Hopf Bifurcation

Yet another example using van der Pol osc.

$$\ddot{x} + (3x^2 - \mu)\dot{x} + x = 0 \qquad \mu \approx 0$$

$$\left(\dot{x} + x^3 - \mu x\right)^{\cdot} + x = 0$$

$$y = \dot{x} + x^3 - \mu x$$

Lienard system ($\mu > 0$)
There is a unique periodic orbit

$$\begin{cases} \dot{x} = y - x^3 + \mu x \\ \dot{y} = -x \end{cases}$$

Use Polar coordinates,

$$\dot{\theta} = \left(\tan^{-1}\left(\frac{y}{x}\right)\right)^{\cdot} = \frac{1}{1 + \frac{y^2}{x^2}} \frac{x\dot{y} - y\dot{x}}{x^2}$$

$$\dot{\theta} = \frac{1}{r^2}\left(x(-x) - y\left(y - x^3 + \mu x\right)\right)$$

$$= \frac{1}{r^2}\left(-r^2 + yx^3 - \mu xy\right)$$

$$\dot{\theta} = -1 + r^2 \sin\theta \cos^3\theta - \mu \cos\theta \sin\theta$$

≈ -1 when r, μ small

$$r\dot{r} = x\dot{x} + y\dot{y} = x(y - x^3 + \mu x) + y(-x)$$

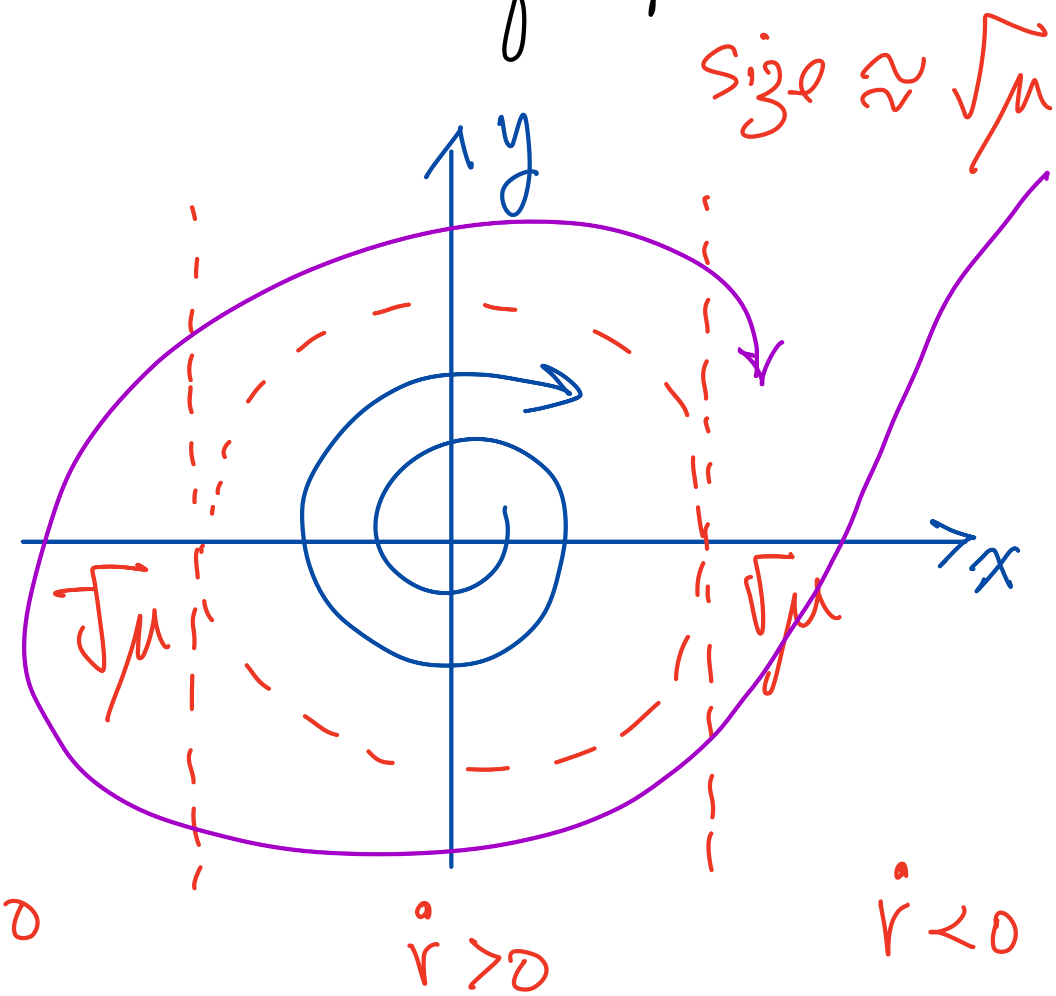
$$= -x^4 + \mu x^2 = x^2(\mu - x^2)$$

$$\boxed{\dot{r} = \frac{1}{r} x^2(\mu - x^2)}$$

$\mu \leq 0 \Rightarrow \dot{r} < 0$, origin is stable, no periodic orbit.

$\mu > 0 \Rightarrow$ Lienard system

$\Rightarrow$ there is a unique per. orbit

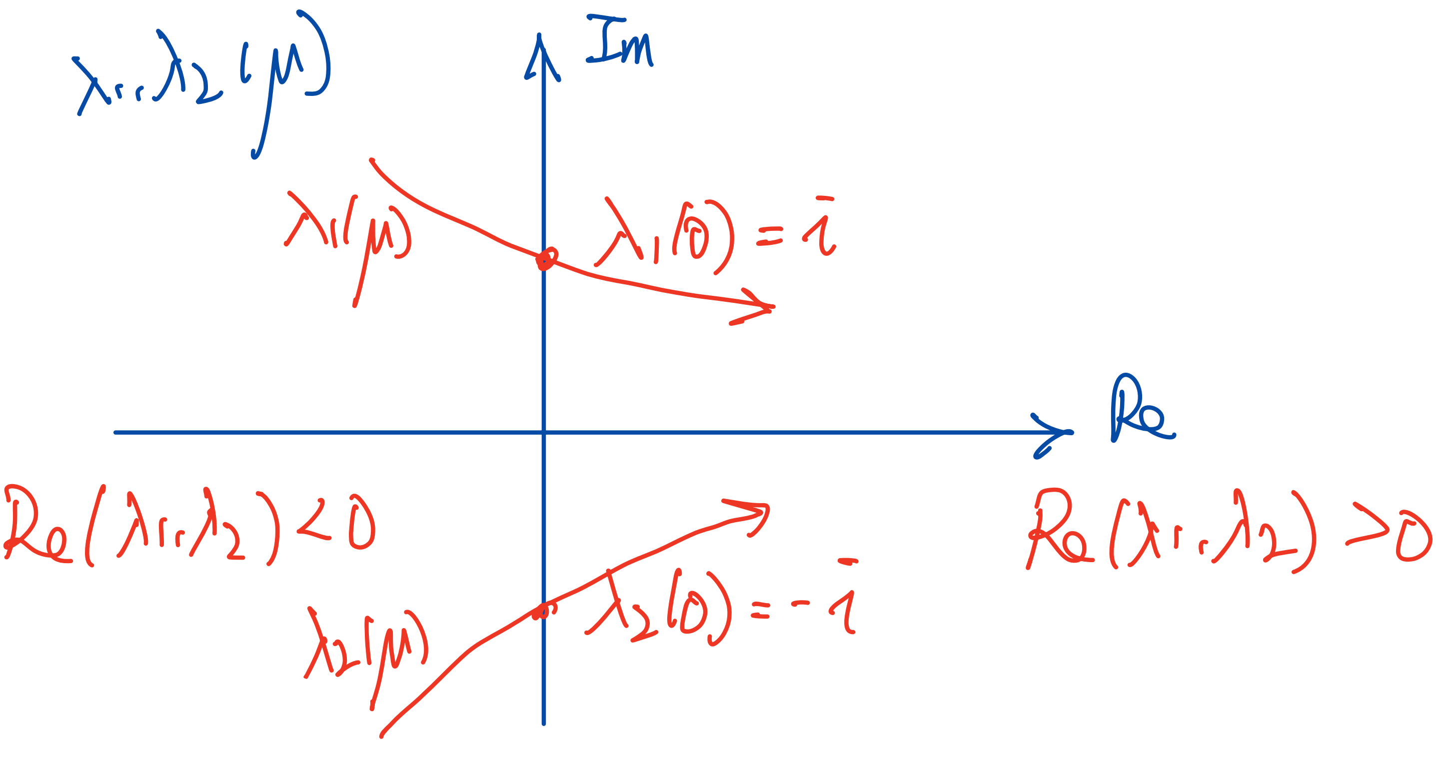


$$\dot{x} = \mu x + y - x^3$$
$$\dot{y} = -x$$

$$\begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \underbrace{\begin{pmatrix} \mu & 1 \\ -1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x^3 \\ 0 \end{pmatrix}$$

$$\det(A - \lambda I) = (\lambda - \mu)\lambda + 1 = \lambda^2 - \mu\lambda + 1 = 0$$

$$\lambda_1, \lambda_2 = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$
$$= \frac{\mu}{2} \pm \frac{\sqrt{4 - \mu^2}}{2} i \qquad |\mu| \ll 1$$



[HSD]

Notice that, along γ_1, $y(t)$ may be regarded as a function of x. Thus we have

$$\delta_1(p) = \int_0^1 \frac{x^2\left(1-x^2\right)}{dx/dt}\, dx$$

$$= \int_0^1 \frac{x^2\left(1-x^2\right)}{y - f(x)}\, dx,$$

where $f(x) = x^3 - x$. As p moves up the y-axis, $y - f(x)$ increases (for (x,y) on γ_1). Thus $\delta_1(p)$ decreases as p increases. Similarly $\delta_3(p)$ decreases as p increases.

On γ_2, $x(t)$ may be regarded as a function of y that is defined for $y \in [y_1, y_2]$ and $x \geq 1$. Therefore, since $dy/dt = -x$, we have

$$\delta_2(p) = \int_{y_2}^{y_1} -x(y)\left(1 - x(y)^2\right) dy$$

$$= \int_{y_1}^{y_2} x(y)\left(1 - x(y)^2\right) dy,$$

so that $\delta_2(p)$ is negative.

As p increases, the domain $[y_1, y_2]$ of integration becomes steadily larger. The function $y \to x(y)$ depends on p, so we write it as $x_p(y)$. As p increases, the curves γ_2 move to the right; thus $x_p(y)$ increases and so $x_p(y)(1 - x_p(y)^2)$ decreases. It follows that $\delta_2(p)$ decreases as p increases, and evidently $\lim_{p\to\infty} \delta_2(p) = -\infty$. Consequently, $\delta(p)$ also decreases and tends to $-\infty$ as $p \to \infty$. This completes the proof of the proposition.

12.4 A Hopf Bifurcation

We now describe a more general class of circuit equations where the resistor characteristic depends on a parameter μ and is denoted by f_μ. (Perhaps μ is the temperature of the resistor.) The physical behavior of the circuit (see Figure12.9) is then described by the system of differential equations on $\mathbb{R}^2$:

$$\frac{dx}{dt} = y - f_\mu(x),$$

$$\frac{dy}{dt} = -x.$$

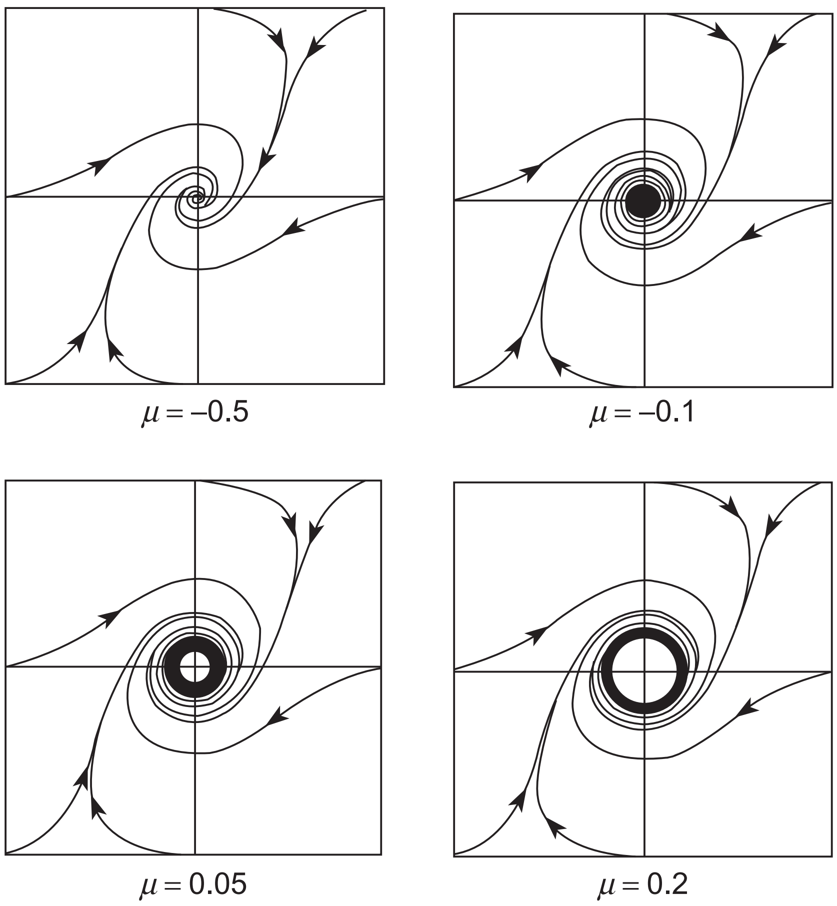


Figure 12.9 Hopf bifurcation in the system $x' = y - x^3 + \mu x,\ y' = -x$.

Consider as an example the special case where f_μ is described by

$$f_\mu(x) = x^3 - \mu x$$

and the parameter μ lies in the interval $[-1, 1]$. When $\mu = 1$ we have the van der Pol system from the previous section. As before, the only equilibrium point lies at the origin. The linearized system is

$$Y' = \begin{pmatrix} \mu & 1 \\ -1 & 0 \end{pmatrix} Y,$$

and the eigenvalues are

$$\lambda_{\pm} = \frac{1}{2}\left(\mu \pm \sqrt{\mu^2 - 4}\right).$$

Thus the origin is a spiral sink for $-1 \le \mu < 0$ and a spiral source for $0 < \mu \le 1$. Indeed, when $-1 \le \mu \le 0$, the resistor is passive as the graph of f_μ lies in the first and third quadrants. Therefore, all solutions tend to the origin in this case. This holds even in the case where $\mu = 0$ and the linearization yields a center. The circuit is physically dead in that, after a period of transition, all the currents and voltages stay at 0 (or as close to 0 as we want).

However, as μ becomes positive, the circuit becomes alive. It begins to oscillate. This follows from the fact that the analysis of Section 12.3 applies to this system for all μ in the interval $(0,1]$. We therefore see the birth of a (unique) periodic solution γ_μ as μ increases through 0 (see Exercise 4 at the end of this chapter). As just shown, this solution attracts all other nonzero solutions. As in Chapter 8, Section 8.5, this is an example of a *Hopf bifurcation*. Further elaboration of the ideas in Section 12.3 can be used to show that $\gamma_\mu \to 0$ as $\mu \to 0$ with $\mu > 0$. Review Figure 12.9 for some phase portraits associated with this bifurcation.

12.5 Exploration: Neurodynamics

One of the most important developments in the study of the firing of nerve cells or neurons was the development of a model for this phenomenon in giant squid in the 1950s by Hodgkin and Huxley [23]. They developed a four-dimensional system of differential equations that described the electrochemical transmission of neuronal signals along the cell membrane, a work for which they later received the Nobel Prize. Roughly speaking, this system is similar to systems that arise in electrical circuits. The neuron consists of a cell body, or *soma*, that receives electrical stimuli. These stimuli are then conducted along the *axon*, which can be thought of as an electrical cable that connects to other neurons via a collection of synapses. Of course, the motion is not really electrical, as the current is not really made up of electrons but rather ions (predominantly sodium and potassium). See Edelstein-Keshet [15] or Murray [34] for a primer on the neurobiology behind these systems.

The four-dimensional Hodgkin–Huxley system is difficult to deal with primarily because of the highly nonlinear nature of the equations. An important breakthrough from a mathematical point of view was achieved by Fitzhugh [18] and Nagumo et. al. [35], who produced a simpler model of the Hodgkin–Huxley model. Although this system is not as biologically accurate as the original system, it nevertheless does capture the essential behavior of nerve impulses, including the phenomenon of *excitability* alluded to in the following.

The Fitzhugh–Nagumo system of equations is given by

$$\begin{aligned} x' &= y + x - \frac{x^3}{3} + I \\ y' &= -x + a - by, \end{aligned}$$

From Lec 20-21 [T, p. 322]

$$\dot{x} = -y + x(\mu + \sigma(x^2+y^2))$$
$$\dot{y} = x + y(\mu + \sigma(x^2+y^2))$$

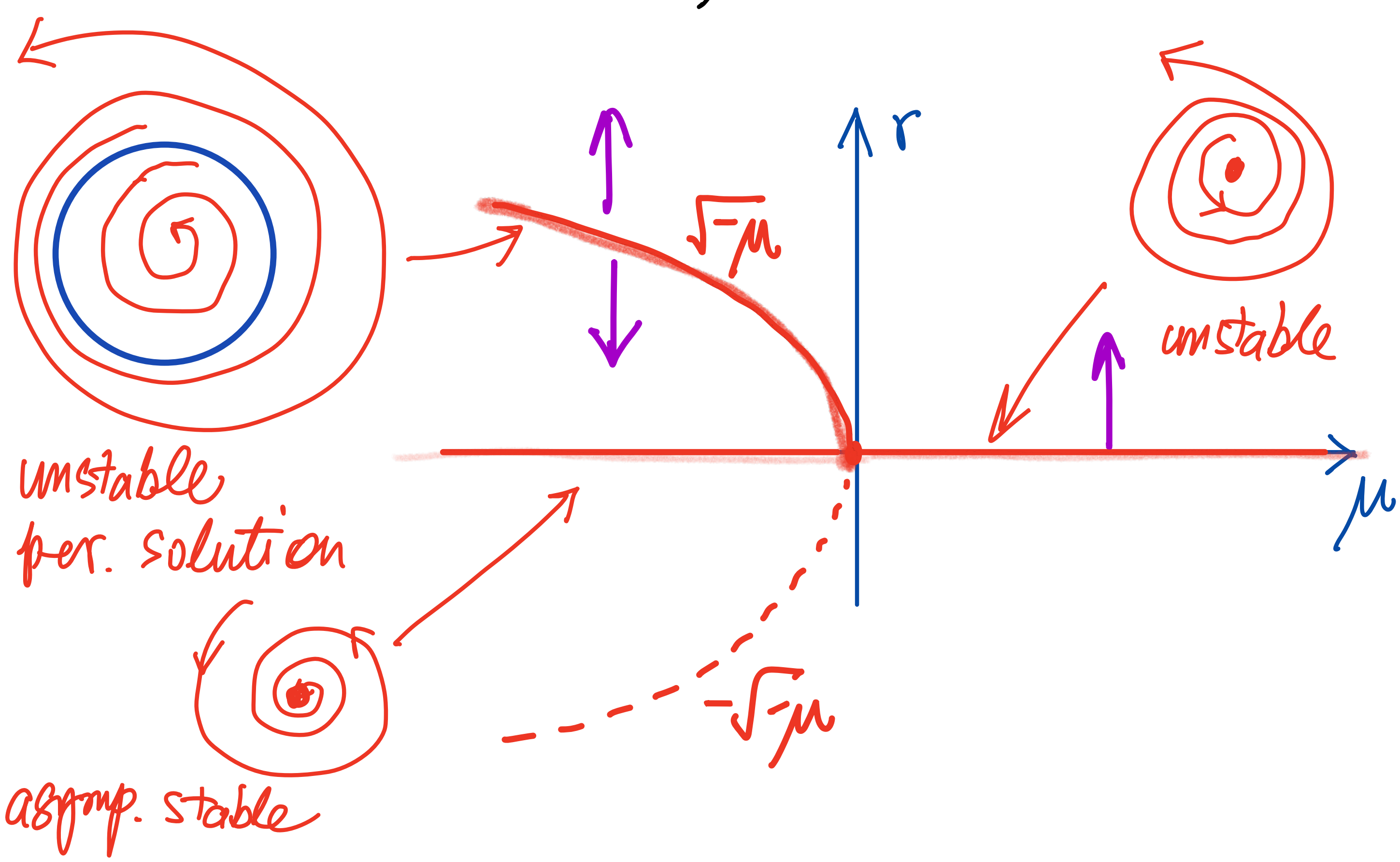
$h(r) = \mu + \sigma r^2$

$$\dot{r} = r h(r) = r(\mu + \sigma r^2)$$
$$\dot{\theta} = 1$$

$\sigma = 1, \ \mu < 0$ $\quad \dot{r} = r(\mu + r^2) = r(r - \sqrt{-\mu})(r + \sqrt{-\mu})$

$r = 0, \ \sqrt{-\mu}$

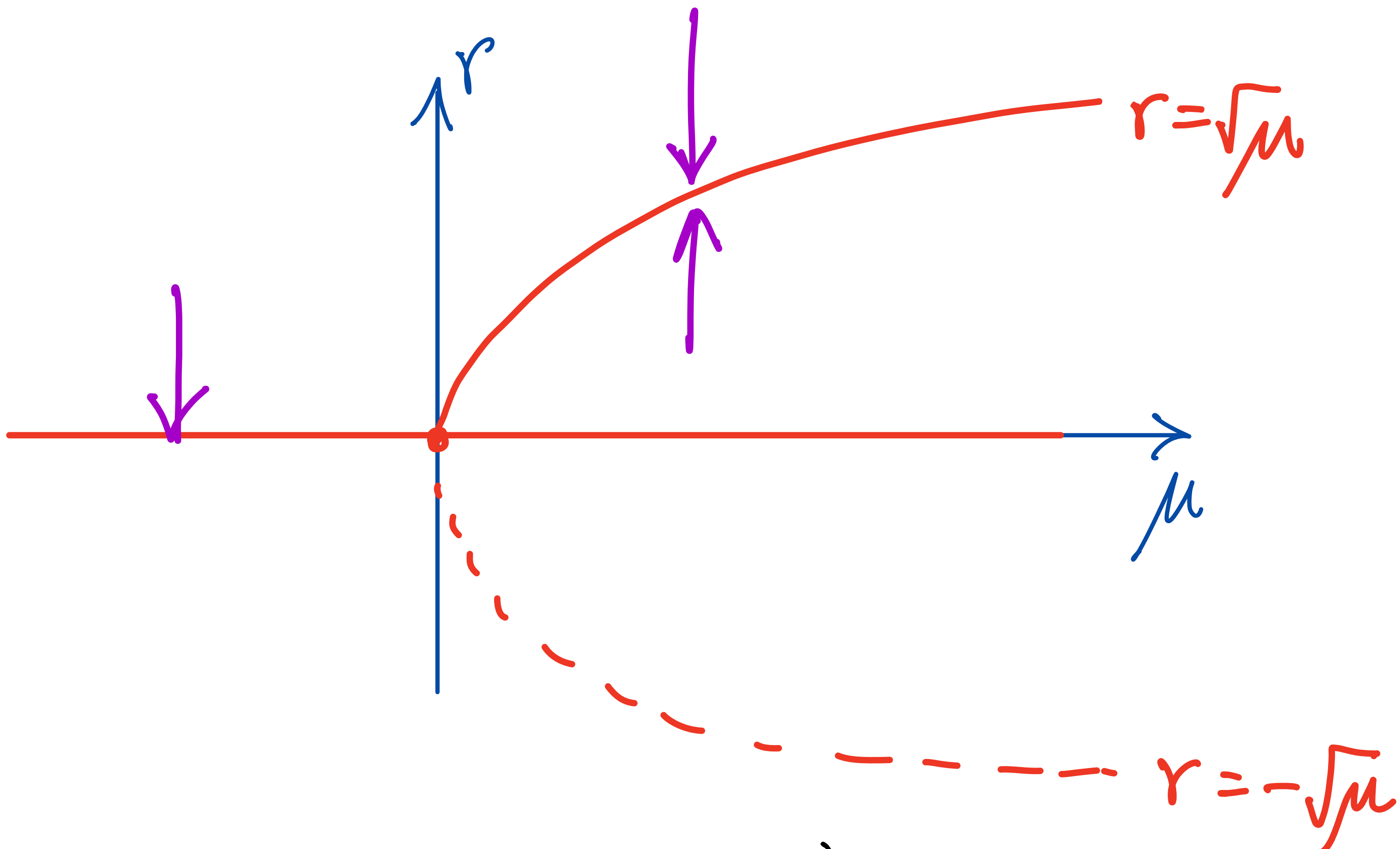
$\sigma = 1, \ \mu > 0$ $\quad \dot{r} = r(\mu + r^2) > 0$ $\quad r = 0$



$\sigma = -1, \mu > 0$ $\quad \dot{r} = r(\mu - r^2) = r(\sqrt{\mu} - r)(\sqrt{\mu} + r)$

$r = 0, \sqrt{\mu}$

$\sigma = -1, \mu < 0,$ $\quad \dot{r} = r(\mu - r^2) < 0$ $\quad r = 0$

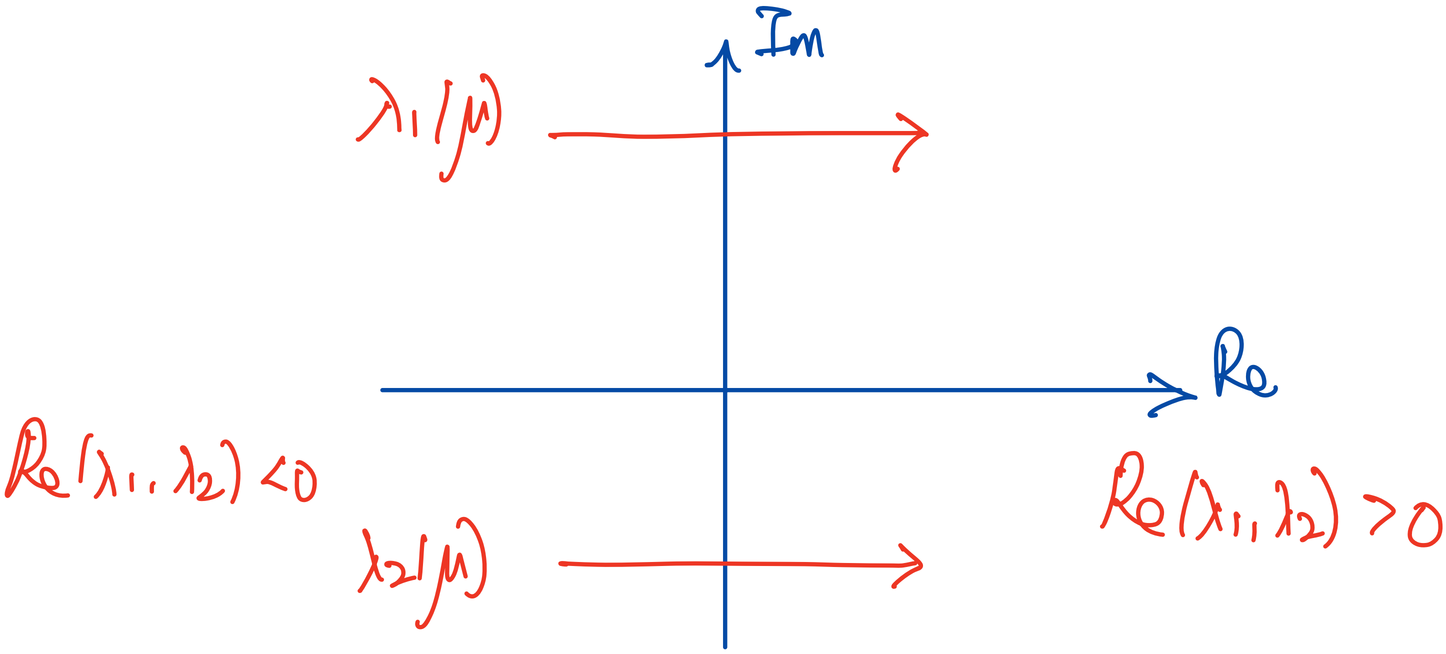


$$\dot{x} = -y + x(\mu - (x^2 + y^2))$$
$$\dot{y} = x + y(\mu - (x^2 + y^2))$$

$$\begin{pmatrix} x \\ y \end{pmatrix}^{\circ} = \underbrace{\begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x(x^2+y^2) \\ -y(x^2+y^2) \end{pmatrix}$$

$$\det(A-\lambda I) = (\lambda-\mu)^2 + 1 = 0$$

$$\lambda_1, \lambda_2 = \mu \pm i$$



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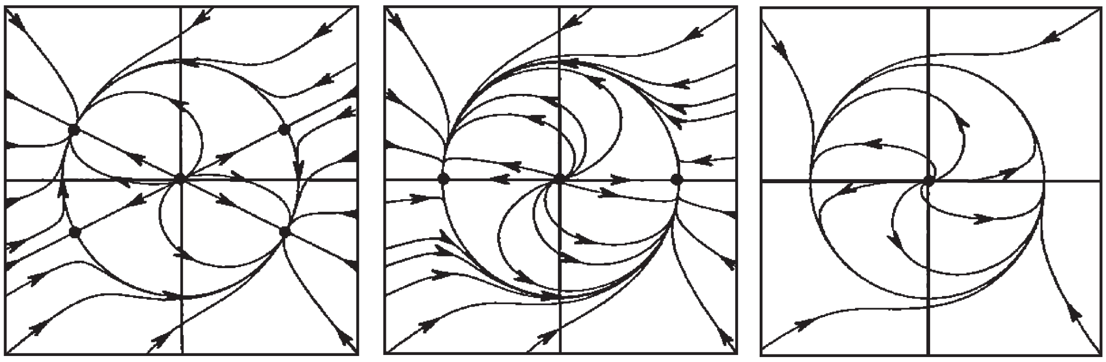

Figure 8.9 Global effects of saddle-node bifurcations when $a < 0$, $a = 0$, and $a > 0$.

and $\theta = \pi + \theta_+$, and the unstable curves of the saddles are given by the unit circle minus the sinks. See Figure 8.9. ■

The previous examples all featured bifurcations that occur when the linearized system has a zero eigenvalue. Another case where the linearized system fails to be hyperbolic occurs when the system has pure imaginary eigenvalues.

Example. (Hopf Bifurcation) Consider the system

$$x' = ax - y - x(x^2 + y^2)$$
$$y' = x + ay - y(x^2 + y^2).$$

There is an equilibrium point at the origin and the linearized system is

$$X' = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} X.$$

The eigenvalues are $a \pm i$, so we expect a bifurcation when $a = 0$.

To see what happens as a passes through 0, we change to polar coordinates. The system becomes

$$r' = ar - r^3$$
$$\theta' = 1.$$

Note that the origin is the only equilibrium point for this system, since $\theta' \neq 0$. For $a < 0$, the origin is a sink since $ar - r^3 < 0$ for all $r > 0$. Thus all solutions tend to the origin in this case. When $a > 0$, the equilibrium becomes a source. What else happens? When $a > 0$, we have $r' = 0$ if $r = \sqrt{a}$. So the circle of radius $\sqrt{a}$ is a periodic solution with period 2π. We also have $r' > 0$ if $0 < r < \sqrt{a}$, while $r' < 0$ if $r > \sqrt{a}$. Thus, all nonzero solutions spiral toward this circular solution as $t \to \infty$.

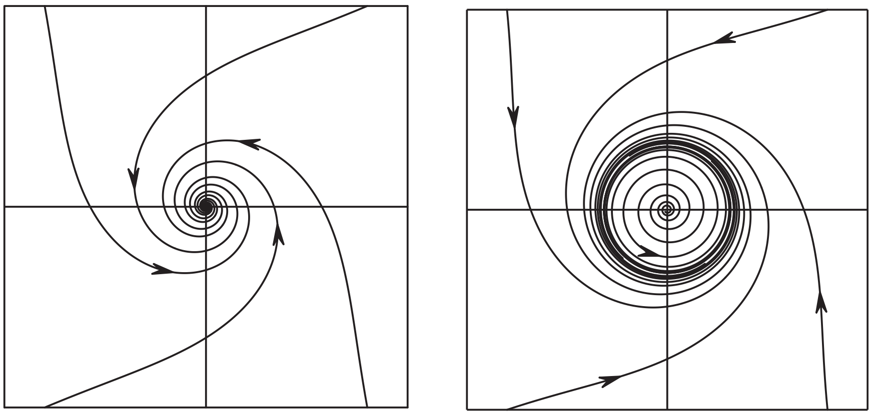

Figure 8.10 Hopf bifurcation for $a < 0$ and $a > 0$.

This type of bifurcation is called a *Hopf bifurcation*. At a Hopf bifurcation, no new equilibria arise. Instead, a periodic solution is born at the equilibrium point as a passes through the bifurcation value. See Figure 8.10. ■

8.6 Exploration: Complex Vector Fields

In this exploration, you will investigate the behavior of systems of differential equations in the complex plane of the form $z' = F(z)$. Throughout this section, z will denote the complex number $z = x + iy$ and $F(z)$ will be a polynomial with complex coefficients. Solutions of the differential equation will be expressed as curves $z(t) = x(t) + iy(t)$ in the complex plane.

You should be familiar with complex functions such as exponential, sine, and cosine, as well as with the process of taking complex square roots, to comprehend fully what you see in the following. Theoretically, you should also have a grasp of complex analysis as well. However, all of the routine tricks from integration of functions of real variables work just as well when integrating with respect to z. You need not prove this, for you can always check the validity of your solutions when you have completed the integrals.

1. Solve the equation $z' = az$ where a is a complex number. What kind of equilibrium points do you find at the origin for these differential equations?
2. Solve each of the following complex differential equations and sketch the phase portrait.

   (a) $z' = z^2$

   (b) $z' = z^2 - 1$

   (c) $z' = z^2 + 1$

[GH]

EXERCISE 3.3.2. Show that the system

$$\dot{x} = y + o(|x|, |y|),$$
$$\dot{y} = o(|x|, |y|)$$

can be transformed to

$$\begin{aligned} \dot{u} &= v + au^2 \\ \dot{v} &= bu^2 \end{aligned} \quad + o(|u|^2, |v|^2),$$

or to

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= au^2 + buv \end{aligned} \quad + o(|u|^2, |v|^2).$$

Find a basis for G_3 and give the normal form up to the third order (cf. Chapter 7).

To end this section we discuss the rôle of parameters in normal form calculations. As in the computation of center manifolds for parametrised systems, we again employ the trick of extending the system $\dot{x} = f(x, \mu)$ to the larger system

$$\begin{aligned} \dot{x} &= f(x, \mu), \\ \dot{\mu} &= 0. \end{aligned} \tag{3.3.16}$$

One can perform the normal form calculations in this larger system while insisting that the coordinate transformations $H(x, \mu)$ all be of the form $H(x, \mu) = (h(x, \mu), \mu)$. These transformations necessarily leave the equation $\dot{\mu} = 0$ invariant and will transform the system $\dot{x} = f(x, \mu)$ in a μ-dependent way. In practice, these calculations proceed as before, but the coefficients are regarded as power series in the parameters μ.

The normal form theorem described in this section is far from being the final word concerning the question of when two vector fields can be transformed into one another by a smooth change of coordinates. Apart from the Siegel and Sternberg linearization theorem referred to above, Takens [1973a] gives results concerning this question which apply to vector fields having a single zero eigenvalue or a single pair of pure imaginary eigenvalues. For equilibria with more degenerate linear parts little appears to be known.

3.4. Codimension One Bifurcations of Equilibria

In this section we describe the simplest bifurcations of equilibria. These are represented by the following four differential equations which depend on a single parameter μ:

$$\dot{x} = \mu - x^2 \quad (\textit{saddle-node}), \tag{3.4.1}$$
$$\dot{x} = \mu x - x^2 \quad (\textit{transcritical}), \tag{3.4.2}$$
$$\dot{x} = \mu x - x^3 \quad (\textit{pitchfork}), \tag{3.4.3}$$

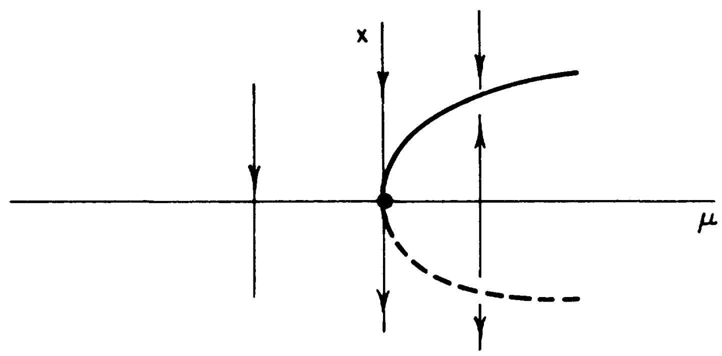

Figure 3.4.1. Saddle-node bifurcation.

and

$$\begin{cases} \dot{x} = -y + x(\mu - (x^2 + y^2)) \\ \dot{y} = x + y(\mu - (x^2 + y^2)) \end{cases} \quad (Hopf). \tag{3.4.4}$$

The bifurcation diagrams for these four equations are depicted in Figures 3.4.1–3.4.4. Each of the equations (3.4.1)–(3.4.4) arises naturally in a suitable context as determining the local qualitative behavior of the *generic* bifurcation of an equilibrium. Our purpose here is to describe in detail how, and under what conditions, one can reduce the study of the general equation (3.1.1) to one of these four specific examples.

The Saddle-Node

Consider a system of equations

$$\dot{x} = f_\mu(x), \tag{3.4.5}$$

with $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}$, and f_μ smooth. Assume that at $\mu = \mu_0$, $x = x_0$, (3.4.5) has an equilibrium at which there is a zero eigenvalue (for the linearization). Usually, this zero eigenvalue will be simple, and the center manifold theorem allows us to reduce the study of this kind of bifurcation problem to one in which x is one dimensional. More precisely, using the ideas of Section 3.2,

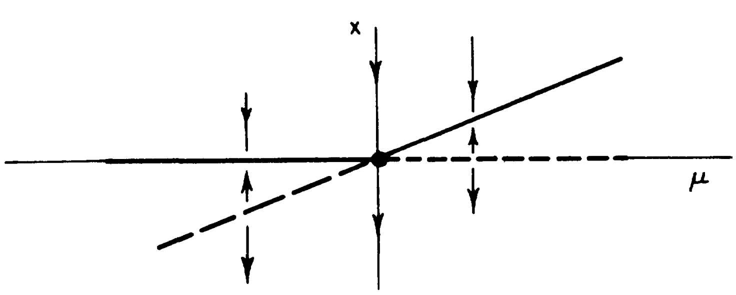

Figure 3.4.2. Transcritical bifurcation.

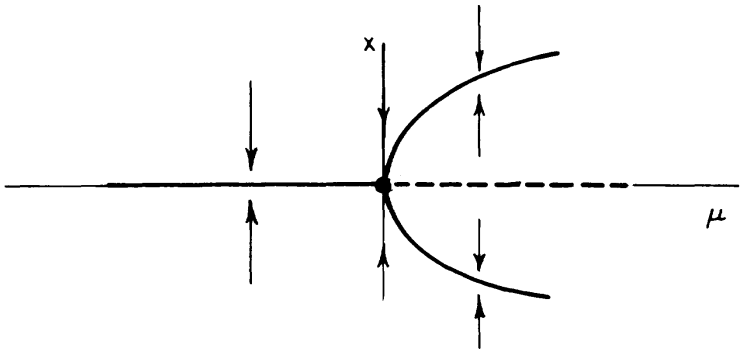

Figure 3.4.3. Pitchfork bifurcation (supercritical).

we can find a two-dimensional center manifold $\Sigma \subset \mathbb{R}^n \times \mathbb{R}$ passing through (x_0, μ_0) such that

(1) The tangent space of Σ at (x_0, μ_0) is spanned by an eigenvector of 0 for $Df_{\mu_0}(x_0)$ and a vector parallel to the μ-axis.
(2) For any finite r, Σ is C^r if restricted to a small enough neighborhood of (x_0, μ_0).
(3) The vector field of (3.4.5) is tangent to Σ.
(4) There is a neighborhood U of (x_0, μ_0) in $\mathbb{R}^n \times \mathbb{R}$ such that all trajectories contained entirely in U for all time lie in Σ.

(Note: The center manifold theorem allows one to formulate stronger properties than (4) which describe the qualitative structure of trajectories which remain close to (x_0, μ_0) in forward time or in backwards time, cf. Carr [1981].)

Restricting (3.4.5) to Σ, we obtain a one-parameter family of equations on the one-dimensional curves Σ_μ in Σ obtained by fixing μ (cf. Figure 3.2.7). This one-parameter family is our reduction of the bifurcation problem.

Let us now formulate transversality conditions for a system (3.4.5) with $n = 1$, which yield the saddle-node bifurcation. We have $(df_{\mu_0}/dx)(x_0) = 0$, but we take $(\partial f_{\mu_0}/\partial \mu)(x_0) \neq 0$ as a transversality condition. The implicit function theorem then implies that the equilibria of (3.4.5) form a curve which will be tangent to the line $\mu = \mu_0$. An additional transversality condition $(d^2 f_{\mu_0}/dx^2)(x_0) \neq 0$ implies that the curve of equilibria has a *quadratic* tangency with $\mu = \mu_0$ and locally lies to one side of this line.

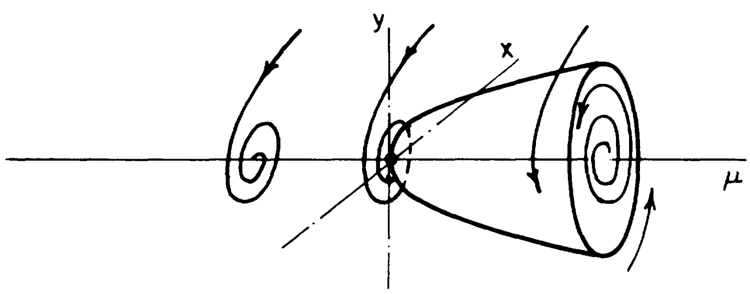

Figure 3.4.4. Hopf bifurcation (supercritical).

This information is already sufficient to imply that the local phase portraits of this system are topologically equivalent to those of a family $\dot{x} = \pm(\mu - \mu_0) \pm (x - x_0)^2$. However, we can also formulate these transversality conditions for an n-dimensional system without recourse to the center manifold reduction. The following theorem states the necessary conditions (cf. Sotomayor [1973]).

Theorem 3.4.1. *Let $\dot{x} = f_\mu(x)$ be a system of differential equations in $\mathbb{R}^n$ depending on the single parameter μ. When $\mu = \mu_0$, assume that there is an equilibrium p for which the following hypotheses are satisfied*:

(SN1) *$D_x f_{\mu_0}(p)$ has a simple eigenvalue 0 with right eigenvector v and left eigenvector w. $D_x f_{\mu_0}(p)$ has k eigenvalues with negative real parts and $(n - k - 1)$ eigenvalues with positive real parts (counting multiplicity).*

(SN2) $w((\partial f_\mu/\partial\mu)(p, \mu_0)) \neq 0$.

(SN3) $w(D_x^2 f_{\mu_0}(p)(v, v)) \neq 0$.

Then there is a smooth curve of equilibria in $\mathbb{R}^n \times \mathbb{R}$ passing through (p, μ_0), tangent to the hyperplane $\mathbb{R}^n \times \{\mu_0\}$. Depending on the signs of the expressions in (SN2) *and* (SN3), *there are no equilibria near (p, μ_0) when $\mu < \mu_0$ ($\mu > \mu_0$) and two equilibria near (p, μ_0) for each parameter value $\mu > \mu_0$ ($\mu < \mu_0$). The two equilibria for $\dot{x} = f_\mu(x)$ near (p, μ_0) are hyperbolic and have stable manifolds of dimensions k and $k + 1$, respectively. The set of equations $\dot{x} = f_\mu(x)$ which satisfy* (SN1)–(SN3) *is open and dense in the space of C^∞ one-parameter families of vector fields with an equilibrium at (p, μ_0) with a zero eigenvalue.*

This formal (and formidable) theorem merely expresses the fact that the "generic" saddle node bifurcation is qualitatively like the family of equations $\dot{x} = \mu - x^2$ in the direction of the zero eigenvector, with hyperbolic behavior in the complementary directions. Hypotheses (SN2) and (SN3) are the transversality conditions which control the nondegeneracy of the behavior with respect to the parameter and the dominant effect of the quadratic nonlinear term.

EXERCISE 3.4.1. Consider the variational van der Pol equations (equation (2.1.14))

$$\dot{u} = u - \sigma v - u(u^2 + v^2),$$
$$\dot{v} = \sigma u + v - v(u^2 + v^2) - \gamma.$$

Find the locus of saddle-node bifurcations in (σ, γ) space. At which point(s) do conditions (SN1) and (SN3) fail? (This involves tedious calculations, since a cubic equation must be solved for the fixed points.)

The results obtained from Theorem 3.4.1 are limited in two different ways. On the one hand, it is possible that more quantitative information about the flows near bifurcation can be extracted. For example, one can use the system $\dot{x} = \mu - x^2$ to give estimates of how rapid the convergence to the various equilibria are. Higher-order terms in the Taylor expansion of an equation can be used to refine these estimates. This is an aspect of the theory of differential equations which we do not pursue further in this book because our attention is to focus on geometric issues rather than analytic ones. In this regard, the reader should be reminded that we often do not strive to state the strongest or most general theorem for a given situation but rather aim to illustrate the phenomena and methods of analysis in the simplest ways.

The second limitation of Theorem 3.4.1 is that there may be global changes in a phase portrait associated with a saddle-node bifurcation. Consider, for example, the flows depicted in Figure 3.4.5, which we have already met in the van der Pol example of Section 2.1 (cf. Figure 2.1.3). Here a saddle-node in a two-dimensional system occurs, with the coalescense of a sink and a saddle. After the bifurcation, there is a new periodic orbit which has appeared because the unstable separatrix at the saddle-node lies in the two-dimensional stable manifold of the bifurcating equilibrium. This is an example of a *global bifurcation* phenomenon that cannot be reduced to the study of a neighborhood of an equilibrium or a fixed point in a return map. We return to global bifurcation problems in Chapter 6.

Transcritical and Pitchfork Bifurcations

The importance of the saddle-node bifurcation is that all bifurcations of one-parameter families at an equilibrium with a zero eigenvalue can be perturbed to saddle-node bifurcations. Thus one expects that the zero eigenvalue bifurcations encountered in applications will be saddle-nodes. If they are not, then there is probably something special about the formulation of the problem which restricts the context so as to prevent the saddle-node from occurring. The *transcritical* bifurcation is one example which illustrates how the setting of the problem can rule out the saddle-node bifurcation.

In classical bifurcation theory, it is often assumed that there is a trivial solution from which bifurcation is to occur. Thus, the systems (3.4.5) are assumed to satisfy $f_\mu(0) = 0$ for all μ, so that $x = 0$ is an equilibrium for all

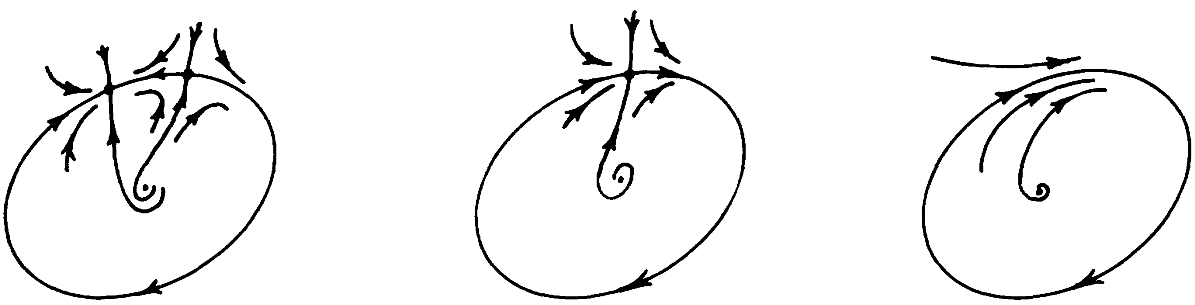

Figure 3.4.5. A saddle node occurring on a closed curve leads to global bifurcation.

parameter values. Since the saddle-node families contain parameter values for which there are *no* equilibria near the point of bifurcation, this situation is qualitatively different. To formulate the appropriate transversality conditions we look at the one-parameter families which satisfy the constraint that $f_\mu(0) = 0$ for all μ. This prevents hypothesis (SN2) of Theorem 3.4.1 from being satisfied. If we replace this condition by the requirement that $w((\partial^2 f/\partial\mu\,\partial x)(v)) \neq 0$ at $(0, \mu_0)$, then the phase portraits of the family near the bifurcation will be topologically equivalent to those of Figure 3.4.2, and we have a *transcritical bifurcation* or *exchange of stability*.

A second setting in which the saddle-node does not occur involves systems which have a symmetry. Many physical problems are formulated so that the equations defining the system do have symmetries of some kind. For example, the Duffing equation is invariant under the transformation $(x, y) \to (-x, -y)$ and the Lorenz equation is symmetric under the transformation $(x, y, z) \to (-x, -y, z)$. In one dimension, a differential equation (3.4.5) is symmetric or *equivariant* with respect to the symmetry $x \to -x$ if $f_\mu(-x) = -f_\mu(x)$. Thus the equivariant vector fields are ones for which f_μ is an odd function of x. In particular, all such equations have an equilibrium at 0. The transcritical bifurcation cannot occur in these systems, however, because an odd function f_μ cannot satisfy the condition $\partial^2 f_\mu/\partial x^2 \neq 0$ required by the transcritical bifurcation (cf. SN3). If this condition is replaced by the transversality hypothesis $\partial^3 f_\mu/\partial x^3 \neq 0$, then one obtains the *pitchfork bifurcation*. At the point of bifurcation, the stability of the trivial equilibrium changes, and a new *pair* of equilibria (related by the symmetry) appear to one side of the point of bifurcation in parameter space, as in Figure 3.4.3. We leave to the reader the formulation of results analogous to Theorem 3.4.1 for the transcritical and pitchfork bifurcations (cf. Sotomayor [1973]).

We note that the direction of the bifurcation and the stability of the branches in these examples is determined by the sign of $\partial^2 f_\mu/\partial x^2$ or $\partial^3 f_\mu/\partial x^3$. In the last case, if $\partial^3 f_\mu/\partial x^3$ is negative, then the branches occur "above" the bifurcation value and we have a *supercritical* pitchfork bifurcation, whereas we have a *subcritical* bifurcation if it is positive.

EXERCISE 3.4.2. Compute the normal form for the Lorenz equations at the bifurcation of the origin when $\rho = 1$, through terms of third degree. Note that the bifurcation is a pitchfork even though the analytic expression of the Lorenz equations involves only quadratic terms (cf. Exercise 3.2.5).

EXERCISE 3.4.3. Analyze the pitchfork bifurcation which takes place for the variational van der Pol equation at $(\sigma, \gamma) = (1/\sqrt{3}, \sqrt{8/27})$.

Hopf Bifurcations

Consider now a system (3.4.5) with a parameter value μ_0 and equilibrium $p(\mu_0)$ at which Df_{μ_0} has a simple pair of pure imaginary eigenvalues, $\pm i\omega$, $\omega > 0$, and no other eigenvalues with zero real part. The implicit function theorem guarantees (since Df_{μ_0} is invertible) that for each μ near μ_0 there will be an equilibrium $p(\mu)$ near $p(\mu_0)$ which varies smoothly with μ. Nonetheless, the dimensions of stable and unstable manifolds of $p(\mu)$ do change if the eigenvalues of $Df(p(u))$ cross the imaginary axis at μ_0. This qualitative change in the local flow near $p(\mu)$ must be marked by some other local changes in the phase portraits not involving fixed points.

A clue to what happens in the generic bifurcation problem involving an equilibrium with pure imaginary eigenvalues can be gained from examining linear systems in which there is a change of this type. For example, consider the system

$$\begin{aligned} \dot{x} &= \mu x - \omega y, \\ \dot{y} &= \omega x + \mu y, \end{aligned} \tag{3.4.6}$$

whose solutions have the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\mu t} \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \tag{3.4.7}$$

When $\mu < 0$, solutions spiral into the origin, and when $\mu > 0$, solutions spiral away from the origin. When $\mu = 0$, all solutions are periodic. Even in a one-parameter family of equations, it is highly special to find a parameter value at which there is a whole family of periodic orbits, but there is still a surface of periodic orbits which appears in the general problem.

The normal form theorem gives us the required information about how the generic problem differs from the system (3.4.6). By smooth changes of coordinates, the Taylor series of degree 3 for the general problem can be brought to the following form (cf. Equation (3.3.15))

$$\begin{aligned} \dot{x} &= (d\mu + a(x^2 + y^2))x - (\omega + c\mu + b(x^2 + y^2))y, \\ \dot{y} &= (\omega + c\mu + b(x^2 + y^2))x + (d\mu + a(x^2 + y^2))y, \end{aligned} \tag{3.4.8}$$

which is expressed in polar coordinates as

$$\begin{aligned} \dot{r} &= (d\mu + ar^2)r, \\ \dot{\theta} &= (\omega + c\mu + br^2). \end{aligned} \tag{3.4.9}$$

Since the $\dot{r}$ equation in (3.4.9) separates from θ, we see that there are periodic orbits of (3.4.8) which are circles $r = \text{const.}$, obtained from the nonzero solutions of $\dot{r} = 0$ in (3.4.9). If $a \neq 0$ and $d \neq 0$ these solutions lie along the parabola $\mu = -ar^2/d$. This implies that the surface of periodic orbits has a quadratic tangency with its tangent plane $\mu = 0$ in $\mathbb{R}^2 \times \mathbb{R}$. The content of the Hopf bifurcation theorem is that the qualitative properties of (3.4.8) near the origin remain unchanged if higher-order terms are added to the system:

Theorem 3.4.2 [Hopf [1942]). *Suppose that the system $\dot{x} = f_\mu(x)$, $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}$ has an equilibrium (x_0, μ_0) at which the following properties are satisfied:*

(H1) $D_x f_{\mu_0}(x_0)$ *has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts.*

Then (H1) *implies that there is a smooth curve of equilibria* $(x(\mu), \mu)$ *with* $x(\mu_0) = x_0$. *The eigenvalues* $\lambda(\mu)$, $\bar{\lambda}(\mu)$ *of* $D_x f_{\mu_0}(x(\mu))$ *which are imaginary at* $\mu = \mu_0$ *vary smoothly with* μ. *If, moreover,*

(H2) $$\frac{d}{d\mu}(\text{Re}\,\lambda(\mu))|_{\mu=\mu_0} = d \neq 0,$$

then there is a unique three-dimensional center manifold passing through (x_0, μ_0) *in* $\mathbb{R}^n \times \mathbb{R}$ *and a smooth system of coordinates (preserving the planes* $\mu =$ const.) *for which the Taylor expansion of degree* 3 *on the center manifold is given by* (3.4.8). *If* $a \neq 0$, *there is a surface of periodic solutions in the center manifold which has quadratic tangency with the eigenspace of* $\lambda(\mu_0)$, $\bar{\lambda}(\mu_0)$ *agreeing to second order with the paraboloid* $\mu = -(a/d)(x^2 + y^2)$. *If* $a < 0$, *then these periodic solutions are stable limit cycles, while if* $a > 0$, *the periodic solutions are repelling.*

This theorem can be proved by a direct application of the center manifold and normal form theorems given above (cf. Marsden and McCracken [1976]).

Exercise 3.4.4. Find the Hopf bifurcations which occur in the variational van der Pol equations (2.1.14).

Exercise 3.4.5. In the Duffing equation,

$$\ddot{x} + \mu\dot{x} + (x - x^3) = 0,$$

a bifurcation with pure imaginary eigenvalues occurs at $\mu = 0$ when the system changes from having negative to positive dissipation, but it is degenerate. Why? Compute the normal form through terms of third degree. What modifications might be made to the system to make the bifurcation "generic"; i.e., satisfy all of the hypotheses of the Hopf theorem?

For large systems of equations, computation of the normal form (3.4.8) and the cubic coefficient a, which determines the stability, can be a substantial undertaking.

In a two-dimensional system of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}, \tag{3.4.10}$$

with $f(0) = g(0) = 0$ and $Df(0) = Dg(0) = 0$, the normal form calculation which we sketch in the appendix to this section, yields

$$a = \frac{1}{16}[f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}] + \frac{1}{16\omega}[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}], \tag{3.4.11}$$