Floquet, Monodromy and Poincare

(Review of Lec 17, 18)

$$(*)_1 \begin{cases} \frac{dX}{dt} = A(t)X, \qquad A(t+T) = A(t) \\ X(0) = X(T) \end{cases}$$

$$X(t) = \Phi(t)X(0)$$

$$X(T) = X(0) \iff \Phi(T)X(0) = X(0)$$

$$\iff (\Phi(T) - I)\,X(0) = 0$$

$(*)_1$ has a non-trivial solution $(X(0) \neq 0)$

$\iff (\Phi(T) - I)$ is not invertible

$\left((\Phi(T) - I)^{-1}\right.$ does not exist$\left.\right)$

$\iff \mu = 1$ is an eigenvalue of $\Phi(T)$

($X(0)$ is an eigenvector of $\Phi(T)$
(wrt. $\mu = 1$))

[H, Thm 1.1, p. 408]

$$(*)_2 \begin{cases} \frac{dX}{dt} = A(t)X + g(t) \\ X(t+T) = X(t) \end{cases}$$

$A(t+T) = A(t)$
$g(t+T) = g(t)$

$$X(t) = \Phi(t)X(0) + \int_0^t \underbrace{\Phi(t,s)}_{=\Phi(t)\Phi(s)^{-1}} g(s)\,ds$$

$$X(T) = X(0)$$

$$\Longleftrightarrow \quad X(0) = \Phi(T)X(0) + \int_0^T \Phi(T,s)g(s)\,ds$$

$$\Longleftrightarrow \quad (\Phi(T) - I)X(0) = -\int_0^T \Phi(T,s)g(s)\,ds$$

If $(\Phi(T) - I)^{-1}$ exists, then $(*)_2$ has a soln. for any $g(t)$ and

$$X(0) = -(\Phi(T) - I)^{-1}\int_0^T \Phi(T,s)g(s)\,ds$$

and $\|X(\cdot)\|_{C^0(0,T)} \leq \alpha \int_0^T \|g(s)\|\,ds$

Otherwise, for $(*)_2$ to have a solution, $g(\cdot)$ needs to satisfy some compatibility condition

[H, Thm 2.1, p. 412]

$$(*)_3 \begin{cases} \dfrac{dX}{dt} = A(t)X + f(t,X) \\ X(t+T) = X(t) \end{cases}$$

$A(t+T) = A(t)$

$f(t+T, x) = f(t,x)$

$(*)_3$ has a unique solution if

(1) $(\Phi(T) - I)^{-1}$ exists

(2) $\|f(t,X) - f(t,Y)\| \leq \theta \|X - Y\|$

(3) $\alpha \theta T < 1$

[H, p. 415, Thm 2.3]

$$(*)_4 \begin{cases} \dfrac{dX}{dt} = F(t, X, \mu) \\ X(t+T) = X(t) \end{cases}$$

$F(t+T, X, \mu) = F(t, X, \mu)$

Suppose (1) $(*)_4 \big|_{\mu=0}$ has a solution $\gamma(t)$

(2) Let $A(t) = D_X F(t, \gamma(t), 0) \Rightarrow \Phi(t)$

$(\Phi(T) - I)^{-1}$ exists

Then $(*)_4$ has a solution for $|\mu| \ll 1$

Floquet Theory [M, Sec 2.8]

$$\frac{dX}{dt} = A(t)X \qquad A(t+T) = A(t)$$

$$\boxed{\Phi(t) = Q(t)e^{tB}, \quad Q(t+T) = Q(t)}$$

$$\Phi(0) = I \Rightarrow Q(0) = I$$

$$\Rightarrow Q(nT) = I$$

$$\Phi(T) = e^{TB} := M \quad (\text{Monodromy matrix})$$

$$\Phi(nT) = e^{nTB} = M^n$$

$$X(t) = Q(t)e^{tB}X(0)$$

$$X(nT) = Q(nT)e^{nTB}X(0)$$

$$= M^n X(0)$$

$$= \mu^n X(0)$$

choose $MX(0) = \mu X(0)$

$$\|X(nT)\| = |\mu|^n \|X(0)\| \to \begin{cases} 0 & \text{if } |\mu| < 1 \\ +\infty & \text{if } |\mu| > 1 \end{cases}$$

Let $X(t) = Q(t)Y(t)$, or $Y(t) = Q^{-1}(t)X(t)$

Then $\dot{X}(t) = \left(Q(t)Y(t)\right)^{\cdot}$

$$= \dot{Q}(t)Y(t) + Q(t)\dot{Y}(t)$$

$$\frac{d}{dt}\Phi(t) = A(t)\Phi(t)$$

i.e. $\left(Q(t)e^{tB}\right)^{\cdot} = A(t)Q(t)e^{tB}$

$$\dot{Q}(t)\cancel{e^{tB}} + Q(t)B\cancel{e^{tB}} = A(t)Q(t)\cancel{e^{tB}}$$

$$\dot{Q}(t) + Q(t)B = A(t)Q(t)$$

$$\dot{X}(t) = \left(A(t)Q(t) - Q(t)B\right)Y + Q(t)\dot{Y}(t)$$

$$= A(t)\underbrace{Q(t)Y}_{X} - Q(t)BY + Q\dot{Y}$$

$$\Longrightarrow \quad \dot{X} - A(t)X = Q(t)(\dot{Y} - BY)$$

Hence $\underset{\Phi(t)}{\dot{X} = A(t)X} \Longleftrightarrow \underset{e^{tB}}{\dot{Y} = BY}$

$$\Phi(T) = M = e^{TB}$$

$$MX = \mu X \qquad BY = \lambda Y$$

- Eigenvalues of B (λ_i) are called characteristic exponents of B
- Eigenvalues of M (μ_i) are called characteristic multipliers of M
- $\mu_i = e^{T\lambda_i}$

$$Re(\lambda_i) < 0 \iff |\mu_i| < 1$$

Autonomous System $\frac{dX}{dt} = F(X)$ (*)

$$\boxed{\text{Suppose (*) has a } T\text{-periodic solution } \gamma(t) \quad (\gamma(t+T) = \gamma(t))}$$

Note: T is not known apriori.

Let $A(t) = D_X F(\gamma(t))$.

Then $A(t+T) = A(t)$

Consider: $\frac{d\gamma(t)}{dt} = F(\gamma(t))$

Then $\frac{d}{dt}\Big(\underbrace{\frac{d}{dt}\gamma(t)}_{X(t)}\Big) = \frac{d}{dt} F(\gamma(t)) = \underbrace{D_X F(\gamma(t))}_{A(t)} \underbrace{\frac{d\gamma}{dt}}_{X(t)}$

Hence $X(t) = \dot{\gamma}(t) \neq 0$ solves

$$\frac{dX}{dt} = A(t) X, \quad X(t+T) = X(t)$$

$$\Longrightarrow \boxed{(\Phi(T) - I)^{-1} \text{ does not exist!}}$$

$$\boxed{\begin{array}{ll} \text{and} & X(0) = \Phi(T)\,X(0) \\ \text{i.e.} & \dot{\gamma}(0) = \Phi(T)\,\dot{\gamma}(0) \\ \text{i.e.} & \dot{\gamma}(0) \text{ is an eigenvector of } \Phi(T) \text{ w.r.t. } \mu = 1 \end{array}}$$

Physical Interpretation : time shift

$$\begin{cases} \dfrac{d\gamma(t)}{dt} = F(\gamma(t)) \\[2ex] \dfrac{d}{dt}\gamma(t+\delta) = F(\gamma(t+\delta)) \end{cases}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\gamma(t+\delta) - \gamma(t)\right) = F(\gamma(t+\delta)) - F(\gamma(t))$$

$$\frac{d}{dt}\left(\frac{\gamma(t+\delta) - \gamma(t)}{\delta}\right) = \frac{F(\gamma(t+\delta)) - F(\gamma(t))}{\delta}$$

$\delta \to 0$

$$\boxed{\frac{d}{dt}\left(\dot{\gamma}(t)\right) = \left[D_x F(\gamma(t))\right] \dot{\gamma}(t)}$$

[H, p.416, Thm 2.4]

Consider $\frac{dx}{dt} = F(x, \mu)$ (*)

Suppose at $\mu = 0$

(1) (*) has a T-periodic solution $\gamma_0(t)$

(2) Let $A(t) = D_x F(\gamma_0(t)) \Longrightarrow \Phi(t)$

$\mu = 1$ is a simple eigenvalue of $\Phi(T)$

i.e. multiplicity of $\mu = 1$ is one

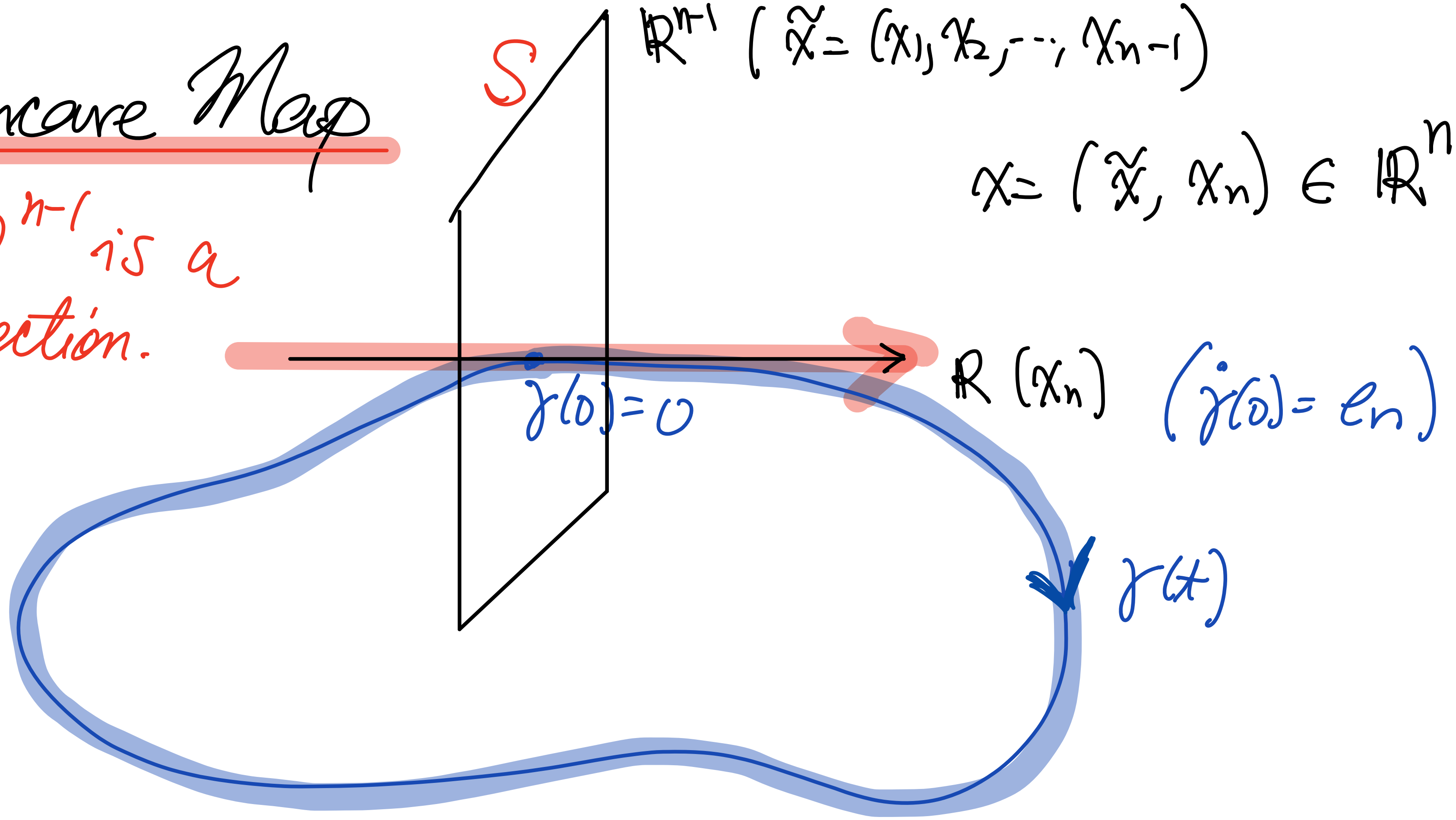
i.e. $\dim(\mathrm{Null}(\Phi(T) - I)) = 1$

Then for $|\mu| \ll 1$, (*) has a solution $\gamma_\mu(\cdot)$ with period $T(\mu)$

Poincare Map

$S \cong \mathbb{R}^{n-1}$ is a cross-section.

$\perp e_n$

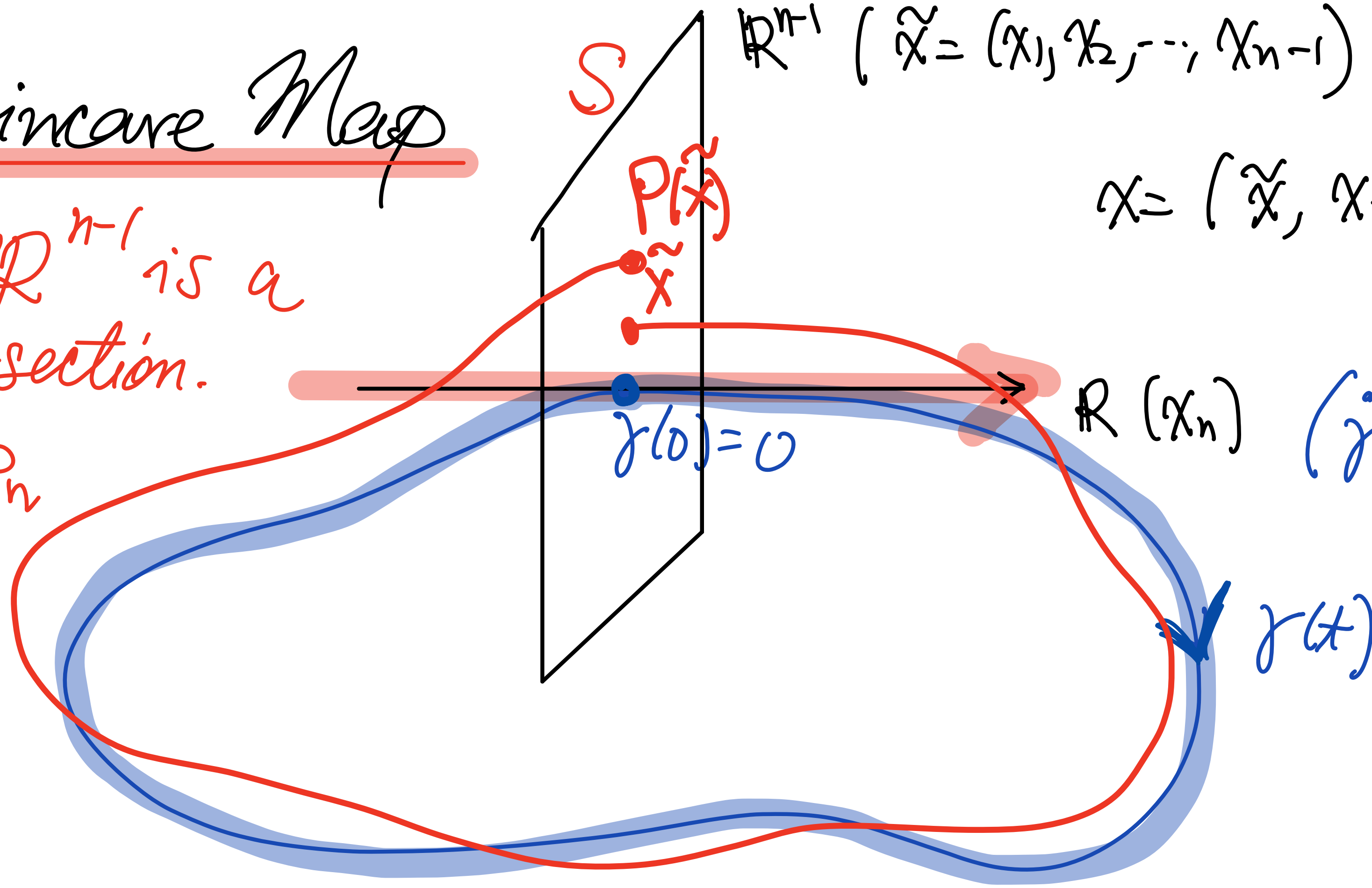


Poincare Map

$S \cong \mathbb{R}^{n-1}$ is a cross-section.
$\perp e_n$

$\mathbb{R}^{n-1}$ $(\tilde{x} = (x_1, x_2, \cdots, x_{n-1}))$

$x = (\tilde{x}, x_n) \in \mathbb{R}^n$



- For any $X(0) = \tilde{x} \in S$, close to $\gamma(0)$, there is a $\tau(x)$ s.t.

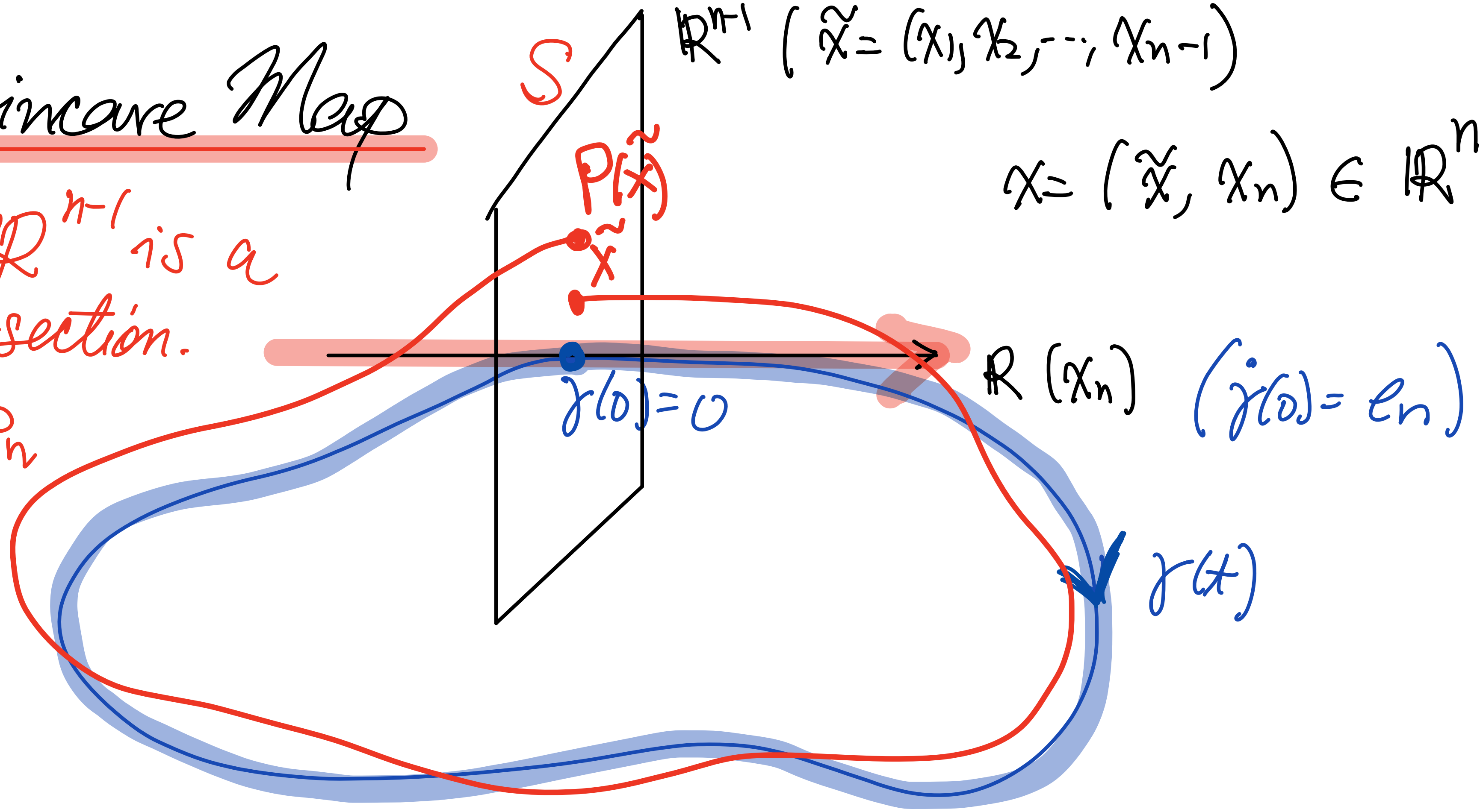
$$\underline{\phi_{\tau(\tilde{x})}(\tilde{x}) \in S}$$

- Poincare Map: $\underline{P(\tilde{x}) = \phi_{\tau(\tilde{x})}(\tilde{x})}$

$P : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$
$(S \to S)$

Poincare Map

$S \cong \mathbb{R}^{n-1}$ is a cross-section.
$\perp e_n$



- The solution $X(t)$ starting at $\tilde{x} \in S$ is a periodic orbit if and only if $\tilde{x}$ is a fixed point of P, i.e. $P(\tilde{x}) = \tilde{x}$ e.g. $P(0)=0$

Relationship between M and $DP(0)$

[M, Thm 4.55]

collection of eigenvalues

$$\mathrm{Spec}(M) = \mathrm{Spec}(DP(0)) \cup \{1\}$$

$n \times n$ matrix $(n-1)\times(n-1)$ matrix Comes from time shift.

$$\begin{array}{c} \\ \left[\begin{array}{c} \\ M \\ \\ \end{array}\right]^{n\times n} \end{array} = \left[\begin{array}{c|c} DP(0)^{(n-1)\times(n-1)} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \times \times \times \times \times \times \times \times & 1 \end{array}\right]$$

- $e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector of M w.r.t. $\mu = 1$

  (time shift)

- $M\big|_S = DP(0)$

  $\mathbb{R}^{n-1}$ $(n-1)\times(n-1)$

Pf (Outline)

$$\tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \in \mathbb{R}^{n-1} = S \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

$e_n \parallel F(0) = \dot{\gamma}(0)$

①

$$\frac{d}{dt}\varphi_t(x) = F(\varphi_t(x))$$

$D_x \downarrow$

$$\frac{d}{dt}\left[D_x\varphi_t(x)\right] = \left[D_xF(\varphi_t(x))\right]\left[D_x\varphi_t(x)\right]$$

$$\frac{d}{dt}\left[D_x\varphi_t(0)\right] = \left[A(t)\right]\left[D_x\varphi_t(0)\right]$$

$$D_x\varphi_t(0) = \Phi(t) \qquad (D_x\varphi_0(0) = I)$$

$$\Longrightarrow \boxed{D_x\varphi_T(0) = \Phi(T) = M}$$

$$[M] = \left[\begin{array}{c|c} \tilde{M} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline * * * * * * & 1 \end{array}\right]$$

Recall: $M\dot{\gamma}(0) = \dot{\gamma}(0)$, i.e. $Me_n = e_n$

(2) Let $Q : \mathbb{R}^n \longrightarrow S \subseteq \mathbb{R}^n$

$$x \to \varphi_{\tau(x)}(x) \in S$$

$$Q(x) = \varphi_{\tau(x)}(x)$$

$$D_x Q(x) = D_x \varphi_{\tau(x)}(x) + \frac{d\varphi_{\tau(x)}(x)}{dt} \nabla_x \tau(x)$$

$x=0$

$$D_x Q(0) = D_x \varphi_{\tau(0)}(0) + \frac{d}{dt} \varphi_{\tau(0)}(0) \nabla_x \tau(0)$$

$$\left(\tau(0) = T,\ D_x \varphi_T(0) = M,\ \frac{d}{dt}\varphi_T(0) = F(0) /\!/ e_n \right)$$

$$D_x Q(0) = M + F(0) \nabla_x \tau$$

$$= \begin{bmatrix} \tilde{M} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ x\,x \cdots x & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x \end{bmatrix} \begin{bmatrix} \nabla_x \tau \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{M} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ x\,x \cdots x & x \end{bmatrix}$$

(3) $P : S \longrightarrow S$

$\tilde{x} \longrightarrow \varphi_{\tau(\hat{x})}(\hat{x})$

$P(\tilde{x}) = Q(\hat{x})$

$$\left[\begin{array}{c} P(\tilde{x}) \\ \hline 0 \end{array}\right] = Q\left(\begin{array}{c} \tilde{x} \\ \hline 0 \end{array}\right)$$

$$\begin{bmatrix} D_{\tilde{x}} P(\tilde{x}) \\ 0 \end{bmatrix} = \left[D_x Q\begin{pmatrix} \tilde{x} \\ 0 \end{pmatrix} \right] \left[D_{\tilde{x}} \begin{pmatrix} \tilde{x} \\ 0 \end{pmatrix} \right]$$

$\tilde{x} = 0$

$$\begin{bmatrix} \underbrace{D_{\tilde{x}} P(0)}_{(n-1)\times(n-1)} \\ 0 \end{bmatrix} = \Big[\underbrace{D_x Q(0)}_{n \times n} \Big] \left[\begin{array}{c} \overbrace{I}^{(n-1)\times(n-1)} \\ \hline 0 \end{array}\right]$$

$$= \underbrace{\left(M + F(0) \nabla_x \tau \right)}_{} \left(\begin{array}{c} I \\ \hline 0 \end{array}\right)$$

$$\left[\begin{array}{ccccc|c} & & \tilde{M} & & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline x & x & \cdots & & x & x \end{array}\right] = \left[\begin{array}{c} \underbrace{\tilde{M}}_{(n-1)\times(n-1)} \\ \hline x\, x\, x \cdots x \end{array}\right]$$

Hence $D_{\tilde{x}} P(0) = \tilde{M}$

④ $$[\,M\,]^{n\times n} = \left[\begin{array}{c|c} \boxed{\tilde{M}} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline x \cdots x & \boxed{1} \end{array}\right] = \left[\begin{array}{c|c} \boxed{D_{\tilde{x}}P(0)} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline x \cdots x & \boxed{1} \end{array}\right]$$

$$\Downarrow$$

$$\boxed{\operatorname{spec}(M) = \operatorname{spec}(D_{\tilde{x}}P(0)) \cup \{1\}}$$

Proof of [H, p.416, Thm 2.4] (Outline)

Need: $\varphi_{\tau(\tilde{x})}(\tilde{x},\mu) - \tilde{x} = 0$

i.e. $P(\tilde{x},\mu) - \tilde{x} = 0$

At $\mu = 0$, $P(0,0) - 0 = 0$

as $\operatorname{Spec} M = \underbrace{\operatorname{Spec} DP(0)}_{1 \notin} \cup \{1\}$

Hence $D_{\tilde{x}}P(0,0) - I$ is invertible

By Implicit Function Thm,

$P(\tilde{x},\mu) - \tilde{x} = 0$ has a solution $\tilde{x}$ for $|\mu| \ll 1$.

(Idea of Implicit Function Thm

Suppose A^{-1} exist.

Then $AX = g(X)$, $g(X) = O(|X|^2)$ has a solution near $X = 0$.

Iteration: $AX_n = g(X_{n-1})$, $X_n = A^{-1} g(X_{n-1})$

Let $Y = A^{-1} g(X)$ ← A contraction

$$\|Y\| = \|A^{-1} g(X)\| \leq \|A^{-1}\| \|g(X)\| \leq C \|A^{-1}\| \|X\|^2$$

Let $\|X\| \leq \delta \ll 1$

Consider $Y_1 = A^{-1} g(X_1)$, $Y_2 = A^{-1} g(X_2)$

Then $Y_1 - Y_2 = A^{-1}(g(X_1) - g(X_2))$

$$\|Y_1 - Y_2\| \leq \underbrace{C(\|X_1\| + \|X_2\|)}_{<1 \text{ if } 2C\delta < 1} (\|X_1 - X_2\|)$$

$\Rightarrow$ Banach Fixed Pt. Thm.)

Pf We have $0 = P(0,0)$. Look for $\tilde{x}$ s.t.

$$\tilde{x} = P(\tilde{x},\mu).$$

$$= \underbrace{\cancel{P(0,0)}}_{=0} + D_{\tilde{x}}P(0,0)\tilde{x} + D_{\mu}P(0,0)\mu + O(\tilde{x}^2+\mu^2)$$

$$\tilde{x} = D_{\tilde{x}}P(0,0)\tilde{x} + D_{\mu}P(0,0)\mu + O(\tilde{x}^2+\mu^2)$$

$$\left(I - D_{\tilde{x}}P(0,0)\right)\tilde{x} = D_{\mu}P(0,0)\mu + O(\tilde{x}^2+\mu^2)$$

$$\tilde{x} = \underbrace{\left(I - D_{\tilde{x}}P(0,0)\right)^{-1}}_{\text{exist as } 1 \notin D_{\tilde{x}}\tilde{P}(\tilde{0},0)}\left(D_{\mu}P(0,0)\mu + O(\tilde{x}^2+\mu^2)\right)$$

Again, we expect $\tilde{x} = O(\mu)$. Let $\tilde{x} = \mu\tilde{Y}$.

Upon dividing by μ $\Longrightarrow$

$$O(\tilde{x}^2+\mu^2) = O(\mu^2\tilde{Y}^2+\mu^2) = \mu^2 O(\tilde{Y}^2+1) = \mu^2 h(\tilde{Y})$$

$\Longrightarrow$ Then $\tilde{Y} = \underbrace{\left(I - D_{\tilde{x}}P(0,0)\right)^{-1}\left(D_{\mu}P(0,0) + \mu h(\tilde{Y})\right)}$

$T\tilde{Y}$

i.e. look for a fixed pt: $\tilde{Y} = T\tilde{Y}$.

Consider

$$T\tilde{Y}_1 - T\tilde{Y}_2 = (I - D_{\tilde{X}} P(0,0))^{-1}(\cancel{D_\mu P(0,0)} + \mu h(\tilde{Y}_1))$$

$$- (I - D_{\tilde{X}} P(0,0))^{-1}(\cancel{D_\mu P(0,0)} + \mu h(\tilde{Y}_2))$$

$$= (I - D_{\tilde{X}} P(0,0))^{-1} \mu (h(\tilde{Y}_1) - h(\tilde{Y}_2))$$

$$\|T\tilde{Y}_1 - T\tilde{Y}_2\| \leq \underbrace{|\mu| \, \|(I - D_{\tilde{X}} P)^{-1}\| \, L}\, \|\tilde{Y}_1 - \tilde{Y}_2\|$$

Lip const. of L

Choose μ small s.t. $|\mu| \, \|(I - D_{\tilde{X}} P)^{-1}\| \, L < 1$

So that T is a contraction.

$\Longrightarrow$ T has a fixed pt.