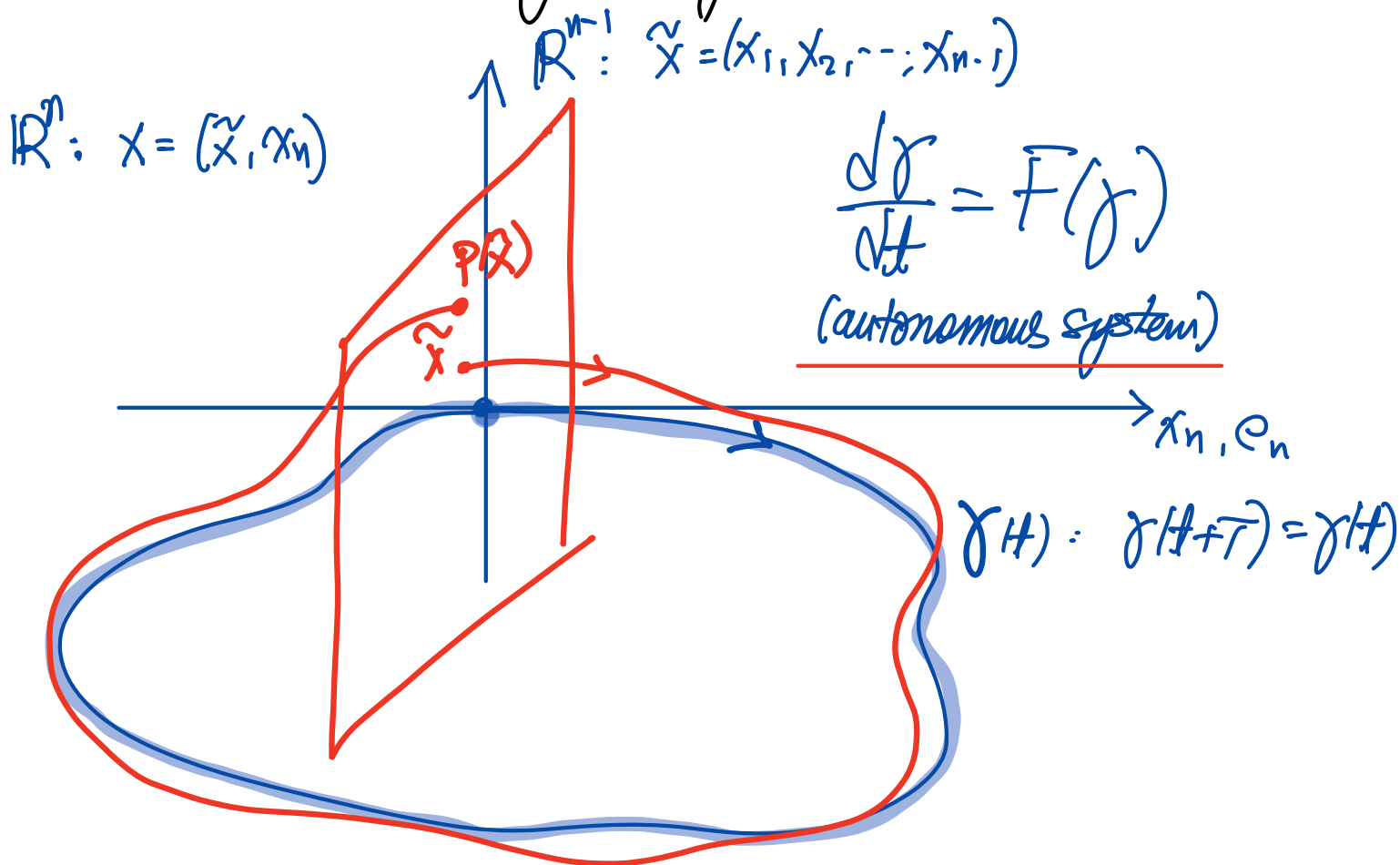


Examples of Stability Analysis of Periodic Solutions



① Monodromy matrix: $M = \Phi(T)^{n \times n}$, $M e_n = e_n$

Poincaré Map: $P: S \cong \mathbb{R}^{n-1} \rightarrow S \cong \mathbb{R}^{n-1}$

$$[DP(0)]^{(n-1) \times (n-1)}$$

time shift

②

$$M^{n \times n} = \begin{bmatrix} \boxed{DP(0)^{(n-1) \times (n-1)}} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \text{xxxxxxxxxx} & \boxed{1} \end{bmatrix}$$

③

$$\text{Spec}\{M\} = \text{Spec}\{DP(0)\} \cup \{1\}$$

④

Eigenvalues of DP $\{|\mu_i|\} < 1 \Rightarrow$ asymptotically stable.
 $|\mu_i| > 1 \Rightarrow$ unstable.

⑤

How to compute μ_i ?

$$\det M = \det \Phi(T) = (\det DP(0))(1) = \det DP(0)$$

Abel Formula (Thm 2.34)

$$\frac{d\Phi(t,s)}{dt} = A(t)\Phi(t,s), \quad \Phi(s,s) = I, \quad t \geq s$$

$$\det \Phi(t,s) = e^{\int_s^t \text{tr} A(r) dr}$$

$$\det \Phi(T) = e^{\int_0^T \text{tr} A(r) dr}$$

$$\textcircled{7} \quad \frac{d\gamma(t)}{dt} = F(\gamma(t)), \quad A(t) = D_x F(\gamma(t))$$

$$\begin{aligned} \underline{\text{tr } A(t)} &= \text{tr } D_x F(\gamma(t)) = \sum_i \partial_{x_i} F_i(\gamma(t)) \\ &= \underline{\text{div } F(\gamma(t))} \end{aligned}$$

$$\det \Phi(T) = e^{\int_0^T \text{div } F(\gamma(r)) dr}$$

$$\textcircled{8} \quad \underline{n=2}$$

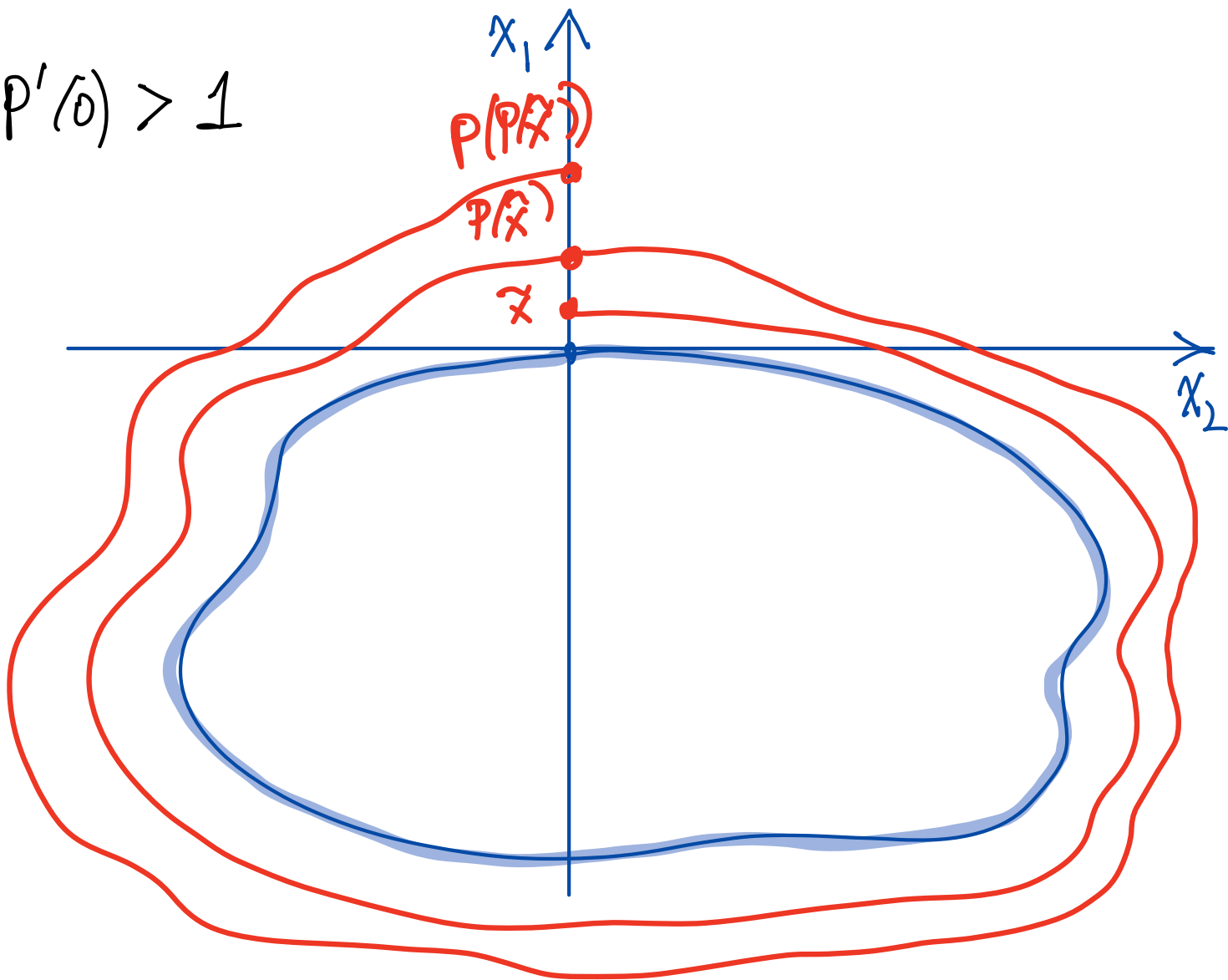
$$\det M = \det P'(0)$$

$$\det P'(0) = e^{\int_0^T \text{div } F(\gamma(t)) dt}$$

$P'(0)$

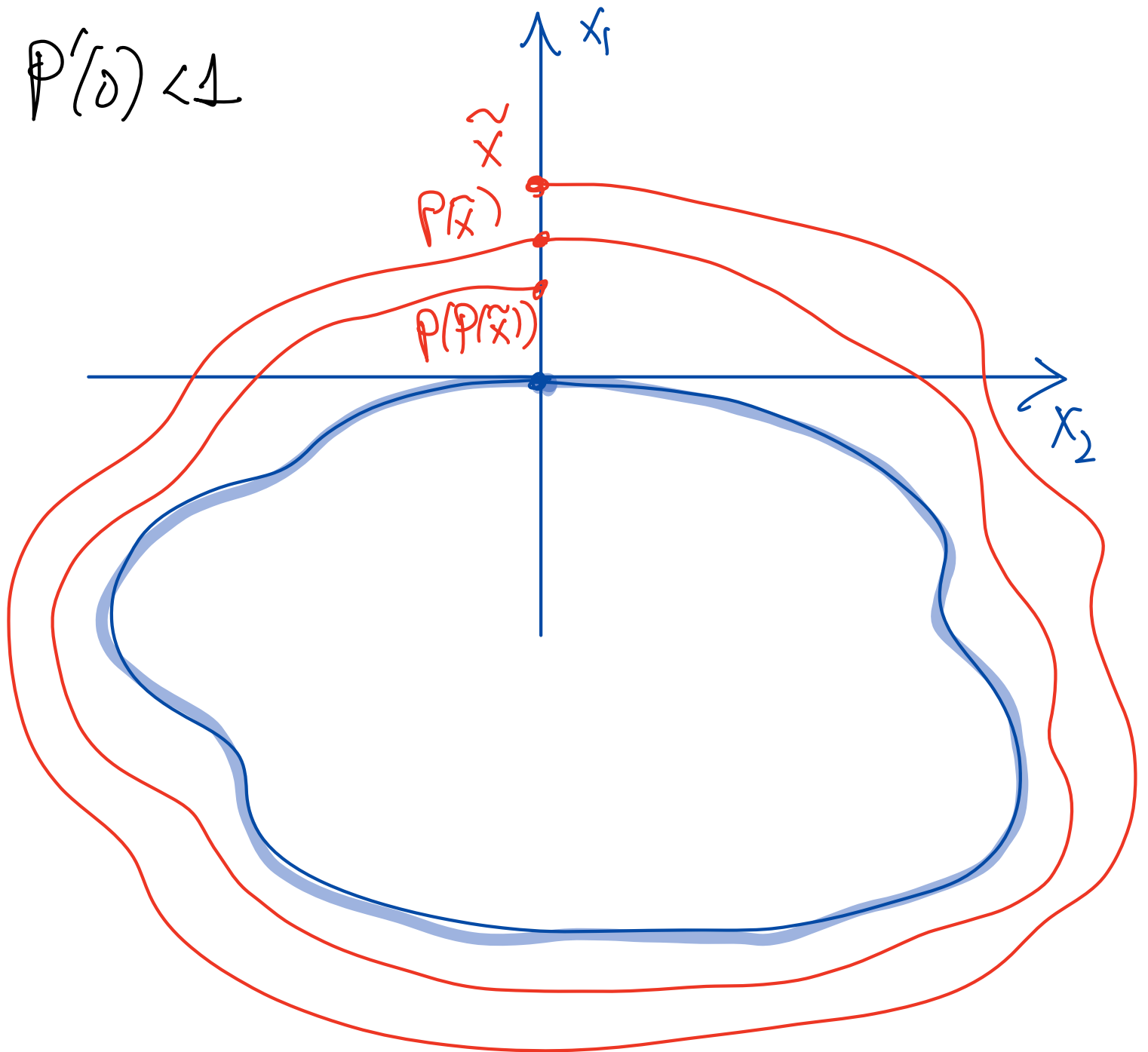
$$P'(0) \begin{cases} > 1 \\ = 1 \\ < 1 \end{cases} \iff \int_0^T \text{div } F(\gamma(t)) dt \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

$$P'(\hat{\theta}) > 1$$



$$\hat{x} < P(\hat{x}) < P^2(\hat{x}) < P^3(\hat{x}) < \dots$$

$$P'(0) < 1$$



$$\tilde{x} > P(\tilde{x}) > P^2(\tilde{x}) > P^3(\tilde{x}) > \dots$$

Examples

$$(1) \quad \begin{aligned} \dot{x} &= \underline{-y + x(1 - x^2 - y^2)} \\ \dot{y} &= \underline{x + y(1 - x^2 - y^2)} \end{aligned}$$

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_A \begin{pmatrix} x \\ y \end{pmatrix} - \underbrace{\begin{pmatrix} x(x^2 + y^2) \\ y(x^2 + y^2) \end{pmatrix}}_{O(x^3 + y^3)}$$

$$A, \lambda_1, \lambda_2 = 1 \pm i$$

(origin hyperbolic, unstable,
 $\operatorname{Re}(\lambda_1, \lambda_2) > 0$, but other solutions?)

$$\begin{aligned} \dot{x} &= -y + x h(r) \\ \dot{y} &= x + y h(r) \end{aligned} \quad r = \sqrt{x^2 + y^2}$$

Use polar coord. $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$

$$\begin{aligned} \dot{r} &= r h(r) \\ \dot{\theta} &= 1 \end{aligned}$$

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}}$$

$$= \frac{x(-y + x(1 - x^2 - y^2)) + y(x + y(1 - x^2 - y^2))}{\sqrt{x^2 + y^2}}$$

$$= \frac{x^2 + y^2 - x^2(x^2 + y^2) - y^2(x^2 + y^2)}{\sqrt{x^2 + y^2}}$$

$$\underline{\dot{r} = r(1 - r^2)}$$

$$\dot{\theta} = \frac{1}{1 + \frac{y^2}{x^2}} \left[\frac{x\dot{y} - y\dot{x}}{x^2} \right]$$

$$= \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}$$

$$= \frac{x(x + y(1 - x^2 - y^2)) - y(-y + x(1 - x^2 - y^2))}{x^2 + y^2}$$

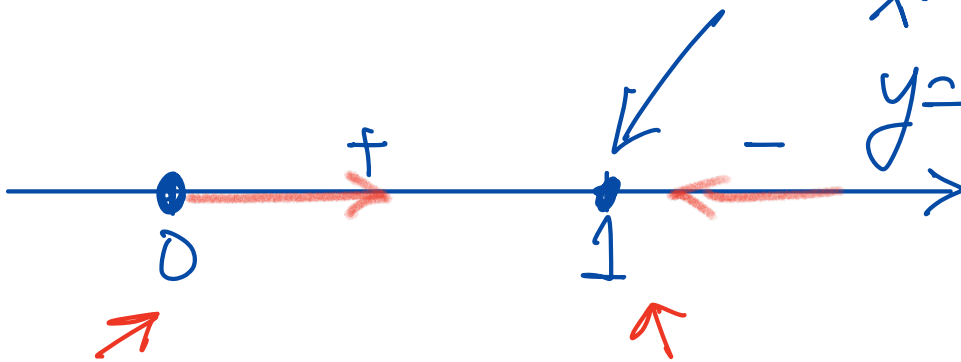
$$= \frac{x^2 + y^2}{x^2 + y^2}$$

$$\underline{\dot{\theta} = 1}$$

$$\dot{r} = r(1-r^2)$$

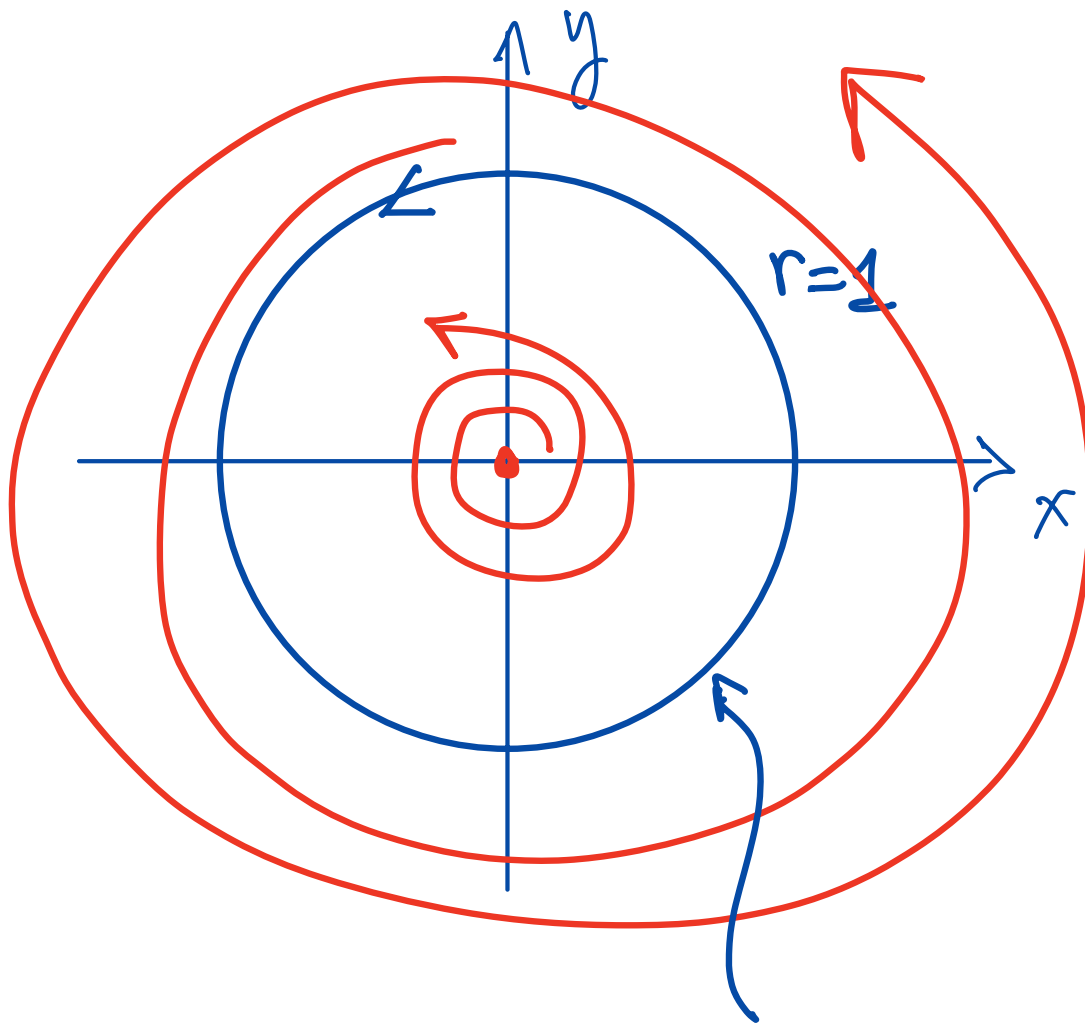
2π -per. solution

$$x = \cos t$$
$$y = \sin t$$



$r=0$, unstable

$r=1$, stable



Asymptotically stable 2π -per. solution

$$(1)_2 \quad F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y + x(1-x^2-y^2) \\ x + y(1-x^2-y^2) \end{pmatrix}$$

$$DF\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-x^2-y^2-2x^2 & -1-2xy \\ 1-2xy & 1-x^2-y^2-2y^2 \end{pmatrix}$$

↙ $x = \cos t, y = \sin t$ (per. solution)

$$= \begin{pmatrix} -2c^2 & -1-2sc \\ 1-2sc & -2s^2 \end{pmatrix}$$

$A(t)$

$$\text{tr } A(t) = -2$$

$$e^{\int_0^{2\pi} \text{tr } A(t) dt} = e^{-4\pi} < 1 \Rightarrow \text{asympt. stable}$$

$$[\Phi(2\pi)] = [M] = \begin{bmatrix} p'(0) & | & 0 \\ \hline \hline & | & 1 \end{bmatrix} = \begin{bmatrix} e^{-4\pi} & 0 \\ \hline & 1 \end{bmatrix}$$

(2) [T, p. 322]

$$\dot{x} = -y + x(\mu + \sigma(x^2 + y^2))$$

$$\dot{y} = x + y(\mu + \sigma(x^2 + y^2))$$

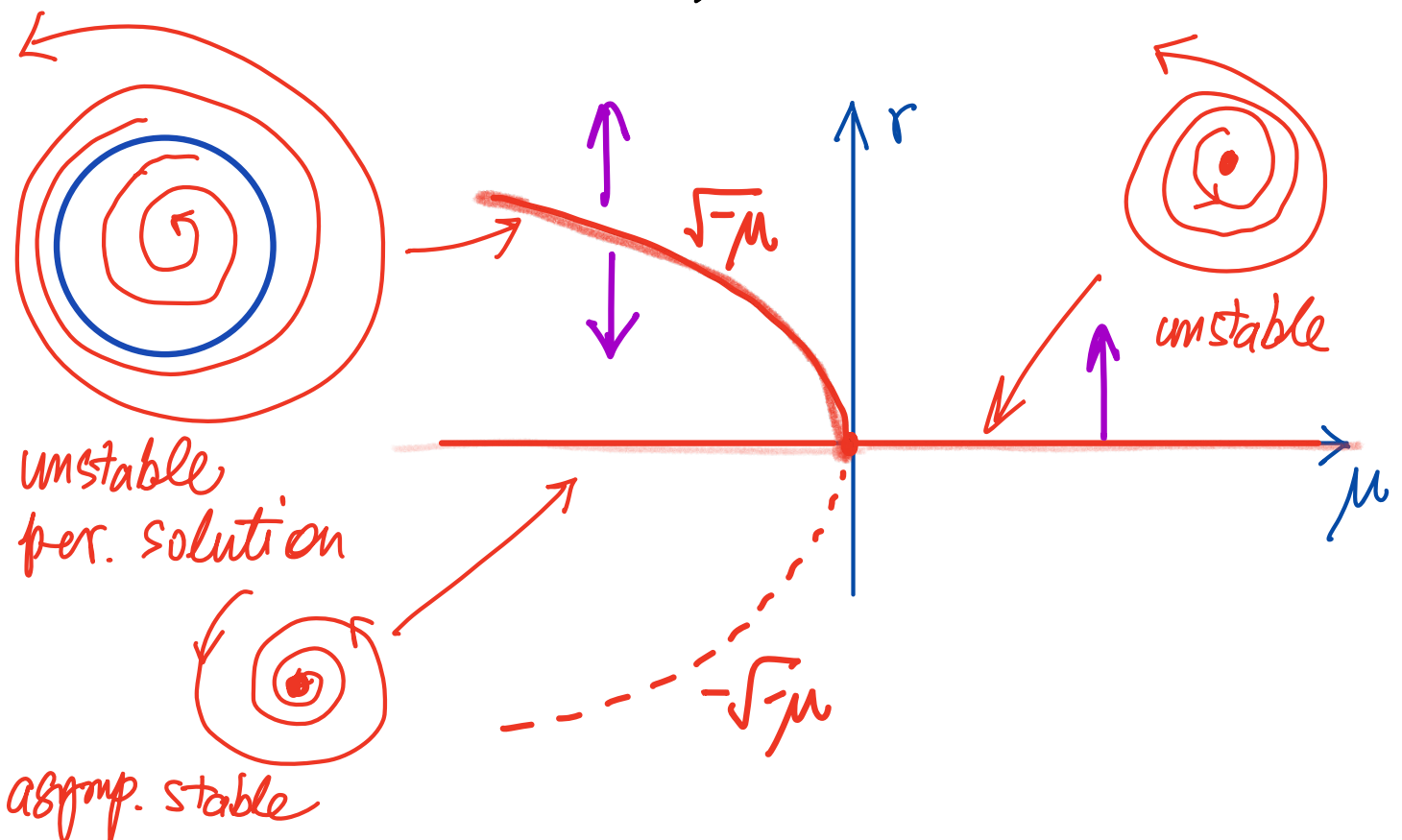
$$h(r) = \mu + \sigma r^2$$

$$\dot{r} = r h(r) = \sigma r(\mu + \sigma r^2)$$

$$\dot{\theta} = 1$$

$\sigma = 1, \mu < 0$ $\dot{r} = r(\mu + r^2) = r(r - \sqrt{-\mu})(r + \sqrt{-\mu})$ $r = 0, \sqrt{-\mu}$

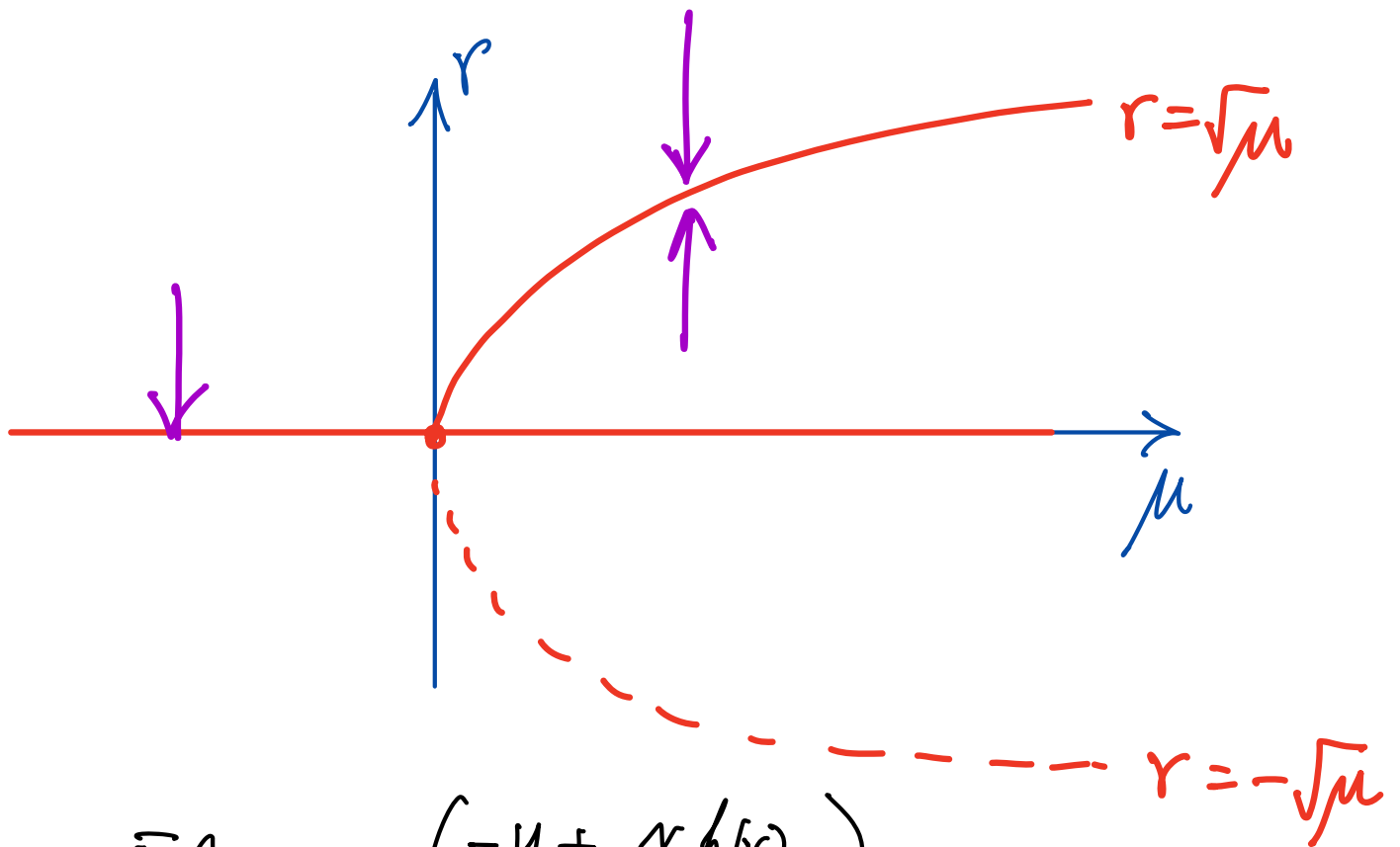
$\sigma = 1, \mu > 0$ $\dot{r} = r(\mu + r^2) > 0$ $r = 0$



$$\underline{\sigma = -1, \mu > 0} \quad \dot{r} = r(\mu - r^2) = r(\sqrt{\mu} - r)(\sqrt{\mu} + r)$$

$r = 0, \sqrt{\mu}$

$$\underline{\sigma = -1, \mu < 0}, \quad \dot{r} = r(\mu - r^2) < 0 \quad \underline{r = 0}$$



$$\vec{F}(x, y) = \begin{pmatrix} -y + x h(r) \\ x + y h(r) \end{pmatrix}$$

$$\operatorname{div} \vec{F} = h(r) + x h'(r) \frac{x}{r} + h(r) + y h'(r) \frac{y}{r}$$

$$= 2h(r) + r h'(r) \quad h(r) = \mu + \sigma r^2$$

$$= 2\mu + 2\sigma r^2 + r 2\sigma r = 2\mu + 4\sigma r^2$$

$$= 2(\mu + 2\sigma r^2)$$

$$\underline{\sigma = 1, \mu < 0} \Rightarrow r = \sqrt{-\mu}$$

$$\text{div}F = 2(\mu + 2(-\mu)) = -2\mu > 0$$

unstable

$$\underline{\sigma = -1, \mu > 0} \Rightarrow r = \sqrt{\mu}$$

$$\text{div}F = 2(\mu - 2\mu) = -2\mu < 0$$

stable

(3) Example from ecology (population growth)

Lotka - Volterra (Predator-Prey)

$$\begin{aligned} \dot{x} &= (A - By)x && \text{— prey (food)} \\ \dot{y} &= (Cx - D)y && \text{— predator} \end{aligned}$$

$$A, B, C, D = 1$$

$$\Rightarrow \underline{\text{eq. pt } x=1, y=1}$$

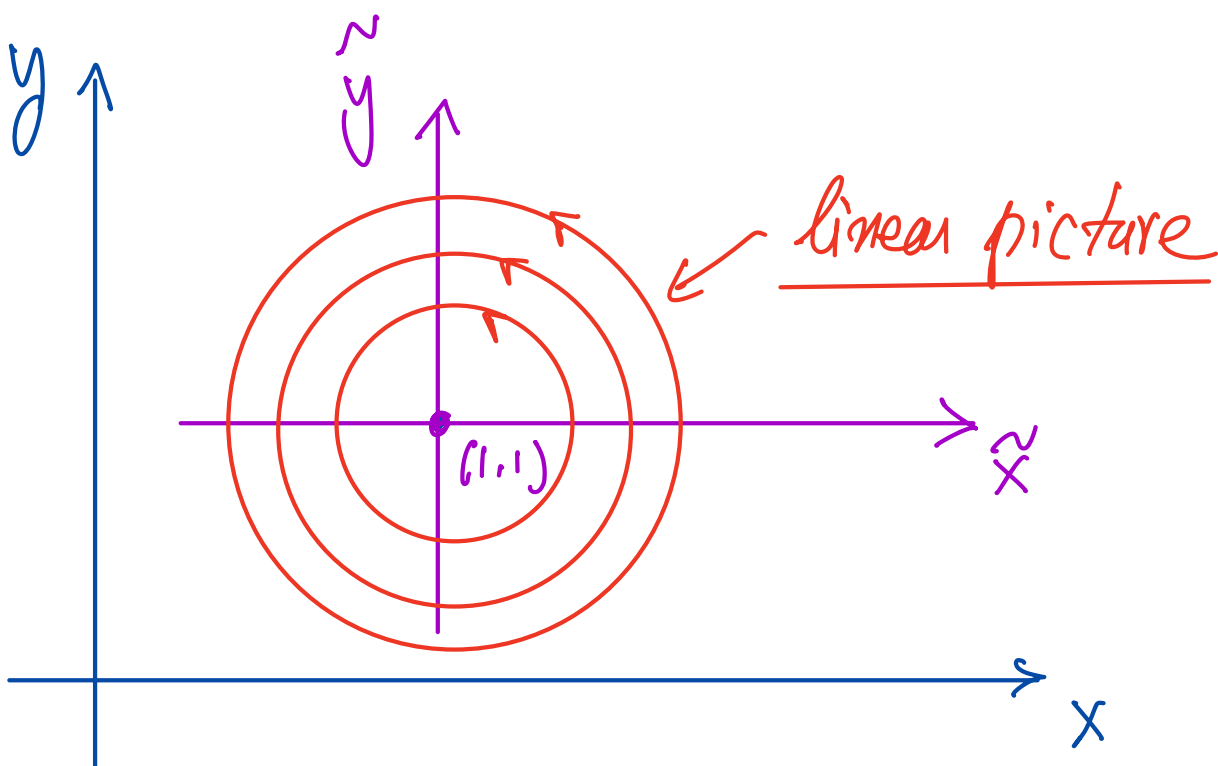
↙ what type?

$$\tilde{x} = x - 1, \quad \tilde{y} = y - 1$$

$$\begin{cases} \dot{x} = (1-y)x \\ \dot{y} = (x-1)y \end{cases} \iff \begin{cases} \dot{\tilde{x}} = -\tilde{y}(\tilde{x}+1) \\ \dot{\tilde{y}} = \tilde{x}(\tilde{y}+1) \end{cases}$$

$$\frac{d}{dt} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\lambda_1, \lambda_2 = \pm i, \text{ center}} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} -\tilde{x}\tilde{y} \\ +\tilde{x}\tilde{y} \end{pmatrix}$$

$\lambda_1, \lambda_2 = \pm i$, center



Nonlinear picture?

$$\dot{x} = (1-y)x$$

$$\dot{y} = (x-1)y$$

$$\frac{dx}{dy} = \frac{(1-y)x}{(x-1)y}$$

$$\frac{x-1}{x} dx = - \left(\frac{y-1}{y} \right) dy$$

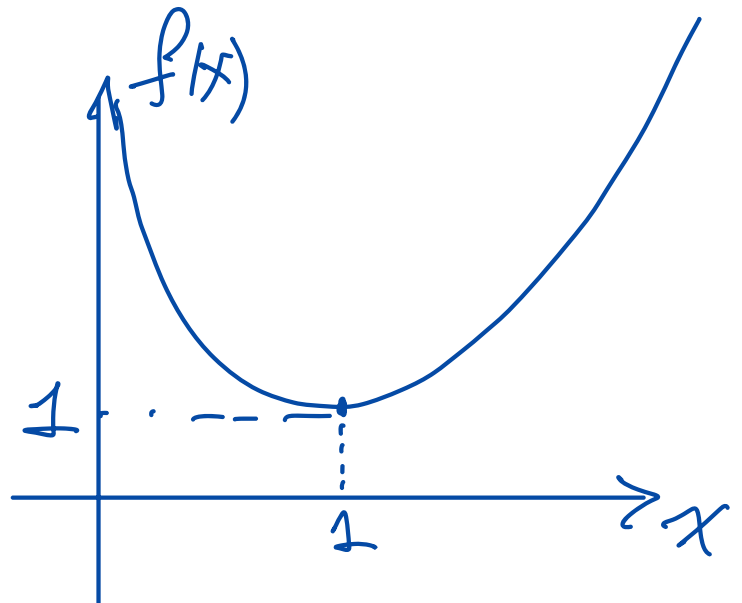
$$\int \left(1 - \frac{1}{x} \right) dx = \int - \left(1 - \frac{1}{y} \right) dy$$

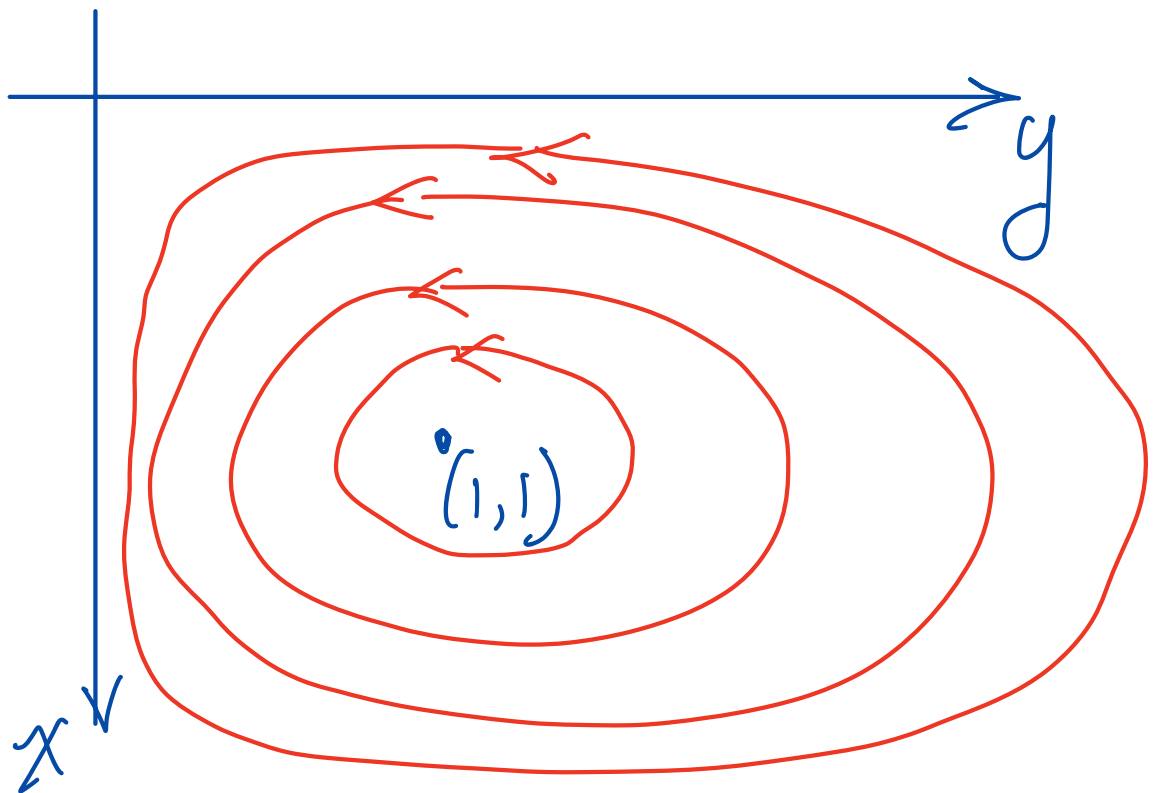
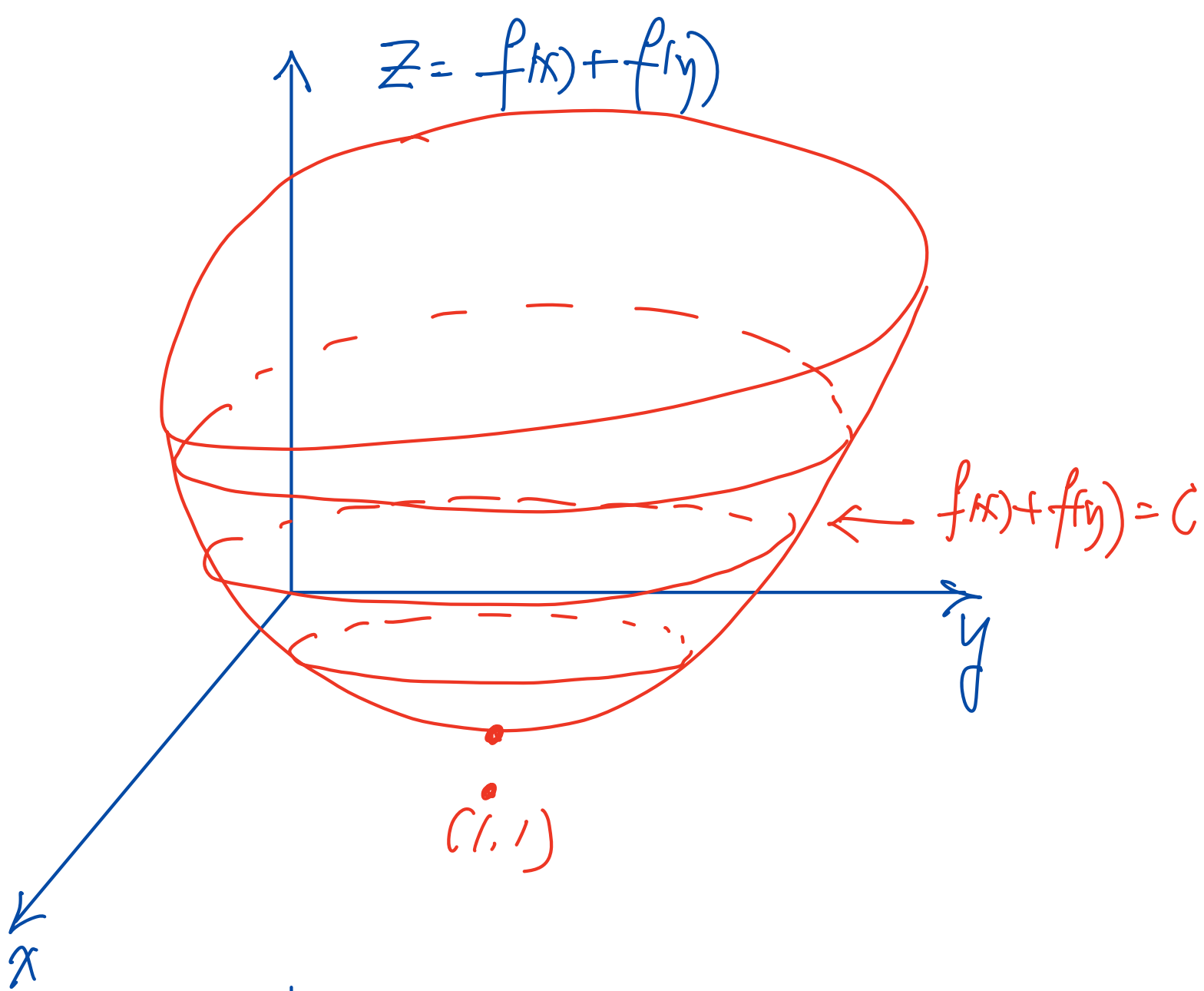
$$x - \log x = - (y - \log y) + C$$

$$\underbrace{(x - \log x)} + (y + \log y) = C$$

$$f(x) = x - \log x$$

$$> 0$$





(4) Hamiltonian System

$$H: (X, Y) \in \mathbb{R}^{2n} \longrightarrow H(X, Y) \in \mathbb{R}$$

$$\dot{X} = \nabla_Y H(X, Y)$$

$$\dot{Y} = -\nabla_X H(X, Y)$$

$$F(X, Y)$$

$$\frac{d}{dt} H(X(t), Y(t)) = 0$$

$$H(X(t), Y(t)) = \text{Const.}$$

$$\text{div}_{(X, Y)} F(X, Y)$$

$$= \partial_{x_i} \partial_{y_i} H - \partial_{y_i} \partial_{x_i} H$$

$$= 0$$

$$n=1 \quad \dot{x} = \partial_y H(x, y)$$

$$\dot{y} = -\partial_x H(x, y)$$

$$\text{div}_{(x, y)} F = \partial_x \partial_y H - \partial_y \partial_x H = 0$$

Hamiltonian system is extremely degenerate!

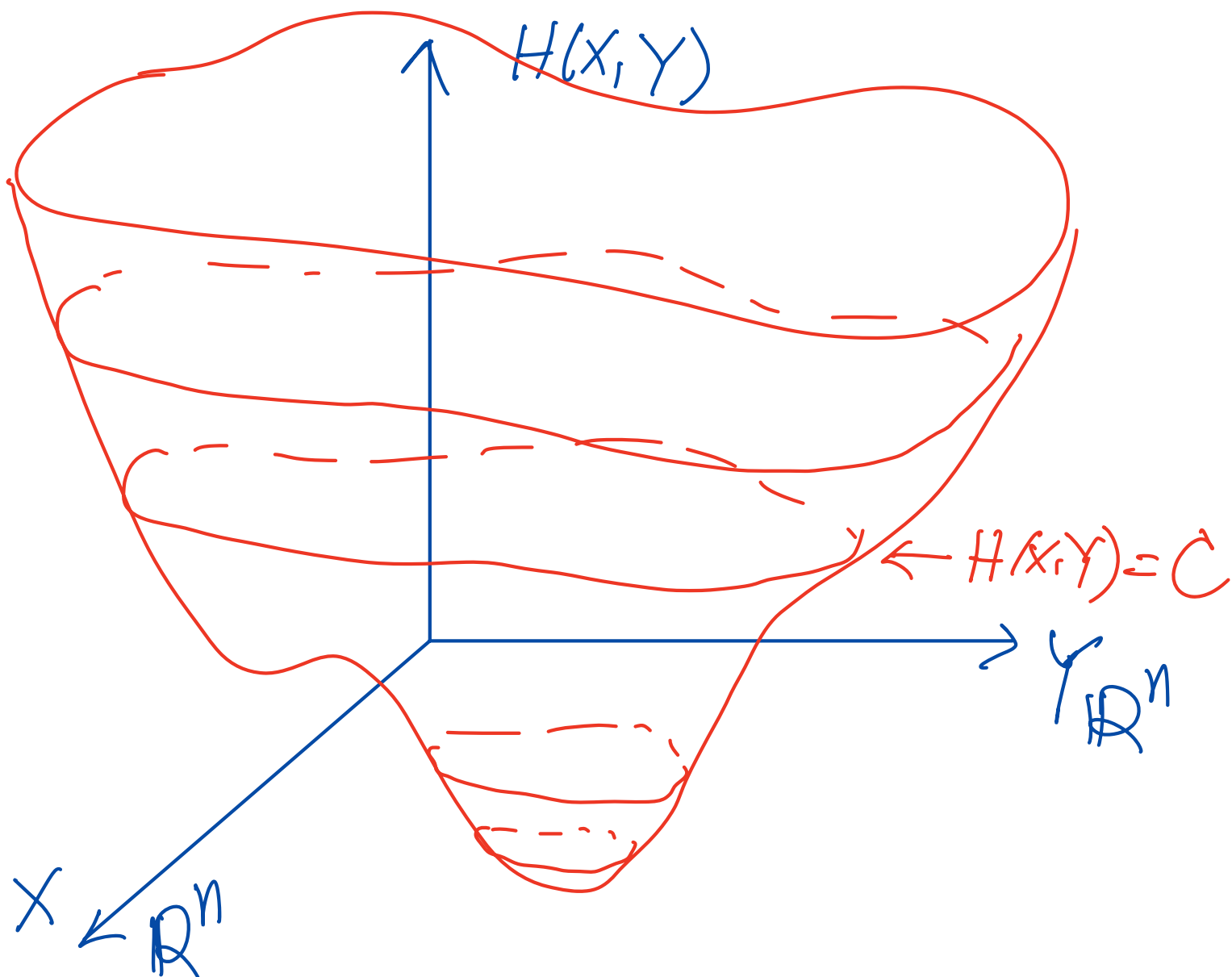
"Usually",

$$H(x, Y) = P.E. + K.E$$

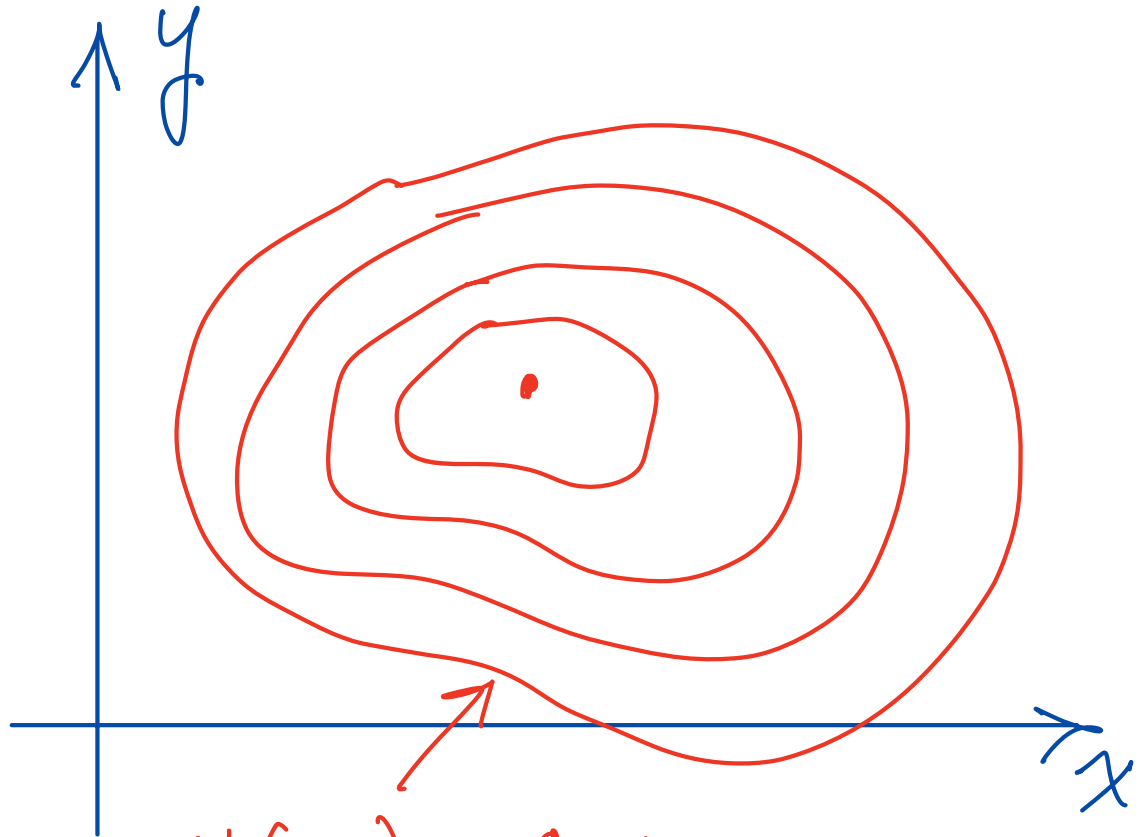
$$= V(x) + \frac{1}{2m} |Y|^2$$

$+\infty$ as $|x| \rightarrow \infty$

$+\infty$ as $|Y| \rightarrow \infty$



$n=1$. one degree of freedom



$H(x,y) = \text{const}$
(periodic solution)

Persistence of Periodic Orbit

$$\frac{dx}{dt} = F(x, \mu) \quad (*)$$

Suppose $(*)$ has a periodic solution $\gamma_0(t)$ at $\mu=0$

A sufficient condition for the existence of periodic solution $\gamma_\mu(t)$ for $0 < |\mu| \ll 1$ is

$\mu=1$ is a simple eigenvalue for $\Phi(T)$

[H, Thm 2.4]

where $\Phi(T)$ is the monodromy matrix for the linearized system at $\gamma_0(t)$:

$$\frac{dZ}{dt} = A(t) Z \quad \leftarrow D_x F(\gamma_0(t), 0)$$

But this is not true for Hamiltonian system.

① Consider $n=1$ (Hamiltonian system with 1 deg. of freedom)

$$\begin{cases} \dot{x} = \partial_y H(x, y) \\ \dot{y} = -\partial_x H(x, y) \end{cases} \leftarrow \text{suppose it has a periodic solution } \gamma(t)$$

$$\Phi(T) = M^{2 \times 2} = \left[\begin{array}{c|c} P'(0) & 0 \\ \hline \text{XXXXXX} & 1 \end{array} \right]$$

$$P'(0) = e^{\int_0^T \text{div } F(\gamma(t)) dt}$$

from time shift

$$F(x, y) = (\partial_y H(x, y), -\partial_x H(x, y))$$

$$\text{div } F = \partial_x \partial_y H - \partial_y \partial_x H = 0$$

Hence $P'(0) = e^0 = 1$

ie. $\mu = 1$ is a repeated eigenvalue of M

[H, Thm 2.4] cannot be "easily" utilized

In fact $\gamma(t)$ can be easily "destroyed."

$$\dot{x} = \partial_y H(x, y) + \varepsilon g_1(x, y)$$

$$\dot{y} = -\partial_x H(x, y) + \varepsilon g_2(x, y)$$

$$\varepsilon \ll 1$$

Consider $\frac{d}{dt} H(x(t), y(t))$

$$= (\partial_x H) \dot{x} + (\partial_y H) \dot{y}$$

$$= (\partial_x H) (\partial_y H + \varepsilon g_1) + (\partial_y H) (-\partial_x H + \varepsilon g_2)$$

$$= \varepsilon [(\partial_x H) g_1 + (\partial_y H) g_2]$$

choose g_1, g_2 s.t.
it is < 0

eg $g_1 \equiv 0, g_2 \equiv -\partial_y H$

$$\dot{x} = \partial_y H(x, y)$$

$$\dot{y} = -\partial_x H(x, y) - \nu \dot{x}$$

friction

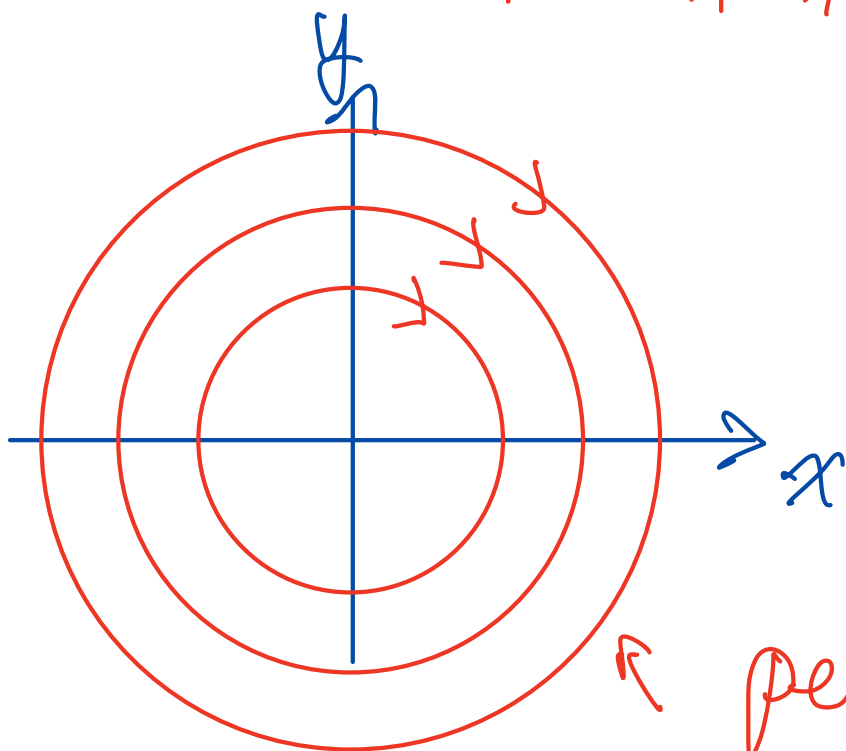
$$-\nu \partial_y H(x, y)$$

②

$$\frac{dX}{dt} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} X = \begin{pmatrix} y \\ -x \end{pmatrix}$$

Harmonic oscillator

$\lambda = \pm i$, non-hyperbolic



periodic solutions
"degenerate"

$$F(x, y) = (y, -x)$$

$$\operatorname{div} F = \partial_x y + \partial_y (-x) = 0 = P'(0)$$