Periodic Solutions (Hartman, ODE, p. 407–418)

Consider the following non-autonomous system

$$\frac{dX}{dt} = F(t, X)$$

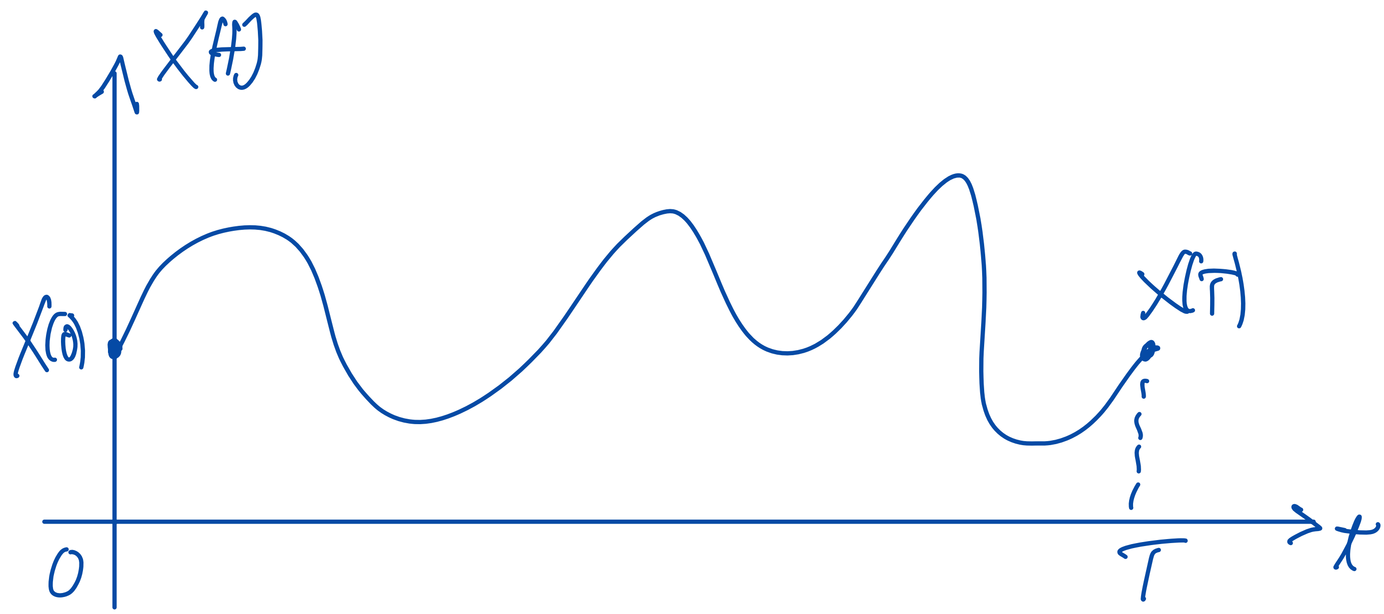
F is T-periodic $F(t+T, X) = F(t, X)$

$$= A(t)X + f(t, X)$$

$A(t+T) = A(t)$, $f(t+T, X) = f(t, X)$

Look for T-periodic solution:

$$X(0) = X(T) \Rightarrow X(t+T) = X(t)$$



(More general boundary condition:

$$M X(0) = N X(T)$$

If $M = N = I$, then $X(0) = X(T)$.)

Linear Homogeneous Equation

[H, p. 407 Lem 1.1]

$$\frac{dX}{dt} = A(t) X, \qquad M X(0) = N X(T)$$

(Existence? $X(0) = ?$)

Recall the fundamental solution:

$$\Phi(t,s), \quad s \le t$$

$$\frac{d}{dt} \Phi(t,s) = A(t) \Phi(t,s)$$

$$\Phi(s,s) = I$$

$$\left(\Phi(t) := \Phi(t,0) \right)$$

(1) $X(0) \Rightarrow X(t) = \Phi(t) X(0)$

(2) $$X(T) = \Phi(T) X(0)$$

(3) $M X(0) = N X(T)$

$$\Longleftrightarrow M X(0) = N \Phi(T) X(0)$$

$$\Longleftrightarrow \big(M - N\Phi(T)\big) X(0) = 0$$

i.e. $X(0) \in Null\{(M - N\Phi(T))\}$ (*)

(4) The solution is completely characterized by $X(0)$ which satisfies (*)

(5) If $\underline{\big(M - N\Phi(T)\big)^{-1} \text{ exists}}$, then $\underline{X(0) = 0}$

i.e. <u>only the trivial solution</u> $(X(t) \equiv 0)$

(6) To have non-trivial solution, $X(t) \neq 0$

we need <u>$M - N\Phi(T)$ to be singular</u>

i.e. <u>$\big(M - N\Phi(T)\big)^{-1}$ does not exist.</u>

(7) $M = N = I$ ie. $X(0) = X(T)$

$\iff X(0) = \Phi(T) X(0)$ $\quad X(0) \neq 0$

ie. $X(0)$ is an eigenvector of $\Phi(T)$ with eigenvalue $\mu = 1$

$$\boxed{\Phi(T) X(0) = \mu X(0) \qquad \mu = 1}$$

If $\mu = 1$ is not an eigenvalue of $\Phi(T)$,

$\iff (\Phi(T) - I)^{-1}$ exists

$\implies X(0) = 0$

Then there is only the trivial solution

$X(t) \equiv 0$

Linear Inhomogeneous Equation [H, p408, Thm 1.1]

(*) $$\frac{dX}{dt} = A(t)X + g(t), \quad MX(0) = NX(T)$$

T- periodic

Suppose Rank$[M, N] = n$, i.e. full rank (eg. $M = N = I$)

Then (*) has a solution for any $g(t)$

if and only if

$$\frac{dX}{dt} = A(t)X, \qquad MX(0) = NX(T)$$

has only the trivial solution $X(t)$

(ie. $(M - N\Phi(T))^{-1}$ exists)

In this case, the solution is unique and there is a constant α s.t.

$$\sup_{t\in[0,T]} \|X(t)\| \leq \alpha \int_0^T \|g(s)\| ds$$

Pf (" $\Leftarrow$ ", i.e. $(M - N\Phi(T))^{-1}$ exists)

$$\frac{dX}{dt} = A(t)X + g(t)$$

$$X(t) = \Phi(t)X(0) + \int_0^t \Phi(t,s)g(s)\,ds$$

$$X(T) = \Phi(T)X(0) + \int_0^T \Phi(T,s)g(s)\,ds$$

Then $\quad MX(0) = NX(T)$

$$\Updownarrow$$

$$N\left[\Phi(T)X(0) + \int_0^T \Phi(T,s)g(s)\,ds\right] = MX(0)$$

$$(M - N\Phi(T))X(0) = N\int_0^T \Phi(T,s)g(s)\,ds$$

$$\boxed{X(0) = (M - N\Phi(T))^{-1} N\int_0^T \Phi(T,s)g(s)\,ds}$$

(unique choice of $X(0)$)

$$\boxed{\begin{aligned} X(t) &= \Phi(t)\left(M - N\Phi(T)\right)^{-1} N \int_0^T \Phi(T,s)\, g(s)\, ds \\ &\quad + \int_0^t \Phi(t,s)\, g(s)\, ds \end{aligned}}$$

$$\Downarrow$$

$$\sup_{t \in [0,T]} \|X(t)\| \leq \alpha \int_0^T \|g(s)\|\, ds$$

(I) Analogy from linear algebra

When is $AX = b$ solvable?

(1) $A^{m\times n} X^{n\times 1} = b^{m\times 1}$ is solvable

iff $b \in \mathrm{Col}(A) = \mathrm{Range}(A)$

$\| \qquad\qquad \|$

Span{ columns of A }

$\{ Y : Y = AX \text{ for some } X \in \mathbb{R}^n \}$

iff $b \perp \mathrm{Null}(A^T)$

$\langle b, Z \rangle = 0$ for all $Z : A^T Z = 0$

(Fredholm alternatives)

$$\underbrace{\mathrm{Col}(A)} \perp \underbrace{\mathrm{Null}(A^T)}$$

$\dim \mathrm{Col}(A) = r$ $\qquad$ $\dim \mathrm{Null}(A^T) = m - r$

$$\mathbb{R}^m = \mathrm{Col}(A^T) \oplus \mathrm{Null}(A^T)$$

(2) $A^{m\times n} X^{n\times 1} = b^{m\times 1}$ is solvable for any b

iff $\text{Rank}(A) = m$ (full rank)

$\dim \text{Col}(A) = m \iff \text{Range}(A) = \mathbb{R}^m$

(3) $A^{n\times n} X^{n\times 1} = b^{n\times 1}$ is solvable for any b

iff A^{-1} exists $\quad (A^{-1}A = AA^{-1} = I)$

(4) $AX = b$ has unique solution (if exists)

iff $\text{Null}(A) = \{0\} \iff (\text{rank}(A) = n)$

iff $\text{Range}(A^T) = \mathbb{R}^n \iff (\text{rank}(A^T) = n)$

(II) Consider 1D Case

$$\frac{dx}{dt} = a(t)x + g(t)$$

$a(t+T) = a(t)$, $g(t+T) = g(t)$

$$x(t) = e^{\int_0^t a(r)\,dr} x_0 + \int_0^t e^{\int_s^t a(r)\,dr} g(s)\,ds$$

$$x(T) = e^{\int_0^T a(r)\,dr} x_0 + \int_0^T e^{\int_s^T a(r)\,dr} g(s)\,ds$$

$=$

$$x(0) = e^{\int_0^T a(r)\,dr} x_0 + \int_0^T e^{\int_s^T a(r)\,dr} g(s)\,ds$$

$$\left(1 - e^{\int_0^T a(r)\,dr}\right) x_0 = \int_0^T e^{\int_s^T a(r)\,dr} g(s)\,ds$$

If $e^{\int_0^T a(r)\,dr} \neq 1$, $\int_0^T a(r)\,dr \neq 0$

then $x_0 = \left(1 - e^{\int_0^T a(r)\,dr}\right)^{-1} \int_0^T e^{\int_s^T a(r)\,dr} g(s)\,ds$

If $e^{\int_0^T a(r)\,dr} = 1$, $\quad \int_0^T a(r)\,dr = 0$

then we need

$$\boxed{\int_0^T e^{\int_s^T a(r)\,dr} g(s)\,ds = 0}$$

Compatibility condition

and x_0 can be anything.

In particular, if $a(t) \equiv a$, a constant.

If $a \neq 0$, then

$$x_0 = \left(1 - e^{Ta}\right)^{-1} \int_0^T e^{(T-s)a} g(s)\,ds$$

If $a = 0$, then we need

$$0 = \int_0^T g(s)\,ds \quad \leftarrow \text{compatibility condition.}$$

and x_0 can be any number.

Nonlinear Equation [H, p. 413, Thm 2.1]

$$(*) \quad \frac{dX}{dt} = A(t)X + f(t,X) \qquad X(0) = X(T)$$

(1) $A(t+T) = A(t), \quad f(t+T, X) = f(t, X)$

(2) $\frac{dX}{dt} = A(t)X, \quad X(0) = X(T)$ has only the trivial solution

$(\Leftrightarrow (\Phi(T) - I)^{-1}$ exists$)$

(3) α from [H, Thm 1.1, p. 408]

(4) $\|f(t,X) - f(t,Y)\| \leq \theta \|X - Y\|$

(5) $\alpha \theta T < 1$

Then $(*)$ has a unique solution

Banach Fixed Point Theorem

$$T: X \longrightarrow Y = TX$$

$$Y(t) = (TX)(t) = \Phi(t) Y_0 + \int_0^t \Phi(t,s) f(s, X(s))\, ds$$

chosen s.t. $(TX)(T) = Y_0$

$$Y_0 = (I - \Phi(T))^{-1} \int_0^T \Phi(T,s) f(s, X(s))\, ds$$

$$= \Phi(t)(I - \Phi(T))^{-1} \int_0^T \Phi(T,s) f(s, X(s))\, ds$$

$$+ \int_0^t \Phi(t,s) f(s, X(s))\, ds$$

$$\| Y_1(t) - Y_2(t) \|$$

$$\leq \left\| \Phi(t)(I - \Phi(T))^{-1} \int_0^T \Phi(T,s)\big(f(s, X_1(s)) - f(s, X_2(s))\big)\, ds \right\|$$

$$+ \left\| \int_0^t \Phi(t,s)\big(f(s, X_1(s)) - f(s, X_2(s))\big)\, ds \right\|$$

$$\leq C_1 \int_0^T C_2 \theta \| X_1(s) - X_2(s) \| ds$$

$$+ \int_0^t C_2 \theta \| X_1(s) - X_2(s) \| ds$$

$$\leq (C_1 C_2 \theta T + C_2 \theta T) \| X_1 - X_2 \|_{C^0}$$

$$\| Y_1 - Y_2 \|_{C_0} \leq \underbrace{(C_1 C_2 \theta T + C_2 \theta T)}_{\text{(required to be } < 1)} \| X_1 - X_2 \|_{C^0}$$

Nonlinear Equation with a parameter [H, p. 415]

$$(*)_\mu \qquad \frac{dx}{dt} = F(t, x, \mu)$$

$F(t+T, x, \mu) = F(t, x, \mu)$

Suppose

(1) $(*)_{\mu=0}$ has a solution $\gamma(t)$

(2) Let $A(t) = \partial_x F(t, \gamma(t), 0)$

$\Phi(t)$ – fundamental solution of $A(t)$

(3) $(\Phi(T) - I)^{-1}$ exists

Then $(*)_\mu$ has a solution for $|\mu| \ll 1$