabla V(x)}$$

also called gradient descent.

$$\boxed{\begin{aligned} \frac{dV(x(t))}{dt} &= \langle \nabla V(x), \dot{x} \rangle \\ &= \langle \nabla V(x), -\nabla V(x) \rangle = -\|\nabla V(x)\| < 0 \end{aligned}}$$

Remark:

(1) V acts like a Lyapunov fct.

(2) Gradient descent is often used in optimization/minimization problems

Hamiltonian Flow

$$H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(X, Y) \qquad\qquad H(X,Y)$$

$$\boxed{\begin{aligned} \frac{d}{dt} X(t) &= \nabla_y H(X(t), Y(t)) \\ \frac{d}{dt} Y(t) &= -\nabla_x H(X(t), Y(t)) \end{aligned}}$$

A key observation:

the value of H is preserved along solutions.

$$\frac{d}{dt} H(X(t), Y(t))$$
$$= \langle \nabla_x H(X,Y), \dot{X} \rangle + \langle \nabla_y H(X,Y), \dot{Y} \rangle$$
$$= \langle \nabla_x H(X,Y), \nabla_y H(X,Y) \rangle$$
$$+ \langle \nabla_y H(X,Y), -\nabla_x H(X,Y) \rangle \quad = 0$$

A Typical Example (from Mechanics)

$$H(x,y) = \underbrace{\frac{1}{2m}|y|^2}_{\text{K.E.}} + \underbrace{V(x)}_{\text{P.E.}} \leftarrow \text{Total energy}$$

position   velocity/momentum

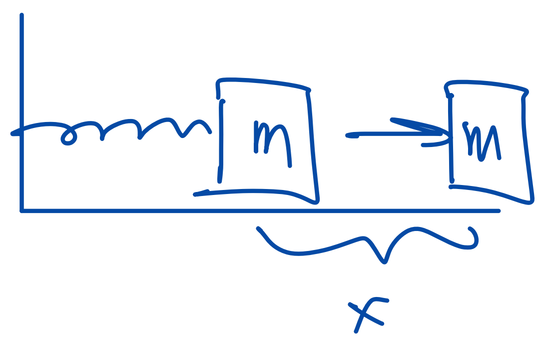
$$\begin{cases} \dot{x} = \nabla_y H(x,y) = \dfrac{y}{m} \\ \dot{y} = -\nabla_x H(x,y) = -\nabla V(x) \end{cases}$$

$$m\ddot{x} = -\nabla V(x) \leftarrow \text{Newton's 2nd Law}$$

Harmonic oscillator

$$m\ddot{x} = -kx$$

$$H(x,y) = \frac{1}{2}\frac{y^2}{m} + \frac{1}{2}kx^2$$



$$\left.\begin{aligned} \dot{x} &= \partial_y H(x,y) \left(= \frac{y}{m}\right) \\ \dot{y} &= -\partial_x H(x,y) \left(= -kx\right) \end{aligned}\right\} \iff m\ddot{x} = -kx$$

$$\boxed{\begin{array}{l} H(x,y) = \frac{1}{2}\frac{y^2}{m} + \frac{1}{2} k x^2 \\ \text{acts as a (weak) Lyapunov fct.} \end{array}}$$

$$H(x,y) \geq 0, \quad = 0 \iff x = 0,\ y = 0$$

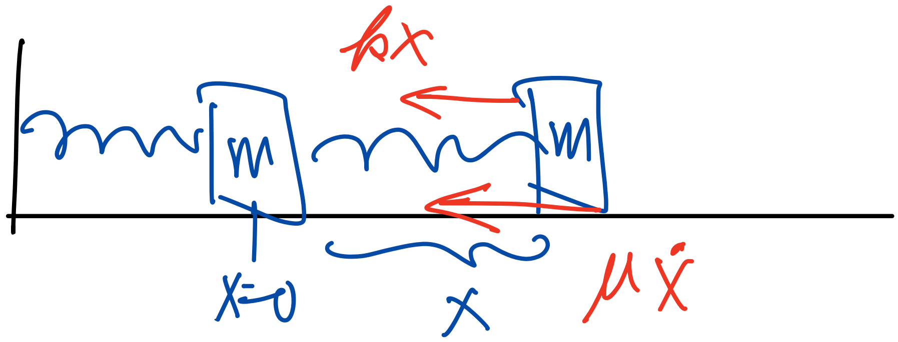
<u>Hamiltonian system with dissipation</u>
(friction)

$$\begin{cases} \dot{x} = \nabla_y H(x,y) \\ \dot{y} = -\nabla_x H(x,y) - \mu \dot{x} \end{cases}$$

frictional coeff.

$$\begin{aligned}
\frac{d}{dt} H(x,y) &= \langle \nabla_x H, \dot{x} \rangle + \langle \nabla_y H, \dot{y} \rangle \\
&= \cancel{\langle \nabla_x H, \nabla_y H \rangle} + \langle \nabla_y H, \cancel{-\nabla_x H} - \mu \dot{x} \rangle \\
&= -\langle \nabla_y H, \dot{x} \rangle \\
&= -\langle \nabla_y H, \nabla_y H \rangle \\
&= -\|\nabla_y H\|^2 < 0 \implies H(x(t), y(t)) \to 0
\end{aligned}$$

Harmonic oscillator with friction



$$m\ddot{x} = -kx - \mu\dot{x}$$

$$H(x,y) = \frac{1}{2}\frac{y^2}{m} + \frac{1}{2}kx^2$$

$$\begin{cases} \dot{x} = \partial_y H(x,y) & \left(= \dfrac{y}{m}\right) \\ \dot{y} = -\partial_x H(x,y) - \mu\dot{x} & \left(= -kx - \mu\dot{x}\right) \end{cases}$$

$H(x(t), y(t)) \downarrow$ in time, $\to 0$ as $t \to +\infty$

$$\Longrightarrow \quad x(t) \to 0, \quad y(t) \to 0$$