Perturbation/Persistence of Periodic Orbits

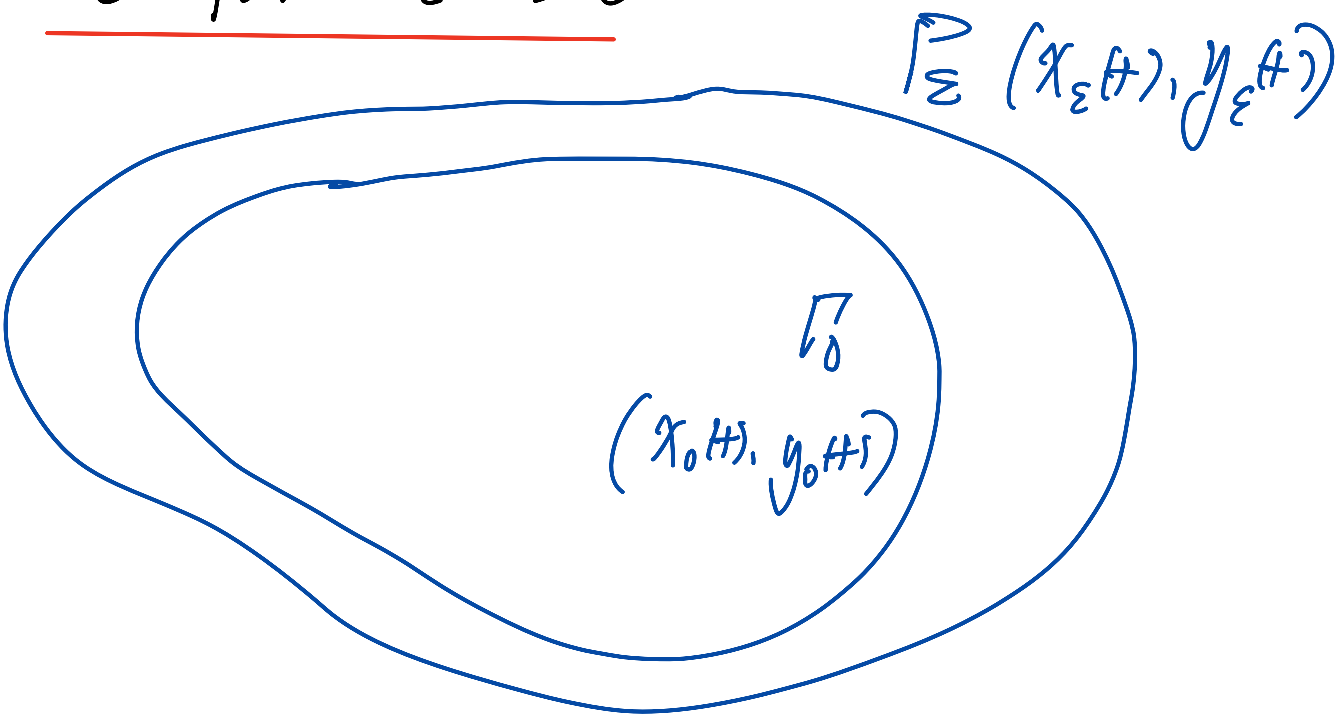
(Hamiltonian System, Melnikov Integral)

$$(I)_0 \quad \begin{cases} \dot{x} = \partial_y H(x,y) =: f_1(x,y) \\ \dot{y} = -\partial_x H(x,y) =: f_2(x,y) \end{cases}$$

H - Hamiltonian function

$$(I)_\varepsilon \quad \begin{cases} \dot{x} = \partial_y H(x,y) + \varepsilon g_1(x,y) \\ \dot{y} = -\partial_x H(x,y) + \varepsilon g_2(x,y) \end{cases}$$

Suppose there are periodic orbits Γ_0 and Γ_ε for all $\varepsilon > 0$



Line Integral of H along the orbits :

$$\int_{\Gamma_0} dH = 0 \quad \text{and} \quad \int_{\Gamma_\varepsilon} dH = 0 \ \text{for all } \varepsilon > 0$$

$$\left(\text{In general} \int_{\substack{\text{Any closed} \\ \text{orbit}}} dH = \int \left(\frac{dH}{dt}\right) dt = 0 \right)$$

$$\int_{\Gamma_0} dH = \int_{\Gamma_0} H_x dx + H_y dy$$

$$= \int_{\Gamma_0} (H_x \dot{x} + H_y \dot{y}) dt$$

$$= \int_{\Gamma_0} \big(H_x (H_y) + H_y (-H_x)\big) dt$$

$$= 0$$

$$\int_{\Gamma_\varepsilon} dH = \int_{\Gamma_\varepsilon} (H_x \dot{x}_\varepsilon + H_y \dot{y}_\varepsilon) dt \quad = 0 \text{ for all } \varepsilon$$

$$= \int_{\Gamma_\varepsilon} H_x(\cancel{H_y} + \varepsilon g_1) + H_y(\cancel{-H_x} + \varepsilon g_2)\, dt$$

$$= \varepsilon \underbrace{\int_{\Gamma_\varepsilon} (H_x g_1 + H_y g_2)\, dt}_{=0} = 0 \text{ for all } \varepsilon$$

i.e. $$\int_{\Gamma_\varepsilon} (H_x g_1 + H_y g_2)\, dt = 0$$

$$\varepsilon \to 0 \Rightarrow \int_{\Gamma_0} (H_x g_1 + H_y g_2)\, dt = 0$$

Melnikov Integral

or $$\boxed{\int_{\Gamma_0} (g_2 f_1 - g_1 f_2)\, dt = 0}$$

Notation: $$g_2 f_1 - g_1 f_2 = \begin{vmatrix} f_1 & g_1 \\ f_1 & g_2 \end{vmatrix} = \vec{f} \wedge \vec{g}$$

$$\underbrace{}_{\vec{f}} \quad \underbrace{}_{\vec{g}}$$

van der Pol Oscillator

$$\ddot{x} + \varepsilon (x^2-1)\dot{x} + x = 0$$

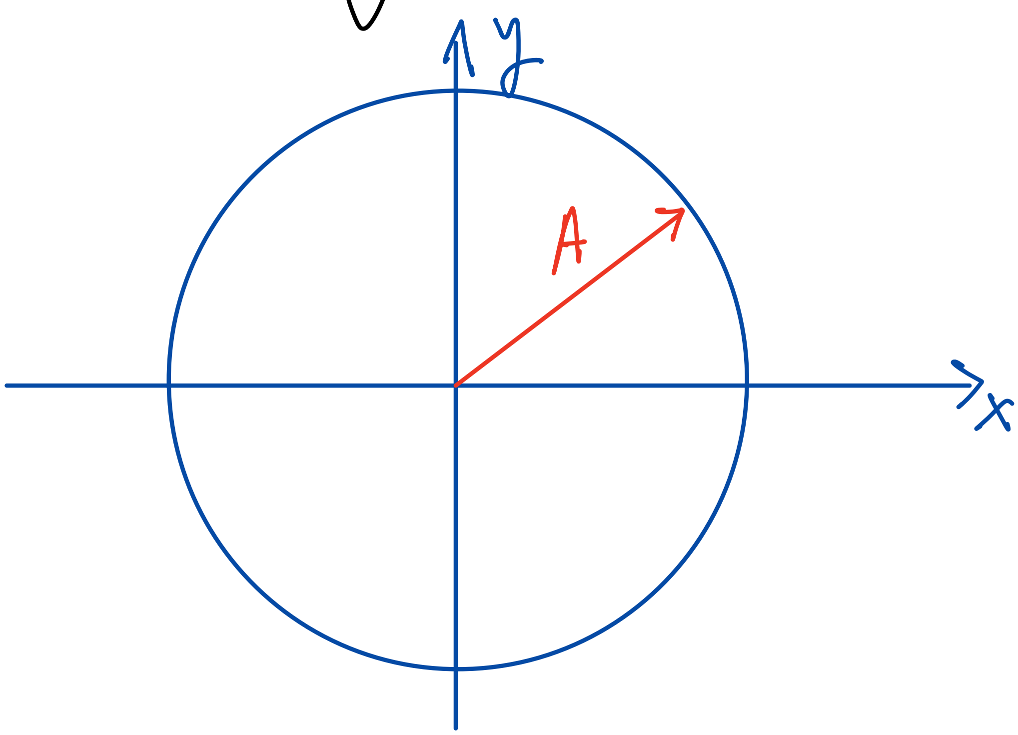
$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \varepsilon (x^2-1) y \end{cases}$$

$f_1 = y$ $\qquad$ $g_1 = 0$

$f_2 = -x$ $\qquad$ $g_2 = -(x^2-1)y$

at $\varepsilon = 0$ $\qquad$ $x = A\cos t$

$y = -A\sin t$



$$\int (f_1 g_2 - f_2 g_1)\, dt$$

$$= \int \left(-y\,(x^2-1)\,y \;-\; (-x)\,(0) \right) dt$$

$$= \int_0^{2\pi} -y^2(t)\left(x^2(t)-1\right) dt$$

$$= \int_0^{2\pi} A^2 \sin^2 t \left(1 - A^2 \cos^2 t\right) dt$$

$$= A^2 \int_0^{2\pi} \sin^2 t \, dt \;-\; A^4 \int_0^{2\pi} \sin^2 t \cos^2 t \, dt$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$\sin^2\alpha\cos^2\alpha = \frac{1}{4}\sin^2 2\alpha = \frac{1}{8}\left(1 - \cos 4\alpha\right)$$

$$= \frac{A^2}{2}\int_0^{2\pi}(1-\cancel{\cos 2t})\,dt - \frac{A^4}{8}\int_0^{2\pi}(1-\cancel{\cos 4t})\,dt$$

$$= \left(\frac{A^2}{2} - \frac{A^4}{8}\right)2\pi$$

$$= \frac{\pi}{4}A^2(4-A^2) = 0 \Rightarrow A = 0 \text{ or } \underline{2}$$

For stability : $\dot{X} = F(X)$

$$\int_0^T \operatorname{div} F(X(t))\,dt$$

$$= \int_0^T \operatorname{div}(f_1 + \varepsilon g_1,\ f_2 + \varepsilon g_2)\,dt$$

$$= \int_0^T (f_1)_x + \varepsilon(g_1)_x + (f_2)_y + \varepsilon(g_2)_y\,dt$$

$$= \int_0^T (\cancel{(H_y)_x + (-H_x)_y})\,dt + \varepsilon\int_0^T ((g_1)_x + (g_2)_y)\,dt$$

$$= \varepsilon \int_0^T (g_1)_x + (g_2)_y \, dt$$

$$g_1 = 0, \quad g_2 = -y(x^2 - 1)$$

$$= \varepsilon \underbrace{\int_0^T (1 - x^2)\, dt}_{\text{// compute at } \varepsilon = 0}$$

$$\int_0^T (1 - x^2)\, dt = \int_0^{2\pi} (1 - A^2 \cos^2 t)\, dt$$

$$= \int_0^{2\pi} (1 - 4\cos^2 t)\, dt$$

$$= \int_0^{2\pi} \left(1 - 2(\cancel{\cos 2t} + 1)\right) dt$$

$$= -2\pi < 0 \qquad \underline{\text{Stable}}$$