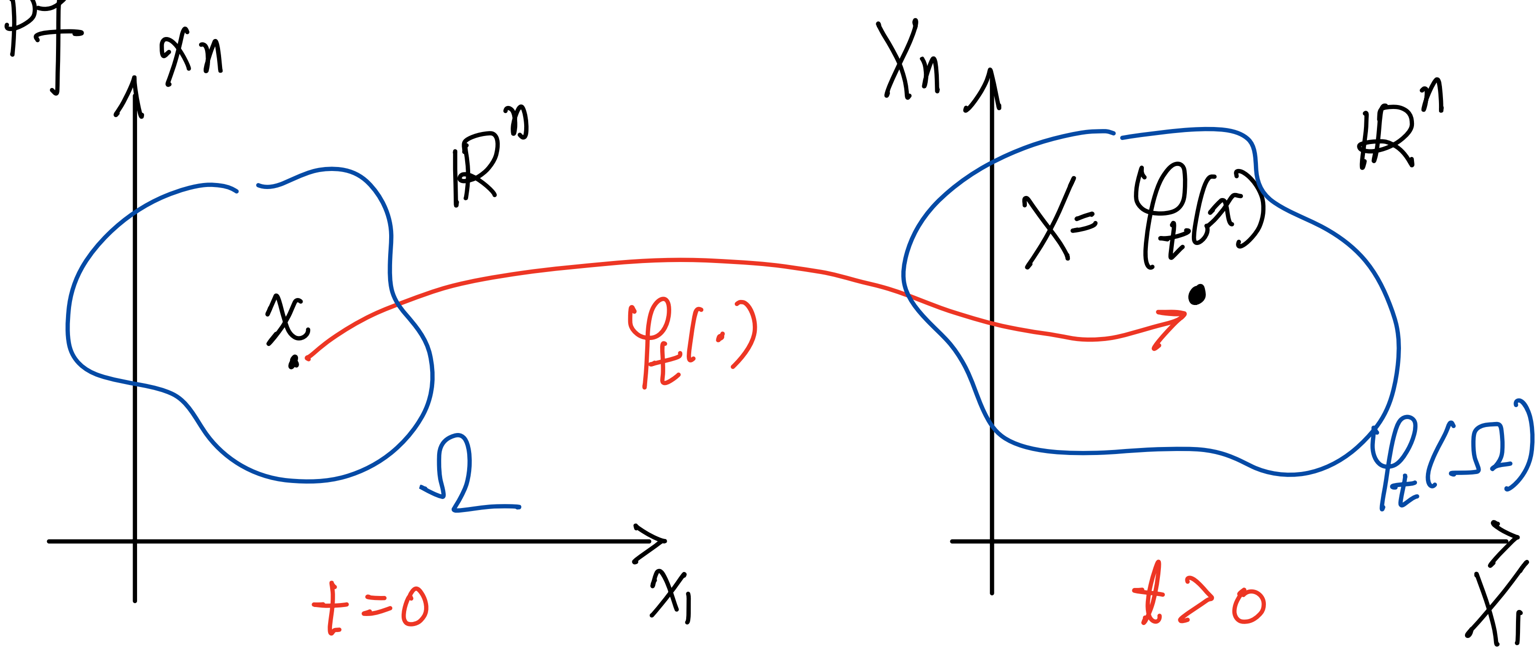
**Abel Thm** Consider $\frac{dX}{dt} = F(X)$, $X(0) = x$.

Let $\varphi : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be the flow map

Then — i.e. $X(t) = \varphi_t(x)$.

$$\boxed{\begin{aligned} \det\left[D_x \varphi_t(x)\right] &= \det\left[D_x \varphi_s(x)\right] e^{\int_s^t \mathrm{div}_X F(\varphi_r(x))\,dr} \quad (\text{for all } 0 < s < t) \\ &= e^{\int_0^t \mathrm{div}_X F(\varphi_r(x))\,dr} \quad (s = 0) \end{aligned}}$$

Pf




Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary domain in $\mathbb{R}^n$ and $\varphi_t(\Omega)$ be its image under the flow map.

Then

$$\int_{\varphi_t(\Omega)} d^n X = \int_{\Omega} \left(\frac{d^n X}{dx^n}\right) d^n x$$

change of variable formula.

$$= \int_{\Omega} \det\left[D_x \varphi_t(x)\right] d^n x$$

Jacobian of flow map

Then

$$\frac{d}{dt}\int_{\varphi_t(\Omega)} d^n X = \frac{d}{dt}\int_{\Omega} \det\left[D_x \varphi_t(x)\right] d^n x$$

$$\parallel$$

$$L.H.S. = \int_{\varphi_t(\Omega)} \left\langle \frac{d}{dt}\varphi_t(x), \hat{n} \right\rangle d^n X = \int_{\partial \varphi_t(\Omega)} F(X)\cdot \hat{n}\, d\sigma_X^{n-1}$$

div thm

$$= \int_{\varphi_t(\Omega)} \operatorname{div}_X F(X)\, d^n X \overset{\text{change of var.}}{=} \int_{\Omega} \operatorname{div}_X F(\varphi_t(x)) \det\left[D_x \varphi_t(x)\right] d^n x$$

$$R.H.S. = \int_{\Omega} \frac{d}{dt} \det\left[D_x \varphi_t(x)\right] d^n x$$

Since $\Omega$ is arbitrary, we have that:

$$\frac{d}{dt}\left(\det\left[D_x \varphi_t(x)\right]\right) = \det\left[D_x \varphi_t(x)\right] \operatorname{div}_X \vec{F}(\varphi_t(x))$$

$$\Downarrow$$

$$\boxed{\det\left[D_x \varphi_t(x)\right] = \det\left[D_x \varphi_s(x)\right] e^{\int_s^t \operatorname{div}_X \vec{F}(\varphi_r(x))dr} \quad 0 \leq s \leq t \qquad = e^{\int_0^t \operatorname{div}_X \vec{F}(\varphi_r(x))dr} \quad s = 0}$$

(Note: $\dot{x}(t) = a(t)\, x(t) \Rightarrow x(t) = x(s) e^{\int_s^t a(r)dr}$)

[M, Thm 2.34] $\dfrac{dX}{dt} = A(t)X, \quad X(s) = x$

The flow map is given by the fundamental Solution:

$$\varphi_{s,t}(x) = \Phi(t,s)x, \quad 0 \le s \le t$$

$$D_x \varphi_{s,t}(x) = \Phi(t,s) \quad (\Phi(s,s) = I)$$

$$\det D_x \varphi_{s,t}(x) = e^{\int_s^t \mathrm{div}_X (A(r)X)\, dr}$$

$$\boxed{\det \Phi(t,s) = e^{\int_s^t (\mathrm{tr}\, A(r))\, dr}}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow AX = \begin{pmatrix} A_{11}X_1 + \cdots + A_{1n}X_n \\ A_{21}X_1 + \cdots A_{2n}X_n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ A_{n1}X_1 + \cdots + A_{nn}X_n \end{pmatrix}$$

$$\mathrm{div}_X AX = \sum_i \partial_{X_i} (AX)_i = \sum_i A_{ii} = \mathrm{tr}\, A$$