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The long time behavior of Brownian motion in tilted periodic potentials

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HIGHLIGHTS

- We study the Langevin equation describing diffusion in tilted periodic potentials.
- In the over-damped limit the long time average velocity converges in probability.
- In the over-damped and vanishing noise limit the convergence rate varies as the tilt crosses threshold.
- In the under-damped limit we recover Risken's results about the bi-stability.
- In the under-damped limit we derive asymptotics of the transition times.

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ABSTRACT

A variety of phenomena in physics and other fields can be modeled as Brownian motion in a heat bath under tilted periodic potentials. We are interested in the long time average velocity considered as a function of the external force, that is, the tilt of the potential. In many cases, the long time behavior – pinning and de-pinning phenomenon – has been observed. We use the method of stochastic differential equation to study the Langevin equation describing such diffusion. In the *over-damped limit*, we show the convergence of the long time average velocity to that of the Smoluchowski–Kramers approximation, and carry out asymptotic analysis based on Risken's and Reimann et al.'s formula. In the *under-damped limit*, applying Freidlin et al.'s theory, we first show the existence of three pinning and de-pinning thresholds of the normalized tilt, corresponding to the bi-stability phenomenon; and second, as noise approaches zero, derive formulas of the mean transition times between the pinning and running states.

1. Introduction

We are concerned with the *pinning* and *de-pinning* behavior of particles moving in inhomogeneous materials when a *driving force* F crosses over some threshold value F^*. That is, as shown in Fig. 1, there exists a threshold F^* of the driving force F such that if F is less than F^*, the *long time average velocity* of the particle V_F (considered as a function of F) is *zero*; whilst if F is greater than F^*, V_F is *positive*.

The study of pinning and de-pinning phenomenon is motivated by many applications, for example, the observations regarding charge-density waves at low temperatures and the concomitant nonlinear conductivity characterized by non-Ohmic behavior above a small threshold electric field. It is believed that the non-Ohmic conduction is caused by the sliding of charge-density wave which is prevented from moving below the threshold field by pinning to impurities and other lattice defects [1]. The pinning effect due to the crystal defects or impurities can also be observed frequently in a type II superconductor with impurities. Analogous pinning phenomena in a Josephson junction – an array of superconductors separated by a thin insulating barrier – in the presence of different types of structural disorder is investigated both analytically and numerically [2–5]. In addition, the pinning and de-pinning (stick–slip) character in the motion of a phase boundary leads to the widely observed rate-independent hysteresis feature in shape-memory alloys [6]. This phenomenon is also related to systems such as dynamics of cracks and geological faults, which can be modeled as front propagation describing the evolution of an interface driven by an external force through an inhomogeneous medium [7,8].

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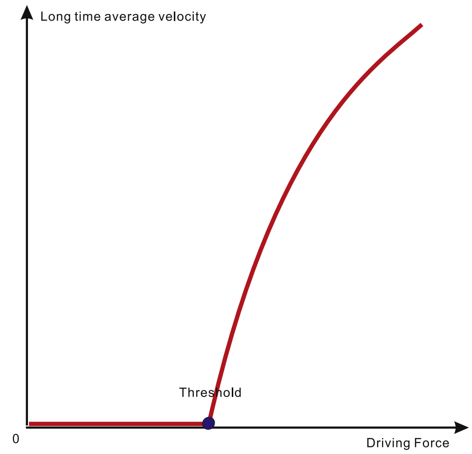


Fig. 1. Pinning and de-pinning phenomena.

In order to obtain more quantitative information about dynamical behaviors, we consider a one dimensional model of particle motion in a *periodic* medium. This model has been applied to study the dynamics of a de-pinned, uniform charge-density wave [9]. We analyze the dynamics in both the *over-damped (dissipation driven)* and *under-damped (inertia driven)* limits. A specific example, in the one-dimensional deterministic case, is given by the following equation:

$$\dot{x} = -\Psi'(x) + F, \qquad x(0) = x_0, \tag{1.1}$$

where $x(\cdot)$ is the position of a particle which evolves on a periodic potential function Ψ. The constant F signifies an external forcing. The *long time average velocity* V_F is defined as

$$V_F := \lim_{t\to\infty} \frac{x(t)}{t}. \tag{1.2}$$

For the above example, if Ψ satisfies some *non-degenerate* condition, then we have [1,9,6,10]:

$$V_F \sim C(F - F^*)^{\frac{1}{2}} \quad \text{for } 0 < F - F^* \ll 1. \tag{1.3}$$

(The above scaling will be explained in page section.)

The goal of this paper is to extend the understanding of (1.1) and (1.3) to incorporate stochastic noise and inertial effects. For this purpose, the following second order *Langevin equation* [11,12] which is an analogue of Newton's Second Law, is often considered:

$$m\ddot{q} = F - \nabla\Psi(q) - \gamma\dot{q} + \sqrt{2\gamma\beta^{-1}}\dot{W}, \qquad q(0) = q_0, \qquad \dot{q}(0) = p_0, \tag{1.4}$$

where the position variable q is a d-dimensional vector in $\mathbb{R}^d$, m is the mass of the particle, F is the driving force, Ψ denotes a smooth periodic potential function depending on the position variable q, γ is the damping coefficient, $\beta = (kT)^{-1}$ is the inverse temperature (k is the Boltzmann constant and T is the absolute temperature), and $\dot{W}$ is a standard d-dimensional white noise. Note that the *fluctuation–dissipation* criterion is imposed in the above form of the equation. It is often useful to consider the above equation in the form of a first order system:

$$\dot{q} = p, \qquad \dot{p} = \frac{F - \nabla\Psi(q)}{m} - \frac{\gamma}{m}p + \frac{\sqrt{2\gamma\beta^{-1}}}{m}\dot{W}, \qquad q(0) = q_0, \qquad p(0) = p_0, \tag{1.5}$$

where q and p are the position and velocity variables. We want to study the *long time average velocity* of the particle diffusion described by (1.4) or (1.5) as a *function of the driving force F* in various regimes. In order to obtain more quantitative results we restrict ourselves to one dimensional case, i.e., $d = 1$. We first give an outline of our results.

For the **over-damped limit**, we will state the convergence of the long time average velocity in the limits of *vanishing noise* and *vanishing mass* in Sections 3 and 4.

- In Section 3, we concentrate on the following *first order* equation

$$\gamma\dot{q} = F - \Psi'(q) + \sqrt{2\gamma\beta^{-1}}\dot{W}, \qquad q(0) = q_0, \tag{1.6}$$

  which is often called the *Smoluchowski* equation. We show in Theorem 3.1 that V_F converges to its deterministic version as the noise goes to zero ($\beta \to \infty$). The convergence rate is exponential (in β) when the driving force is below the pinning and de-pinning threshold (Region I), but only algebraic when the driving force is at or above the threshold (Regions II and III). This is illustrated in Fig. 2. The main technique used is Laplace's method applied to an explicit formula for V_F.
- In Section 4, we will consider the second order Langevin equation (1.5) which incorporates inertia effects. We show the convergence of V_F in the vanishing mass limit ($m \to 0$) in both the deterministic and stochastic versions. For the deterministic case (Theorem 4.1), we make use of the classifications of the ω-limit sets of the dynamics (see for example [13]). For the stochastic case (Theorem 4.2) we make use of the ergodicity of systems (1.5) and (1.6). The ergodicity of the former is nontrivial, since the system is driven by a *degenerate noise* — the noise is directly applied only to the velocity variable $\dot{q}$ (but not the spatial variable q). This is proved by M. Hairer and G.A. Pavliotis in [14] which makes use of the *hypoellipticity* of the Fokker–Planck operator for system (1.5).

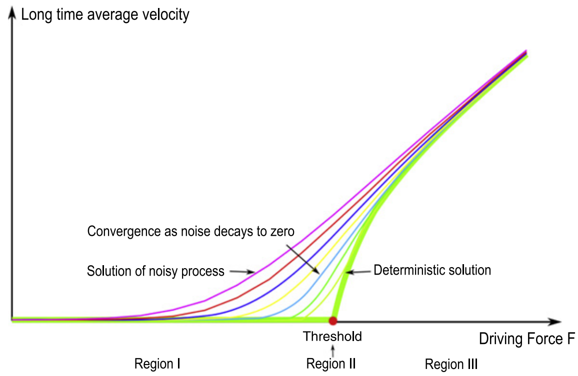
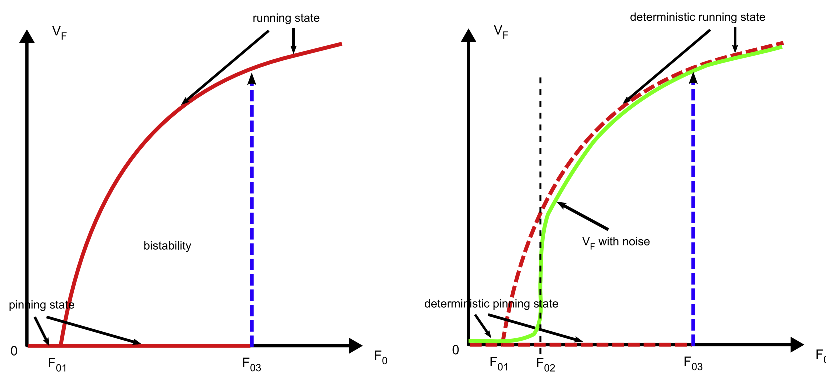


Fig. 2. Convergence of long time average velocity V_F for the Smoluchowski equation (1.6) as noise approaches zero.



(a) Deterministic process ($\gamma = 0, \beta = \infty$). (b) Noisy process ($\gamma \ll 1, \beta \gg 1$).

Fig. 3. Long time average velocity for the Langevin equation (1.4) in the limit $\gamma \to 0$.

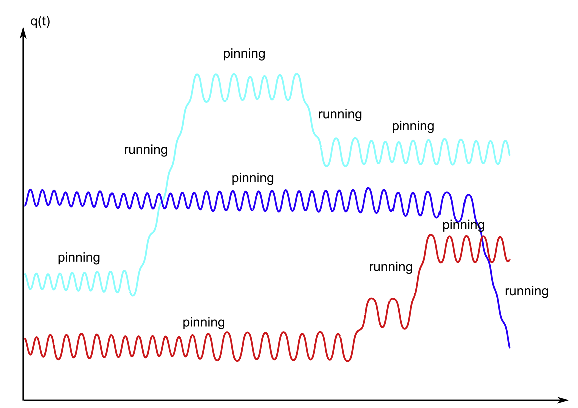


Fig. 4. Trajectories of the Langevin equation (1.4) in the under-damped limit. It clearly demonstrates the switching between pinning and running states.

In Section 5, we concentrate on the **under-damped limit** ($\gamma \to 0$) of the second order equation (1.4). In this case, the inertia effects become important. The interesting phenomenon of *bi-stability* arise. It refers to the fact that the particle can stay in either the pinning or the running state. We only consider F of the same order as γ, i.e. the *scaled tilt* $F_0 := \frac{F}{\gamma}$ being finite. Otherwise, the particle velocity will become unbounded over time. For the deterministic process, there are two thresholds, F_{01} and F_{03} for F_0. For $F_0 < F_{01}$, only the pinning state exists whilst for $F_0 > F_{03}$, only the running state exists. For F_0 between F_{01} and F_{03}, the initial condition determines which state the particle will stay and it is not possible to switch between these two states (see Fig. 3(a)). However, for the noisy process, transitions can occur. See Fig. 4. This phenomenon can be intuitively explained in terms of the (relative) stability of the pinning and de-pinning states. For $F_0 < F_{01}$, only the former is stable while for $F_0 > F_{03}$, only the latter is stable. In addition, there arises a new threshold F_{02} in between F_{01} and F_{03} at which the stability properties of the two states change (Fig. 3(b)).

To make the above description rigorous, we apply Freidlin et al.'s theory [15] to a properly time re-scaled version of (1.4) so that the position and momentum become the fast variables while the Hamiltonian becomes the slow variable in the under-damped limit. In this regime, the distribution of the re-scaled process converges weakly to a diffusion on a graph Γ, called the *Hamiltonian graph*. Making use of the invariant measure on the configuration space (position and momentum), we can calculate the long time average velocity V_F.

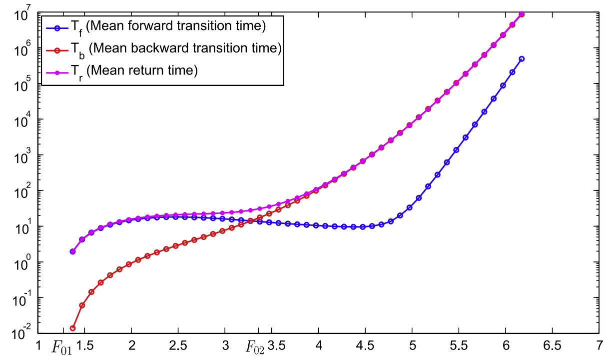


Fig. 5. The graphs of the forward transition time T_f (from pinned to running states) and backward transition time T_b (from running to pinning states). Note the switching of the order at F_{02}. The quantity $T_r = T_f + T_b$ represents the mean return time from the pinning state back to itself.

- In Theorem 5.1, we obtain the asymptotics of V_F as $\beta \to \infty$ (vanishing noise limit) in the under-damped regime ($\gamma \to 0$). This leads to a precise formula for V_F and also the bi-stability thresholds. The results show a rapid change of the long time average velocity as F_0 passes across F_{02} (Fig. 3(b)).
- In Theorem 5.2, we derive asymptotics in the *mean transition time* T_f from the pinning to the running state and T_b for the reverse transition. They can better illustrate the behavior of the trajectories and in fact indicate the fluctuation from the average long time velocity. It also gives another property of F_{02}. As $\beta \to \infty$, F_{02} is exactly the tilt at which T_f and T_b switch their order (Fig. 5).

Next we give some survey of existing literature about the question we are considering. The Langevin equation (1.4) has been widely studied. An extensive bibliography can be found in [16,17]. Among others, the most important quantities are the diffusion coefficient D and long time average velocity V of the position process $q(t)$:

$$D = \lim_{t\to\infty} \frac{\left\langle \left(q(t) - \langle q(t)\rangle\right)^2 \right\rangle}{t} \quad \text{and} \quad V = \lim_{t\to\infty} \frac{q(t)}{t}$$

where $\langle\cdot\rangle$ refers to ensemble average. In the case of potentials without tilt (so that $V = 0$), there are quite a few results concerning the computation of D. The underlying formula involves the invariant measure of the process $q(t)$ and the Green–Kubo Formula. See [18,19] for some recent accounts. These works also consider various long-time/large-spatial scaling to demonstrate that the process converge to a Brownian motion. The relationship between D and the dissipation coefficient γ are analyzed in various regimes. When external force F or tilt of the potential is present, the computation of V has also been performed extensively. Most results concern *linear response* theory in the regime $F \to 0$, characterizing the value of the mobility μ:

$$\mu = \lim_{F\to 0} \frac{V}{F},$$

leading to the celebrated Einstein–Smoluchowski relation connecting the value of D to μ. See [20] for a discussion and refinement of this relation. In contrast to the above works, we emphasize the *nonlinear response* of V to F, in particular the behavior of V near the pinning and de-pinning threshold of F. We also investigate the long time behaviors for different regimes of the mass m, friction γ and the noise strength β^{-1}. To the best of our knowledge, these have not been investigated collectively together. We do wish to point out some closely related works. In [21,22], the behavior of D as a function of F is analyzed, leading to some enhancement of D – giant diffusion phenomena – near the threshold value. Their works essentially quantify the mean square fluctuation of the position process from the linear motion determined by V. This is a step toward central limit theorem type of results in comparison with our strong law of large number statements. However, their results are only for the first order equation (1.6) while our work also touches upon the second order equation (1.4) in various parameter regimes. The motivation for all these works is to capture critical phenomena as the external forcing passes through some threshold values. The work [23] analyzes (1.4) demonstrating that under stochastic perturbation, only the locked (pinned) state can survive even though running states are possible without the presence of noise. This phenomenon is proved in some sub-exponential time scale (in terms of the noise strength β^{-1}) and is only applicable at certain specific value of the tilt (in fact at F_{01} using our notation). Our results work for a whole range of F but the time scale is exponentially large.

We also point out here some specific works which we rely on very much for their results and techniques. In the *over-damped limit*, it was discovered that a Markov process $q(t)$ satisfying the following *Smoluchowski equation*:

$$\gamma\dot{q} = F - \Psi'(q) + \sqrt{2\gamma\beta^{-1}}\dot{W}, \quad q(0) = q_0, \tag{1.7}$$

is a good approximation to the position process of (1.4) or (1.5). More precisely, in this regime ($\gamma \to \infty$ or $m \to 0$), the position process $q(t)$ of the Langevin equation (1.4) converges in probability to that of the Smoluchowski equation (1.6), uniformly over every finite time interval [24,11,25] [26, Ch. 10]. This is called the *Smoluchowski–Kramers approximation*. However, over a long time interval of scale $O\left(\frac{\gamma}{m}\right)$, the convergence of the long time average velocity of (1.4) has not yet been proved rigorously. For (1.6), Risken [17] and Reimann et al. [22] derived formulas for the long time average velocity through different approaches. On the other hand, in the vanishing noise limit, a regime of physical interest, the convergence rates of the long time average velocity of (1.6) have not been fully discussed. In the *under-damped limit* ($\gamma \to 0$), the bi-stability phenomenon and its three thresholds have been discussed by Risken in [17, Ch.11]. In spite of this, rigorous mathematical treatment is lacking.

For convenience, we list here the notations to be used in the remaining of this paper:

N1 • $\Psi(q) = -\cos q$ denotes the periodic potential

N2 • $\Phi(q) = -\cos q - Fq$ denotes the tilted periodic potential
N3 • $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ denotes the Gamma function [27]
N4 • $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ denotes the beta function [27]
N5 • $\mathbb{T} = \mathbb{R}/[0, 2\pi]$ denotes the one dimensional torus
N6 • $C^r_{per}(\mathbb{T})$, $r = 1, 2$, denotes the space of the restriction to $\mathbb{T}$ of functions in $C^r(\mathbb{R})$ that are 2π-periodic
N7 • $C^r_{per}(\mathbb{T}\times\mathbb{R})$, $r = 1, 2$, denotes the space of the restriction to $\mathbb{T}\times\mathbb{R}$ of functions in $C^r(\mathbb{T}\times\mathbb{R})$ that are 2π-periodic in the first variable
N8 • $a \sim b$ means that $\frac{a}{b}$ converges to 1 in some limiting process (stated in context).

We remark here that the assumptions made in notations N_1 and N_2 on the potential do not undermine the generality of our results. They are made mainly for the convenience of computation.

2. Preliminary remarks

2.1. The scaling law

Here we briefly explain how the scaling law (1.3) arises in the deterministic case. The potential function Ψ is assumed to be smooth, periodic and *non-degenerate* in the sense that $\Psi'''(q) \neq 0$ if $\Psi''(q) = 0$. Another assumption which simplifies the following discussion is that Ψ' is "*single-peaked*", i.e., it has one unique local maximum within each period. (Note that both of these assumptions are satisfied by notation N1.) Removing the noise term from Smoluchowski equation (1.6) leads to the following deterministic equation

$$\dot{q} = \frac{F - \Psi'(q)}{\gamma}, \qquad q(0) = q_0. \tag{2.1}$$

We define the *long time average velocity* V_F of the solution $q(t)$ of the above as follows:

$$V_F := \lim_{t\to\infty} \frac{q(t)}{t}. \tag{2.2}$$

The existence of the above limit follows from the periodicity of $\Psi(q)$. If $F \le \|\Psi'\|_{L^\infty(\mathbb{R})}$, the process gets pinned whenever $F = \Psi'$, leading to $V_F = 0$. Whereas, if $F > \|\Psi'\|_{L^\infty(\mathbb{R})}$, then $V_F > 0$. This leads us to define the *pinning and de-pinning threshold* as $F^* = \|\Psi'\|_{L^\infty(\mathbb{R})}$.

The asymptotics of V_F as $F \to (F^*)^+$ can be obtained in the following way. We assume that Ψ' reaches its maximum at $q^* \in [0, 2\pi]$, i.e., $\Psi'(q^*) = F^*$. By the non-degeneracy assumption on Ψ, $\Psi''(q^*) = 0$ and $\Psi'''(q^*) < 0$ (since Ψ' peaks at q^*). Let $\eta = F - F^*$, we have

$$\gamma\dot{q} = (F - F^*) + (F^* - \Psi'(q)) \approx \eta - \tfrac{\Psi'''(q^*)(q-q^*)^2}{2} + o((q-q^*)^2). \tag{2.3}$$

Let ϵ $(0 < \epsilon \ll 1)$ be some fixed number independent of η, and let T be the time needed for $q(t)$ to extend from 0 to 2π. Then we obtain

$$\begin{aligned}
T &= \gamma\int_0^{q^*-\epsilon} \frac{dq}{F-\Psi'(q)} + \gamma\int_{q^*-\epsilon}^{q^*} \frac{dq}{\eta - \frac{\Psi'''(q^*)(q-q^*)^2}{2} + o((q-q^*)^2)} \\
&\quad + \gamma\int_{q^*}^{q^*+\epsilon} \frac{dq}{\eta - \frac{\Psi'''(q^*)(q-q^*)^2}{2} + o((q-q^*)^2)} + \gamma\int_{q^*+\epsilon}^{2\pi} \frac{dq}{F-\Psi'(q)} \\
&\sim \gamma\frac{2\sqrt{2}}{\sqrt{\eta}\sqrt{-\Psi'''(q^*)}} \arctan\left(\sqrt{\frac{-\Psi'''(q^*)}{2\eta}}\,\epsilon\right) \\
&\sim \gamma\frac{2\sqrt{2}}{\sqrt{\eta}\sqrt{-\Psi'''(q^*)}}\frac{\pi}{2} \sim \gamma\frac{\sqrt{2}\pi}{\sqrt{\eta}\sqrt{-\Psi'''(q^*)}} \sim \gamma\, \mathrm{O}\left((F-F^*)^{-\frac{1}{2}}\right), \quad \text{as } F \to (F^*)^+.
\end{aligned}$$

We therefore obtain the following *scaling law*:

$$V_F = \frac{2\pi}{T} \sim \frac{1}{\gamma}\sqrt{-2\Psi'''(q^*)}(F-F^*)^{\frac{1}{2}} \quad \text{for } 0 < F - F^* \ll 1. \tag{2.4}$$

2.2. Ergodicity

Here we introduce the *infinitesimal generator* and its *adjoint* for the stochastic differential equation we are considering. The notations are mainly taken from [28].

The *generator* $\mathcal{L}_1$ and its formal L^2-*adjoint* operator $\mathcal{L}_1^*$ of the Smoluchowski equation (1.6) are defined as:

$$\mathcal{L}_1 f = \frac{1}{\gamma}\cdot(F - \Psi'(q))\frac{df(q)}{dq} + \frac{\beta^{-1}}{\gamma}\cdot\frac{d^2f(q)}{dq^2}, \tag{2.5}$$

and

$$\mathcal{L}_1^* f = -\frac{1}{\gamma}\cdot\frac{d\big((F-\Psi'(q))f(q)\big)}{dq} + \frac{\beta^{-1}}{\gamma}\cdot\frac{d^2f(q)}{dq^2}. \tag{2.6}$$

Similarly, the *generator* $\mathcal{L}_2$ and its formal L^2-*adjoint* operator $\mathcal{L}_2^*$ for (1.5) are given as:

$$\mathcal{L}_2 f = p\cdot\frac{\partial f(q,p)}{\partial q} + \frac{F-\Psi'(q)-\gamma p}{m}\cdot\frac{\partial f(q,p)}{\partial p} + \frac{\gamma\beta^{-1}}{m^2}\cdot\frac{\partial^2 f(q,p)}{\partial p^2}, \tag{2.7}$$

and

$$\mathcal{L}_2^* f = -\frac{\partial\big(pf(q,p)\big)}{\partial q} - \frac{1}{m}\cdot\frac{\partial\big((F-\Psi'(q)-\gamma p)f(q,p)\big)}{\partial p} + \frac{\gamma\beta^{-1}}{m^2}\cdot\frac{\partial^2 f(q,p)}{\partial p^2}. \tag{2.8}$$

For stochastic dynamical systems, *ergodicity* is characterized by the existence of a unique (up to normalization) function in the null space of the adjoint of the generator. We refer to this function as the *invariant distribution*, *stationary distribution* or the *density function of the invariant measure* [28, Sec. 6.4]. More specifically, for the Smoluchowski equation (1.6), ergodicity means that the following *stationary Fokker–Planck equation*

$$\mathcal{L}_1^*\rho^\infty = 0, \qquad \inf_{q\in\mathbb{T}}\rho^\infty(q) > 0, \qquad \int_{\mathbb{T}}\rho^\infty(q)\,dq = 1, \tag{2.9}$$

admits a unique invariant density function $\rho^\infty(q)$ on $\mathbb{T}$. Then it holds for all $\phi\in C^r_{per}(\mathbb{T})$ and the solution process $q(t)$ of (1.6) that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \phi(q(s))\,ds = \textstyle\int_{\mathbb{T}}\phi(q)\rho^\infty(q)\,dq, \quad \text{a.s.} \tag{2.10}$$

It can be shown that the Smoluchowski process equipped with periodic drift and diffusion coefficients is ergodic [28].

Similarly, for Eq. (1.5), ergodicity means the process admits a unique invariant density $\rho^\infty(q,p)$ on the configuration space $\mathbb{T}\times\mathbb{R}$ which solves the following stationary Fokker–Planck equation

$$\mathcal{L}_2^*\rho^\infty = 0, \qquad \inf_{(q,p)\in\mathbb{T}\times\mathbb{R}}\rho^\infty(q,p) > 0, \qquad \int_{\mathbb{T}\times\mathbb{R}}\rho^\infty(q,p)\,dq\,dp = 1. \tag{2.11}$$

Then for all $\phi\in C^r_{per}(\mathbb{T}\times\mathbb{R})$ and the solution process $(q(t),p(t))$ of (1.5), it holds that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \phi(q(s),p(s))\,ds = \int_{\mathbb{T}\times\mathbb{R}}\phi(q,p)\rho^\infty(q,p)\,dq\,dp, \quad \text{a.s.} \tag{2.12}$$

The ergodicity of (1.5) is nontrivial since it is driven by *degenerate* noise. Making use of the works [29,30], it is shown by M. Hairer and G. A. Pavliotis [14, Thm. 3.1] that (1.5) admits a unique invariant distribution on the configuration space $\mathbb{T}\times\mathbb{R}$. The argument makes use of the *hypoellipticity* of the parabolic operator $\mathcal{L}_2$.

3. The first order process (Smoluchowski equation)

We start with the study of the *long time average velocity* of the first order *Smoluchowski equation* (1.6). It can be considered as the over-damped limit of the Langevin equation (1.4) with $m = 0$.

3.1. Smoothened transition

Comparing to the transition (see Fig. 1) for the deterministic process (2.1), due to the white noise term, the transition between the *pinning* and *running* states for the Smoluchowski process (1.6) is smoothened (as illustrated by Fig. 2). It also demonstrates the convergence of the long time average velocity of the noisy process (1.6) to that of the deterministic process (2.1) as noise approaches zero. Here we investigate the convergence rate.

Note that the solution process $q(t)$ for (1.6) is *Markovian*. In the one-dimensional case, the *invariant density* $\rho^\infty(q)$ can be derived explicitly (see for example [17, Ch. 11]). By the ergodicity of the Smoluchowski process (1.6) and formula (2.10), the long time average velocity V_F of (1.6) is equal to the expectation of the function $\phi(q) = \frac{F-\Psi'(q)}{\gamma}$ with respect to the invariant density $\rho^\infty(q)$ on $\mathbb{T}$. In this way, Risken obtained in [17] the following formula for V_F:

$$V_F = \frac{2\pi\beta^{-1}}{\gamma}(1-e^{-2\pi F\beta})\left(\int_0^{2\pi} e^{\Phi(q)\beta}\,dq\int_0^{2\pi} e^{-\Phi(q')\beta}\,dq' - (1-e^{-2\pi F\beta})\int_0^{2\pi} e^{-\Phi(q)\beta}\int_0^q e^{\Phi(q')\beta}\,dq'\,dq\right)^{-1}. \tag{3.1}$$

Using the concept of *first passage time*, Reimann et al. derived in [22] another formula for V_F:

$$V_F = \frac{1-e^{-2\pi F\beta}}{\frac{1}{2\pi}\int_{q_0}^{q_0+2\pi} I_\pm(q)\,dq}, \tag{3.2}$$

where

$$I_+(q) = \beta\gamma e^{\Phi(q)\beta}\int_{q-2\pi}^{q} e^{-\Phi(q')\beta}\,dq', \qquad I_-(q) = \beta\gamma e^{-\Phi(q)\beta}\int_{q}^{q+2\pi} e^{\Phi(q')\beta}\,dq',$$

and "$I_\pm$" means one can choose either "I_+" or "I_-" (see N_2 for definition of Φ). With I_+ in (3.2), the above formula reads

$$V_F = \frac{1}{\gamma}\cdot\frac{2\pi\beta^{-1}(1-e^{-2\pi F\beta})}{\int_0^{2\pi}\int_{q-2\pi}^{q} e^{-\beta(\Phi(q')-\Phi(q))}\,dq'\,dq}. \tag{3.3}$$

One can show easily by switching the order of integration that formulas (3.1) and (3.2) are equivalent. It turns out that it is easier to apply asymptotic analysis to the latter.

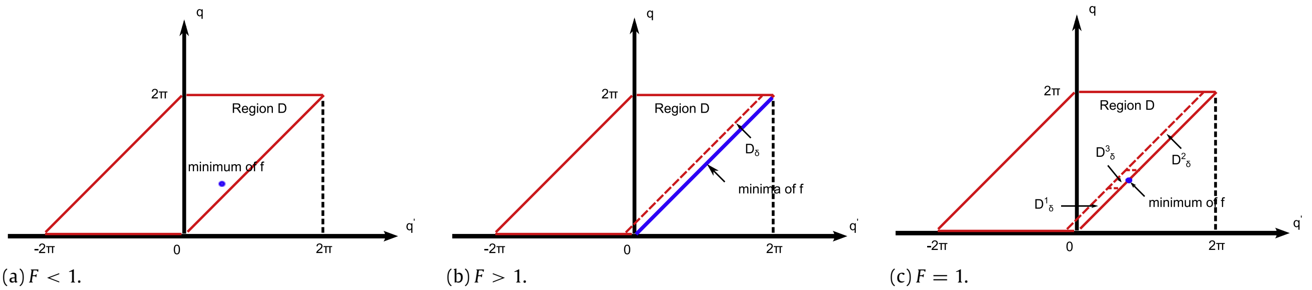


Fig. 6. Global minima of $f(q', q)$ in D.

3.2. Asymptotics of V_F

Here we apply Laplace's method [31] to (3.2) to obtain the asymptotics of V_F as noise approaches zero ($\beta \to \infty$). The result is stated as follows.

Theorem 3.1. *For the long time average velocity V_F of the Smoluchowski process (1.6), for $\beta \gg 1$, it holds that:*

(1) *for* $0 \le F < 1$,

$$V_F \sim \frac{1}{\gamma} \cdot \frac{\sqrt{1-F^2}(1-e^{-2\pi\beta F})}{e^{\beta(2\sqrt{1-F^2}-\pi F+2F\arcsin F)}}; \tag{3.4}$$

(2) *for* $F = 1$,

$$V_F \sim C \cdot \frac{1}{\gamma} \cdot \beta^{-\frac{1}{3}}(1-e^{-2\pi\beta F}) \tag{3.5}$$

(with the constant C explicitly given in the proof (A.1));

(3) *for* $F > 1$,

$$V_F \sim \frac{1}{\gamma} \cdot \frac{\sqrt{F^2-1}(1-e^{-2\pi F\beta})}{1-\frac{3(1+4F^2)}{8(F^2-1)^{\frac{5}{2}}}\beta^{-2}+o(\beta^{-3})}. \tag{3.6}$$

The above result shows that for F below $F^* = 1$, V_F converges *exponentially* fast as $\beta \to \infty$, whilst for F at and above $F^* = 1$, V_F converges *algebraically*.

In the proof of this result, we use the following notations:

N9 • $f(q', q) = \Phi(q') - \Phi(q) = \cos q - \cos q' + F(q - q')$

N10 • $J(\beta) = \int_0^{2\pi}\int_{q-2\pi}^{q} e^{-\beta f(q',q)}\,dq'\,dq = \int_0^{2\pi}\int_{q-2\pi}^{q} e^{-\beta(\Phi(q')-\Phi(q))}\,dq'\,dq$ which is the double integral in the denominator of (3.2).

Recall formula (3.3) for V_F. Its asymptotics for $\beta \gg 1$ is determined by that of $J(\beta)$, which is in turn related to the global minimum of $f(q', q)$ within the integration region D of $J(\beta)$:

$$D := \{(q', q) : q - 2\pi \le q' \le q, 0 \le q \le 2\pi\}.$$

See Fig. 6. It can be checked that when F is below the threshold value 1, $f(q', q)$ has an interior global minimum. Then a direct usage of Laplace method is possible. When F is equal to 1, the global minimum of $f(q', q)$ is degenerate and located on ∂D — the boundary of D; and when F is above 1, the global minima of $f(q', q)$ accumulate on ∂D. In these two latter cases, more refined calculation in some appropriate neighborhood of the minima is needed in order to determine the asymptotics of $J(\beta)$.

3.3. Proof of Theorem 3.1

In the following we prove (3.4) and (3.6). The proof for (3.5) is postponed to the appendix as it requires more computations.

Proof of (3.4). For $0 < F < 1$, f attains its global minimum $-2\sqrt{1-F^2} + F\pi - 2F\arcsin F$ at $(\arcsin F, \pi - \arcsin F)$ which is strictly inside D, the domain of integration of $J(\beta)$ (Fig. 6(a)). Thus by [32, Sec. VIII.10, (10.10), pp. 461], we get that for $\beta \gg 1$,

$$J(\beta) \sim 2\pi\beta^{-1}\left[\det\left(D^2 f(\arcsin F, \pi - \arcsin F)\right)\right]^{-\frac{1}{2}} e^{-\beta f(\arcsin F, \pi-\arcsin F)},$$

where $D^2 f$ and $\det(D^2 f)$ are given as:

$$D^2 f := \begin{bmatrix} f_{q'q'} & f_{q'q} \\ f_{qq'} & f_{qq} \end{bmatrix} = \begin{bmatrix} \cos q' & 0 \\ 0 & -\cos q \end{bmatrix}, \qquad \det(D^2 f) = \begin{vmatrix} f_{q'q'} & f_{q'q} \\ f_{qq'} & f_{qq} \end{vmatrix} = -\cos q' \cos q.$$

We immediately obtain that for $\beta \gg 1$,

$$J(\beta) \sim \frac{2\pi\beta^{-1}}{\sqrt{1-F^2}} e^{\beta(2\sqrt{1-F^2}-F\pi+2F\cdot\arcsin F)}.$$

The assertion then follows. □

Proof of (3.6). For $F > 1$, $f(q', q)$ has no interior minima in D. Instead, all the global minima of $f(q', q)$ accumulate on the line segment $q = q'$, $q' \in [0, 2\pi]$ (Fig. 6(b)). As $\beta \to \infty$, the asymptotics of $J(\beta)$ depends on the asymptotics of the following integral

$$I_\delta(\beta) = \iint_{D_\delta} e^{-f(q',q)\beta}\, dq'\, dq \quad \text{where } D_\delta = \{(q', q)\big| 0 \le q \le 2\pi, q - \delta \le q' \le q\}. \tag{3.7}$$

For each fixed q, we expand $f(q', q)$ near $q' = q$,

$$\begin{aligned} f(q', q) &= f(q, q) + \sum_{s=0}^{\infty} a_s (q' - q)^{s+\mu} \\ &= (\sin q - F)(q' - q) + \frac{\cos q}{2}(q' - q)^2 - \frac{\sin q}{6}(q' - q)^3 + O\left((q' - q)^4\right), \end{aligned}$$

with $\mu = 1$, $a_0 = \sin q - F$, $a_1 = \frac{\cos q}{2}$ and $a_2 = -\frac{\sin q}{6}$. Next set $1 = \sum_{s=0}^{\infty} b_s (q' - q)^{s+\alpha-1}$ with $\alpha = 1$, $b_0 = 1$ and $0 = b_1 = b_2 = \cdots$. Then [32, Thm. II.1, pp. 58] gives:

$$\begin{aligned} \int_{q-\delta}^{q} e^{-f(q',q)\beta}\, dq' &= -\int_{q}^{q-\delta} e^{-f(q',q)\beta}\, dq' \sim -e^{-f(q,q)\beta} \sum_{s=0}^{\infty} \Gamma(s+1) C_s \beta^{-(s+1)} \\ &= -\Gamma(1) C_0 \beta^{-1} - \Gamma(2) C_1 \beta^{-2} - \Gamma(3) C_2 \beta^{-3} + O\left(\beta^{-4}\right), \end{aligned}$$

where

$$\begin{aligned} C_0 &= \frac{b_0}{\mu a_0^{\alpha/\mu}} = \frac{1}{\sin q - F}, \\ C_1 &= \left(\frac{b_1}{\mu} - \frac{(\alpha+1) a_1 b_0}{\mu^2 a_0}\right) \frac{1}{a_0^{(\alpha+1)/\mu}} = -\frac{\cos q}{(\sin q - F)^3}, \\ C_2 &= \left(\frac{b_2}{\mu} - \frac{(\alpha+2) a_1 b_1}{\mu^2 a_0} + \left((\alpha+\mu+2) a_1^2 - 2\mu a_0 a_2\right) \frac{(\alpha+2) b_0}{2\mu^3 a_0^2}\right) \frac{1}{a_0^{(\alpha+2)/\mu}} \\ &= \frac{3\cos^2 q}{2(\sin q - F)^5} + \frac{\sin q}{2(\sin q - F)^4}. \end{aligned}$$

Hence

$$\int_{q-\delta}^{q} e^{-f(q',q)\beta}\, dq' \sim \frac{\beta^{-1}}{F - \sin q} + \frac{4\cos q}{(\sin q - F)^3}\beta^{-2} - \left(\frac{9\cos^2 q}{(\sin q - F)^5} + \frac{3\sin q}{(\sin q - F)^4}\right)\beta^{-3} + O\left(\beta^{-4}\right),$$

so that as $\beta \to \infty$,

$$\begin{aligned} I_\delta(\beta) &= \int_0^{2\pi} \int_{q-\delta}^{q} e^{-f(q',q)\beta}\, dq'\, dq \\ &\sim \int_0^{2\pi} \left[\frac{\beta^{-1}}{F - \sin q} + \frac{4\cos q}{(\sin q - F)^3}\beta^{-2} - \left(\frac{9\cos^2 q}{(\sin q - F)^5} + \frac{3\sin q}{(\sin q - F)^4}\right)\beta^{-3} + O\left(\beta^{-4}\right)\right] dq \\ &= \frac{2\pi\beta^{-1}}{\sqrt{F^2 - 1}} - \frac{3(1 + 4F^2)\pi}{4(F^2 - 1)^{\frac{7}{2}}}\beta^{-3} + O\left(\beta^{-4}\right). \end{aligned}$$

The claim then follows. □

4. The over-damped limit

We show in this section that the *long time average velocity* of the Langevin equation (1.4) converges in the *over-damped limit* ($m \to 0$) to that of the Smoluchowski process. The results are stated for both the deterministic and stochastic versions.

First, removing the noise term from the Langevin equation (1.4) gives

$$m\ddot{q} = F - \Psi'(q) - \gamma\dot{q}, \quad q(0) = q_0, \quad \dot{q}(0) = p_0, \tag{4.1}$$

which can be recast as:

$$\dot{q} = p, \quad \dot{p} = -\tfrac{\gamma}{m} p + \tfrac{1}{m}(F - \Psi'(q)), \quad q(0) = q_0, \quad p(0) = p_0. \tag{4.2}$$

The *long time average velocity* V_F^q of (4.1) (or (4.2)) is defined as

$$V_F^q := \lim_{t\to\infty} \tfrac{q(t)}{t}. \tag{4.3}$$

In the over-damped limit, it is natural to neglect the inertial term in (4.1) and consider the following equation:

$$\dot{Q} = \tfrac{1}{\gamma}(F - \Psi'(Q)), \quad Q(0) = Q_0. \tag{4.4}$$

The *long time average velocity* V_F^Q is similarly defined as

$$V_F^Q := \lim_{t\to\infty} \frac{Q(t)}{t}. \tag{4.5}$$

As explained in Section 2.1, the *pinning* and *de-pinning threshold* F^* of the process $Q(t)$ solving (4.4) is 1.

We first state the convergence result for the deterministic case.

Theorem 4.1. *For (4.2) and (4.4), it holds that*

$$\lim_{m\to 0} V_F^q = V_F^Q. \tag{4.6}$$

For the stochastic case, we can again recast Eq. (1.4) into the following system:

$$\dot{\tilde{q}} = \tilde{p}, \quad \dot{\tilde{p}} = \frac{1}{m}(F - \Psi'(\tilde{q})) - \frac{\gamma}{m}\tilde{p} + \frac{1}{m}\sqrt{2\gamma\beta^{-1}}\dot{W}, \quad \tilde{q}(0) = \tilde{q}_0, \quad \tilde{p}(0) = \tilde{p}_0. \tag{4.7}$$

On the other hand, neglecting the inertial term of (1.4) gives:

$$\dot{\tilde{Q}} = \frac{1}{\gamma}(F - \Psi'(\tilde{Q})) + \frac{1}{\gamma}\sqrt{2\gamma\beta^{-1}}\dot{W}, \quad \tilde{Q}(0) = \tilde{Q}_0. \tag{4.8}$$

It has been shown that for any fixed finite time $T > 0$, we have

$$\lim_{m\downarrow 0} \max_{0\le t\le T} |\tilde{q}(t) - \tilde{Q}(t)| = 0, \quad \text{in probability.}$$

This statement is called the *Smoluchowski–Kramers approximation* [24,11] [26, Ch. 10] [25]. However, this approximation is not enough to guarantee the convergence of the long time behavior.

We similarly define the *long time average velocities* $V_F^{\tilde{q}}$ and $V_F^{\tilde{Q}}$ of (4.7) and (4.8) as:

$$V_F^{\tilde{q}} := \lim_{t\to\infty} \frac{\tilde{q}(t)}{t} \quad \text{and} \quad V_F^{\tilde{Q}} := \lim_{t\to\infty} \frac{\tilde{Q}(t)}{t}. \tag{4.9}$$

By the *ergodicity* of the processes (see Section 2.2), the above quantities exist almost surely and their values can be computed by using $\phi(q, p) = p$ in (2.12) and $\phi(q) = \frac{F-\Psi'(q)}{\gamma}$ in (2.10) respectively.

Next we state the convergence result for the stochastic case.

Theorem 4.2. *For (4.7) and (4.8), it holds that*

$$\lim_{m\to 0} V_F^{\tilde{q}} = V_F^{\tilde{Q}} \quad \textit{in probability}.$$

We now proceed to the proofs of the above theorems.

4.1. Proof of Theorem 4.1 (Deterministic case)

We first describe the idea of the proof. We consider three cases: $F < 1$, $F = 1$ and $F > 1$. For $F < 1$, we apply a result of Andronov, et al. in [13] (stated in page 11) about the ω-limit set of (4.1). For $F > 1$, we use pieces of the first order process of (4.4) to approximate the position process of the second order (4.2). Each piece of (4.4) starts from a prescribed Q, which is the position of (4.2) at that moment, and stops at $Q + 2\pi$. The *periodicity* of Ψ is used to connect these pieces together to form a "complete" first order process of (4.4) starting at q_0. This "complete" first order process is the one used to approximate the position process of the second order equation (4.2). The case $F = 1$ is taken to be the limit of $F \to 1^+$.

Now we present the proof rigorously. Let

$$\eta(q) = \frac{1}{\gamma}(F - \Psi'(q)). \tag{4.10}$$

The above expression works as an approximation of $p(\cdot)$, the velocity of (4.2). Note that there exists a constant $C_1 > 0$ such that

$$|\eta(q)| \le C_1, \quad |\eta'(q)| \le C_1. \tag{4.11}$$

Following the idea of [28, Ch. 15], we give some estimates for $p(t)$ and $q(t)$.

Lemma 4.3. *For (4.2) and (4.4), there exist positive constants C_2, C_3, C_4, K independent of m such that the following statements hold:*

$$|p(t)| < C_2 \quad \textit{for all } t \in [0, \infty), \tag{4.12}$$

$$\limsup_{t\to\infty} |p(t) - \eta(q(t))| \le \frac{mC_3}{\gamma}, \tag{4.13}$$

$$|q(t) - Q(t)|^2 \le C_4 e^{Kt} \left(\frac{m}{\gamma} |p(0) - \eta(q(0))|^2 + \frac{m^2}{\gamma^2} \right). \tag{4.14}$$

Proof. Estimate (4.12) follows by first integrating (4.2). Then

$$|p(t)| = \left| e^{-\frac{\gamma}{m}t} p(0) + \frac{1}{m} e^{-\frac{\gamma}{m}t} \int_0^t e^{\frac{\gamma}{m}s} (F - \Psi'(q(s)))\, ds \right| \le |p(0)| + \frac{F+1}{\gamma}.$$

Next we write $p(t) = \eta(q(t)) + r(t)$. Then

$$\begin{aligned} \frac{dr(t)}{dt} &= \frac{dp}{dt} - \eta'(q)\frac{dq}{dt} = \left(-\frac{\gamma}{m} p + \frac{1}{m}(F - \Psi'(q)) \right) - \eta'(q)\frac{dq}{dt} \\ &= -\frac{\gamma}{m}(p - \eta(q)) - p\eta'(q) = -\frac{\gamma}{m} r(t) - p\eta'(q). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|r(t)|^2 &= r(t)\cdot\frac{dr(t)}{dt} = r(t)\cdot\left(-\frac{\gamma}{m} r(t) - p\eta'(q) \right) \le -\frac{\gamma}{m}|r(t)|^2 + (\max\{C_1, C_2\})^2 |r(t)| \\ &\le -\frac{\gamma}{m}|r(t)|^2 + \frac{m}{2\gamma}(\max\{C_1, C_2\})^4 + \frac{\gamma}{2m}|r(t)|^2 \le -\frac{\gamma}{2m}|r(t)|^2 + \frac{m}{2\gamma}(\max\{C_1, C_2\})^4 . \end{aligned}$$

By Gronwall's lemma, we get

$$|r(t)|^2 \le e^{-\frac{\gamma}{m}t}|r(0)|^2 + \left(1 - e^{-\frac{\gamma}{m}t}\right) \frac{m^2 (\max\{C_1, C_2\})^4}{\gamma^2}. \tag{4.15}$$

Statement (4.13) can be obtained by letting t tend to infinity in (4.15).

For (4.14), note that $\frac{d}{dt}(q - Q) = \eta(q) + r(t) - \eta(Q)$. Then

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|q - Q|^2 &= (q - Q)\cdot\frac{d}{dt}(q - Q) = (q - Q)\cdot(\eta(q) + r(t) - \eta(Q)) \\ &\le |r(t)||q - Q| + C_1|q - Q|^2 \le C_1|q - Q|^2 + \frac{1}{2}|r(t)|^2 + \frac{1}{2}|q - Q|^2. \end{aligned}$$

Thus

$$\frac{d}{dt}|q - Q|^2 \le (2C_1 + 1)|q - Q|^2 + e^{-\frac{\gamma}{m}t}|r(0)|^2 + \left(1 - e^{-\frac{\gamma}{m}t}\right) \frac{m^2 (\max\{C_1, C_2\})^4}{\gamma^2}.$$

By Gronwall's lemma again, we get

$$\begin{aligned} |q - Q|^2 &\le e^{(2C_1+1)t} \left(\int_0^t e^{-(2C_1+1)s} \left(e^{-\frac{\gamma}{m}s}|r(0)|^2 + (1 - e^{-\frac{\gamma}{m}s}) \frac{m^2 (\max\{C_1, C_2\})^4}{\gamma^2} \right) ds \right) \\ &\le \max\{(\max\{C_1, C_2\})^4, 1\} e^{(2C_1+1)t} \left(|p(0) - \eta(q(0))|^2 \frac{m}{\gamma} + \frac{m^2}{\gamma^2} \right) \end{aligned}$$

which gives (4.14). □

The proof of Theorem 4.1 is divided into the following three cases.

Lemma 4.4. *When $F > 1$, for (4.2) and (4.4), it holds that*

$$\lim_{m\to 0} V_F^q = V_F^Q.$$

Proof. Let

$$T_0 = 0, \quad T_i = \int_{q(\sum_{j=1}^{j=i-1} T_j)}^{q(\sum_{j=1}^{j=i-1} T_j) + 2\pi} \frac{F - \Psi'(q)}{\gamma}\, dq, \quad \text{for } i = 1, 2, \ldots .$$

By periodicity, all the T_i's $(i = 1, 2, \ldots)$ are equal to some fixed finite number $T > 0$ which gives the time it takes for the first order process to travel from any point q to $q + 2\pi$.

Let $Q^i(t)$ be the piece of the first order process started at $Q(0) = q((i-1)T)$ and terminated at $Q(0) + 2\pi$, for $i = 1, 2, \ldots$. Define the deviation of the second order process from the first order process as $E_i := q(iT) - Q^i(T)$. Then by Lemma 4.3,

$$\begin{aligned} \limsup_{t\to\infty} \left| \frac{q(t)}{t} - V_F^Q \right| &= \limsup_{n\to\infty} \left| \frac{1}{n}\sum_{i=1}^n \frac{Q^i(T)}{T} + \frac{1}{n}\sum_{i=1}^n \frac{E_i}{T} - V_F^Q \right| \le \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \frac{|E_i|}{T} \\ &\le \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \frac{1}{T}\left(C_4 e^{KT}\cdot\left(|p(iT) - \eta(q(iT))|^2 \frac{m}{\gamma} + \frac{m^2}{\gamma^2} \right) \right)^{\frac{1}{2}} \\ &\le \frac{1}{T}\left(C_4 e^{KT} \left(\frac{m^3 C_3^2}{\gamma^3} + \frac{m^2}{\gamma^2} \right) \right)^{\frac{1}{2}}. \end{aligned}$$

Letting m go to 0 gives $\lim_{m\to 0}\limsup_{t\to\infty} \left| \frac{q(t)}{t} - V_F^Q \right| = 0$, i.e. $\lim_{m\to 0} V_F^q = V_F^Q$ for $F > 1$. □

We now state the result from [13] which is used in the proof for $F < 1$. By time re-scaling $\tau = \frac{t}{\sqrt{m}}$ and setting $\gamma^* = \frac{\gamma}{\sqrt{m}}$, (4.1) can be normalized into:

$$\frac{dq}{d\tau} = p, \quad \frac{dp}{d\tau} = F - \Psi'(q) - \gamma^* p, \quad q(0) = q_0, \quad p(0) = \sqrt{m}p_0. \tag{4.16}$$

In [13, Sec. VII.3], it is shown that there exists a threshold $F^{**} = 1$ for Eq. (4.16) with the following properties:

(i) above F^{**}, no equilibria exist and the ω-limit set (see [33, Ch. 10]) of $(q(\tau), p(\tau))$ in (4.16) is a single *stable limit cycle* encircling the phase cylinder $\mathbb{T} \times \mathbb{R}$ independent of initial data. The uniqueness of this limit cycle can be seen by Poincaré–Bendixson's criteria ([13, Sec. V.9, V.11]). Its stability is due to the fact that (4.16) is a dissipative system ($\gamma^* > 0$) [13, Sec. V.6];

(ii) below F^{**}, for small γ^* ($\gamma^* < \gamma_0^*$), depending on the initial data, the ω-limit set of $(q(\tau), p(\tau))$ in (4.16) can be either a single *stable limit cycle* encircling the phase cylinder $\mathbb{T} \times \mathbb{R}$ or a single *fixed point*; whereas for γ^* big enough ($\gamma^* > \gamma_0^*$), the ω-limit set of $(q(\tau), p(\tau))$ in (4.16) is a single *fixed point* independent of initial data.

Lemma 4.5. *When $F < 1$, for (4.2) and (4.4), it holds that*

$$\lim_{m \to 0} V_F^q = V_F^Q = 0.$$

Proof. Consider the re-scaled (4.16). When $F < 1$, by (ii) above, there exists a threshold $\gamma_0^* > 0$ for γ^* such that when $\gamma^* > \gamma_0^*$ the long time average velocity $\lim_{\tau \to \infty} \frac{q(\tau)}{\tau}$ of (4.16) is zero. We can first choose m small enough such that $\gamma^* = \frac{\gamma}{\sqrt{m}} > \gamma_0^*$. Note $\lim_{\tau \to \infty} \frac{q(\tau)}{\tau} = 0$ for each such m. This gives $\lim_{m \to 0} V_F^q = 0$. □

Lemma 4.6. *When $F = F^{**} (= 1)$, for (4.2) and (4.4), it holds that*

$$\lim_{m \to 0} V_F^q = V_F^Q = 0.$$

Proof. On one hand, $V_F^q \geq 0$ due to the tilted potential. To see this, let the total energy $\mathscr{E}$ be the sum of the kinetic energy and potential energy, that is, $\mathscr{E} = \frac{1}{2}mp^2 + (-Fq + \Psi(q))$. Since the system is dissipative ($\gamma > 0$), it follows that $\frac{d\mathscr{E}}{dt} = -\gamma p^2 < 0$. Suppose $V_F^q < 0$. Then $\mathscr{E}(t) \geq -Fq(t) + \Psi(q(t)) \to \infty$ as $t \to \infty$ since $q(t) \to -\infty$ as $t \to \infty$. But $\mathscr{E}(t) \leq \mathscr{E}(0) < \infty$, leading to a contradiction.

On the other hand, by (4.13) of Lemma 4.3, the second order position process $q(t)$ satisfies that

$$\dot{q} = \frac{1}{\gamma}(F - \Psi'(q) + \mathrm{O}(m)),$$

where by (4.13), we have $\mathrm{O}(m) < C_3 m$ for some constant C_3. Now consider the following first order process

$$\dot{Q}_1 = \frac{1}{\gamma}((F + C_3 m) - \Psi'(Q_1)).$$

Let T_q and T_{Q_1} denote the traveling time from 0 to 2π of $q(t)$ and $Q_1(t)$ respectively. Let $V_{F+C_3m}^{Q_1}$ denote the long time average velocity of Q_1. Then we have

$$T_q = \int_0^{2\pi} \frac{\gamma\, dq}{F - \Psi'(q) + \mathrm{O}(m)}, \quad \text{and} \quad T_{Q_1} = \int_0^{2\pi} \frac{\gamma\, dQ_1}{(F + C_3 m) - \Psi'(Q_1)}.$$

It follows from $\mathrm{O}(m) < C_3 m$ that $T_q > T_{Q_1}$ leading to $V_F^q < V_{F+C_3m}^{Q_1}$. Hence for $F = 1$, $\lim_{m \to 0} V_F^q \leq \lim_{m \to 0} V_{F+C_3m}^{Q_1} = V_F^Q = 0$. □

4.2. Proof of Theorem 4.2 (Stochastic Case)

The idea of proof, as illustrated by Fig. 7, is still to use pieces of the first order process of (4.8) to approximate the position process of the second order equation (4.7). Each piece of the first order process of (4.8) starts from a prescribed $\tilde{Q}$, which is the position of the second order process of (4.7) at that moment, and terminates after running for time T_m. The deviation of each second order piece from the first order process gives rise to an error term. We will let T_m tend to infinity as m tends to zero. On one hand, with increasing T_m, the average velocity of each such first order piece gets closer and closer to the long time average velocity of $\tilde{Q}(t)$. On the other hand, with properly chosen T_m (which does not tend to infinity too fast), the error term generated by deviation vanishes as m goes to zero. We will show that the approximation error converges to zero in $L^2(\Omega)$, which leads to the convergence in probability for $V_F^{\tilde{q}}$ to $V_F^{\tilde{Q}}$ in Theorem 4.2. We now proceed to the rigorous proof.

First we let $\tilde{Q}$ have the same initial value as $\tilde{q}$, i.e. $\tilde{q}(0)$. Then we define

$$\tilde{r}(t) = \tilde{q}(t) - \tilde{Q}(t) \tag{4.17}$$

to be the *deviation* at time t between the two processes. We have the following estimate:

Lemma 4.7. *Given $\tilde{p}(0)$, for $T > 0$ and $0 \leq t \leq T$, it holds that*

$$E(|\tilde{r}(t)|^2) \leq \left(|\tilde{p}(0)|^2 \frac{4m^2}{\gamma^2} + \frac{4m^2}{\gamma^4}\left(F + \|\Psi'\|_{L^\infty}\right)^2 + \frac{4C_1 m}{\beta\gamma^2}\right) e^{\frac{4T\|\Psi''\|_{L^\infty}^2}{\gamma^2} \cdot t}. \tag{4.18}$$

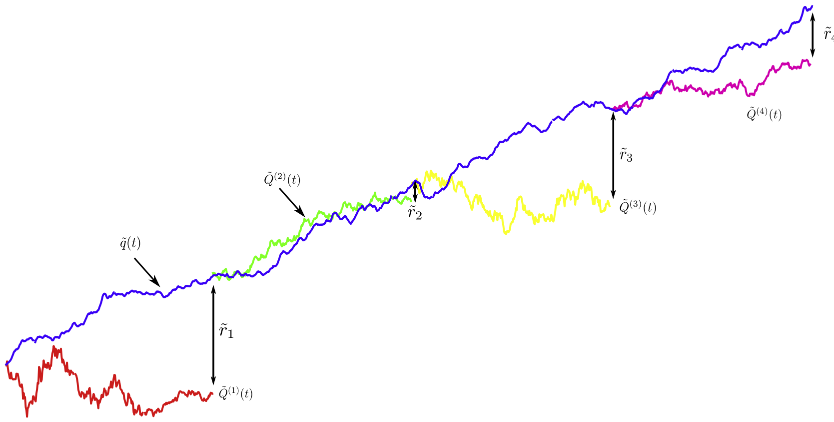


Fig. 7. Approximate the second order process $\tilde{q}(t)$ of (4.7) by pieces of the first order process $\tilde{Q}^{(i)}(t), i = 1, 2, 3 \ldots$. The first order pieces $\tilde{Q}^{(i)}$ are defined in (4.24) and their deviations from the second order process $\tilde{q}$ are denoted by $\tilde{r}_i$ in this graph.

In particular, taking $t = T$ *in* (4.18) *yields*

$$E(|\tilde{r}(T)|^2) \le \left(|\tilde{p}(0)|^2 \tfrac{4m^2}{\gamma^2} + \tfrac{4m^2}{\gamma^4}\left(F + \|\Psi'\|_{L^\infty}\right)^2 + \tfrac{4C_1 m}{\beta\gamma^2}\right) e^{\frac{4T^2\|\Psi''\|^2_{L^\infty}}{\gamma^2}}. \tag{4.19}$$

Proof. By direct computation, we have

$$\begin{aligned}\tilde{r}(t) &= \tilde{p}(0)\frac{m}{\gamma}(1 - e^{-\frac{\gamma}{m}t}) - \frac{1}{\gamma}e^{-\frac{\gamma}{m}t}\int_0^t e^{\frac{\gamma}{m}s}(F - \Psi'(\tilde{q}(s)))\,ds \\ &\quad + \frac{1}{\gamma}\int_0^t (\Psi'(\tilde{Q}(s)) - \Psi'(\tilde{q}(s)))\,ds - \frac{\sqrt{2\gamma\beta^{-1}}}{\gamma}e^{-\frac{\gamma}{m}t}\int_0^t e^{\frac{\gamma}{m}s}\,dW(s).\end{aligned}$$

It yields that

$$\begin{aligned}|\tilde{r}(t)|^2 \le 4\Bigg(&(\tilde{p}(0))^2\frac{m^2}{\gamma^2} + \frac{1}{\gamma^2}e^{-\frac{2\gamma}{m}t}\left(\int_0^t e^{\frac{\gamma}{m}s}(F - \Psi'(\tilde{q}(s)))\,ds\right)^2 \\ &+ \frac{1}{\gamma^2}\left(\int_0^t (\Psi'(\tilde{Q}(s)) - \Psi'(\tilde{q}(s)))\,ds\right)^2 + \frac{2}{\beta\gamma}e^{-\frac{2\gamma}{m}t}\left(\int_0^t e^{\frac{\gamma}{m}s}\,dW(s)\right)^2\Bigg).\end{aligned}$$

Note that

$$\left(\int_0^t e^{\frac{\gamma}{m}s}(F - \Psi'(\tilde{q}(s)))\,ds\right)^2 \le \left(F + \|\Psi'\|_{L^\infty}\right)^2 \frac{m^2}{\gamma^2}\left(e^{\frac{\gamma}{m}t} - 1\right)^2, \tag{4.20}$$

$$\left(\int_0^t (\Psi'(\tilde{Q}(s)) - \Psi'(\tilde{q}(s)))\,ds\right)^2 \le \|\Psi''\|^2_{L^\infty}\left(\int_0^t \left|\tilde{r}(s)\right|\,ds\right)^2 \le \|\Psi''\|^2_{L^\infty} T\int_0^t \left|\tilde{r}(s)\right|^2\,ds. \tag{4.21}$$

In the above, we have used Hölder inequality to establish the second inequality. In addition, by martingale moment inequalities [34, Sec. 3.3.D], we get

$$E\left(\left|\int_0^t e^{\frac{\gamma}{m}s}\,dW(s)\right|^2\right) \le C_1 E\left(\int_0^t e^{\frac{2\gamma}{m}s}\,ds\right) = C_1\tfrac{m}{2\gamma}\left(e^{\frac{2\gamma t}{m}} - 1\right) \le C_1\tfrac{m}{2\gamma}e^{\frac{2\gamma t}{m}}. \tag{4.22}$$

We thus obtain

$$E(|\tilde{r}(t)|^2) \le |\tilde{p}(0)|^2\frac{4m^2}{\gamma^2} + \frac{4m^2}{\gamma^4}\left(F + \|\Psi'\|_{L^\infty}\right)^2 + \frac{4T\|\Psi''\|^2_{L^\infty}}{\gamma^2}\int_0^t E\left(|\tilde{r}(s)|^2\right)\,ds + \frac{4C_1 m}{\beta\gamma^2}.$$

By Gronwall inequality, we get

$$E(|\tilde{r}(t)|^2) \le \left(|\tilde{p}(0)|^2\frac{4m^2}{\gamma^2} + \frac{4m^2}{\gamma^4}\left(F + \|\Psi'\|_{L^\infty}\right)^2 + \frac{4C_1 m}{\beta\gamma^2}\right) e^{\frac{4T\|\Psi''\|^2_{L^\infty}}{\gamma^2}\cdot t},$$

which is the desired result. □

To supplement the above result which depends on the initial velocity $p(0)$, the following estimate gives some uniform estimate for $p(t)$.

Lemma 4.8. *For the velocity process* $\tilde{p}(t)$ *of* (4.7), *we have for all* $t > 0$

$$E(\left|\tilde{p}(t)\right|^2) \le 3\left(|\tilde{p}(0)|^2 + \tfrac{(F + \|\Psi'\|_{L^\infty})^2}{\gamma^2} + \tfrac{C_1}{m\beta}\right). \tag{4.23}$$

Proof. By straightforward computation, we have

$$\tilde{p}(t) = e^{-\frac{\gamma}{m}t}\tilde{p}(0) + \frac{1}{m}e^{-\frac{\gamma}{m}t}\int_0^t e^{\frac{\gamma}{m}s}(F - \Psi'(\tilde{q}(s)))\,ds + \frac{1}{m}e^{-\frac{\gamma}{m}t}\int_0^t e^{\frac{\gamma}{m}s}\sqrt{2\gamma\beta^{-1}}\,dW(s).$$

It follows by Cauchy–Schwarz inequality that

$$\left|\tilde{p}(t)\right|^2 \le 3\left[e^{-\frac{2\gamma}{m}t}|\tilde{p}(0)|^2 + \frac{1}{m^2}e^{-\frac{2\gamma}{m}t}\left(\int_0^t e^{\frac{\gamma}{m}s}(F - \Psi'(\tilde{q}(s)))\,ds\right)^2 + \frac{1}{m^2}e^{-\frac{2\gamma}{m}t}\left(\int_0^t e^{\frac{\gamma}{m}s}\sqrt{2\gamma\beta^{-1}}\,dW(s)\right)^2\right].$$

By (4.20) and (4.22), we obtain

$$E\left(|\tilde{p}(t)|^2\right) \le 3\left(|\tilde{p}(0)|^2 + \frac{\left(F + \|\Psi'\|_{L^\infty}\right)^2}{\gamma^2} + \frac{C_1}{m\beta}\right),$$

where C_1 is the same as in (4.22). □

Note that the above estimate is uniform in time but blows up as $m \to 0$. This is not surprising as in the vanishing mass limit, the velocity process is driven by Brownian motion so that the velocity process is not defined in the point-wise manner.

We now start the proof of Theorem 4.2. Assume that the second order process $\tilde{q}(t)$ of (4.7) starts from $\tilde{q}_0 \in \mathbb{R}$. Construct below pieces $\tilde{Q}^{(k)}(t)$, $k = 1, 2, \dots$ of the first order process of (4.8):

$$\dot{\tilde{Q}}^{(k)} = \frac{F - \Psi'(\tilde{Q}^{(k)})}{\gamma} + \frac{\sqrt{2\gamma\beta^{-1}}}{\gamma}\dot{W}, \qquad \tilde{Q}^{(k)}(t_0^k) = \tilde{q}(t_0^k), \tag{4.24}$$

where

$$T_m > 0 \quad \text{and} \quad t_0^k = (k-1)T_m \quad \text{for} \quad k = 1, 2, \dots.$$

In words, we start $\tilde{Q}^{(k)}(t)$ at time t_0^k with initial data $\tilde{q}(t_0^k)$ and terminate it at time $t_0^k + T_m$. We will use these first order pieces to approximate the second order process. We let $T_m = \sqrt{\alpha|\ln m|}$ with $\alpha > 0$. Hence $T_m \to \infty$ as $m \to 0$. The choice of α will be specified later.

Now we analyze the long time average velocity of $\tilde{q}(t)$ using pieces $\tilde{Q}^{(k)}(t)$ as follows:

$$\begin{aligned}
\lim_{t\to\infty}\frac{\tilde{q}(t)}{t} &= \lim_{n\to\infty}\frac{\sum_{k=1}^n\left(\tilde{Q}^{(k)}(t_0^{k+1}) - \tilde{q}(t_0^k) + \tilde{q}(t_0^{k+1}) - \tilde{Q}^{(k)}(t_0^{k+1})\right)}{nT_m}\\
&= \lim_{n\to\infty}\left(\frac{\sum_{k=1}^n\frac{\tilde{Q}^{(k)}(t_0^{k+1}) - \tilde{Q}^{(k)}(t_0^k)}{T_m}}{n} + \frac{\sum_{k=1}^n\frac{\tilde{q}\left(t_0^{k+1}\right) - \tilde{Q}^{(k)}(t_0^{k+1})}{T_m}}{n}\right),\\
&= \lim_{n\to\infty}\left(\frac{\sum_{k=1}^n A_{k,m}}{n} + \frac{\sum_{k=1}^n B_{k,m}}{n}\right)
\end{aligned}\tag{4.25}$$

where

$$A_{k,m} = \frac{\tilde{Q}^{(k)}(t_0^{k+1}) - \tilde{Q}^{(k)}(t_0^k)}{T_m}, \qquad B_{k,m} = \frac{\tilde{q}\left(t_0^{k+1}\right) - \tilde{Q}^{(k)}(t_0^{k+1})}{T_m} \tag{4.26}$$

i.e. $A_{k,m}$ and $B_{k,m}$ respectively represent the displacement of $\tilde{Q}^{(k)}$ and the difference between $\tilde{q}$ and $\tilde{Q}^{(k)}$ during the time interval $[t_0^k, t_0^{k+1}]$.

Next let $\rho(Q)$ be the invariant density of $\tilde{Q}(t)$ and introduce

$$V_F^{\tilde{Q},\infty} = \int_{\mathbb{T}}\frac{F - \Psi'(Q)}{\gamma}\rho(Q)\,dQ.$$

Then by the ergodicity of $\tilde{Q}$, we have that $\lim_{t\to\infty}\frac{\tilde{Q}(t)}{t} = V_F^{\tilde{Q},\infty}$ almost surely. Theorem 4.2 follows if we can show that

$$\lim_{m\to 0} E\left|\lim_{n\to\infty}\frac{\sum_{k=1}^n A_{k,m}}{n} - V_F^{\tilde{Q},\infty}\right|^2 = 0 \quad \text{and} \quad \lim_{m\to 0} E\left|\lim_{n\to\infty}\frac{\sum_{k=1}^n B_{k,m}}{n}\right|^2 = 0.$$

Furthermore, combining the ergodicity of the processes $\tilde{q}$ and $\tilde{Q}$, the following limits

$$\lim_{n\to\infty}\frac{\sum_{k=1}^n A_{k,m}}{n}, \qquad \lim_{n\to\infty}\frac{\sum_{k=1}^n B_{k,m}}{n}$$

also exist almost surely. Then we have

$$E\left|\lim_{n\to\infty}\frac{\sum_{k=1}^{n}A_{k,m}}{n}-V_F^{\tilde{Q},\infty}\right|^2=E\lim_{n\to\infty}\left|\frac{\sum_{k=1}^{n}A_{k,m}}{n}-V_F^{\tilde{Q},\infty}\right|^2$$

$$=E\liminf_{n\to\infty}\left|\frac{\sum_{k=1}^{n}A_{k,m}}{n}-V_F^{\tilde{Q},\infty}\right|^2\le\liminf_{n\to\infty}E\left|\frac{\sum_{k=1}^{n}A_{k,m}}{n}-V_F^{\tilde{Q},\infty}\right|^2$$

where the last inequality is by Fatou's lemma. Similar argument applies for $E\left|\lim_{n\to\infty}\frac{\sum_{k=1}^{n}B_{k,m}}{n}\right|^2$. Hence it suffices to show the following two statements:

Claim #1.

$$\lim_{m\to 0}\liminf_{n\to\infty}E\left|\frac{\sum_{k=1}^{n}A_{k,m}}{n}-V_F^{\tilde{Q},\infty}\right|^2=0 \tag{4.27}$$

Claim #2.

$$\lim_{m\to 0}\liminf_{n\to\infty}E\left|\frac{\sum_{k=1}^{n}B_{k,m}}{n}\right|^2=0. \tag{4.28}$$

The proof of the above relies on the convergence rate of transition probability density to the invariant density. Let $P(Q,t|Q_0,0)$ be the transition probability and $\rho(Q)$ be the invariant density of $\tilde{Q}(t)$. Then we note that following rate of convergence of P to ρ [28, Theorem 6.4.1]: there exist positive constants C_1 and C_2 such that

$$\|P(Q,t|Q_0,0)-\rho(Q)\|_{L^1(\mathbb{T})}\le C_1e^{-C_2t}. \tag{4.29}$$

Proof of Claim #1. Let $\tilde{A}_{k,m}=A_{k,m}-E(A_{k,m})$ be the fluctuation of $A_{k,m}$ around its mean value. This leads to the following rewriting

$$\frac{\sum_{k=1}^{n}A_{k,m}}{n}=\frac{\sum_{k=1}^{n}E(A_{k,m})}{n}+\frac{\sum_{k=1}^{n}\tilde{A}_{k,m}}{n}$$

so that

$$E\left|\frac{\sum_{k=1}^{n}A_{k,m}}{n}-V_F^{\tilde{Q},\infty}\right|^2=E\left|\frac{\sum_{k=1}^{n}\tilde{A}_{k,m}}{n}+\frac{\sum_{k=1}^{n}E(A_{k,m})}{n}-V_F^{\tilde{Q},\infty}\right|^2$$

$$\le 2E\left(\left(\frac{\sum_{k=1}^{n}\tilde{A}_{k,m}}{n}\right)^2\right)+2\left(\frac{\sum_{k=1}^{n}E(A_{k,m})}{n}-V_F^{\tilde{Q},\infty}\right)^2.$$

The proof of this claim will be completed by showing

$$\lim_{m\to 0}\lim_{n\to\infty}E\left(\left(\frac{\sum_{k=1}^{n}\tilde{A}_{k,m}}{n}\right)^2\right)=0\quad\text{and}\quad\lim_{m\to 0}\lim_{n\to\infty}\frac{\sum_{k=1}^{n}E(A_{k,m})}{n}=V_F^{\tilde{Q},\infty}.$$

We divide the proof into the following four steps.

Step I (Computation of $E\left[(\tilde{A}_{k,m})^2\right]$). Note that $E((\tilde{A}_{k,m})^2)=E(A_{k,m}^2)-E(A_{k,m})^2$ represents the variance of $A_{k,m}$. We make use of the dynamics of the first order process to get

$$A_{k,m}=\frac{\int_{t_0^k}^{t_0^{k+1}}\frac{F-\Psi'(\tilde{Q}^{(k)}(s))}{\gamma}\,ds}{T_m}+\frac{\sqrt{2\gamma\beta^{-1}}}{\gamma T_m}\cdot W(T_m).$$

Taking expectation of its square yields

$$E(A_{k,m}^2) = E\left(\left(\frac{\int_{t_0^k}^{t_0^{k+1}} \frac{F-\Psi'(\tilde{Q}^{(k)}(s))}{\gamma}\,ds}{T_m}\right)^2\right) + 2E\left(\frac{\sqrt{2\gamma\beta^{-1}}}{\gamma}\cdot\frac{\int_{t_0^k}^{t_0^{k+1}} \frac{F-\Psi'(\tilde{Q}^{(k)}(s))}{\gamma}\,ds}{T_m^2}\cdot W(T_m)\right) + E\left(\frac{2}{\gamma\beta}\cdot\frac{(W(T_m))^2}{(T_m)^2}\right). \tag{4.30}$$

For convenience of presentation, we denote

$$D_{k,m} = \frac{\int_{t_0^k}^{t_0^{k+1}} \frac{F-\Psi'(\tilde{Q}^{(k)}(s))}{\gamma}\,ds}{T_m}.$$

For the last two terms on the right hand side of (4.30), we have the following estimates

$$\left|E\left(\frac{\sqrt{2\gamma\beta^{-1}}}{\gamma}\cdot\frac{\int_{t_0^k}^{t_0^{k+1}} \frac{F-\Psi'(\tilde{Q}^{(k)}(s))}{\gamma}\,ds}{T_m^2}\cdot W(T_m)\right)\right| \le \frac{\sqrt{2\gamma\beta^{-1}}}{\gamma}\cdot\frac{F+\|\Psi'\|_{L^\infty}}{\gamma}\cdot\frac{\sqrt{2}}{\sqrt{\pi T_m}},$$

$$\left|E\left(\frac{2}{\gamma\beta}\cdot\frac{(W(T_m))^2}{(T_m)^2}\right)\right| \le \frac{2}{\gamma\beta T_m}.$$

In the above inequality, $E(|W(T_m)|) = \sqrt{\frac{2T_m}{\pi}}$ is used. Furthermore, by the mean zero property of Brownian motion, we have

$$E(A_{k,m}) = E\left(D_{k,m}\right). \tag{4.31}$$

Immediately, we get

$$E(\tilde{A}_{k,m}^2) = E(A_{k,m}^2) - (E(A_{k,m}))^2 \le C_3(m) + C_4 T_m^{-\frac{1}{2}} + C_5 T_m^{-1}, \tag{4.32}$$

where

$$C_3(m) = \left|E\left(D_{k,m}^2\right) - E\left(D_{k,m}\right)^2\right|, \qquad C_4 = \frac{2\sqrt{2\gamma\beta^{-1}}}{\gamma}\cdot\frac{F+\|\Psi'\|_{L^\infty}}{\gamma}\cdot\frac{\sqrt{2}}{\sqrt{\pi}}, \qquad C_5 = \frac{2}{\gamma\beta}. \tag{4.33}$$

Note that C_4 and C_5 are positive constants independent of n and m, whereas $C_3(m)$ depends on m.

To obtain estimate of $C_2(m)$ as $m \to 0$, we compute its two components $E(D_{k,m}^2)$ and $(E(D_{k,m}))^2$ in **Steps II** and **III** below.

Step II (Estimation of $E(D_{k,m}^2)$). We compute

$$\begin{aligned}
E(D_{k,m}^2) &= \frac{1}{T_m^2}E\left(\left(\int_{t_0^k}^{t_0^{k+1}} \frac{F-\Psi'(\tilde{Q}^{(k)}(s))}{\gamma}\,ds\right)^2\right)\\
&= \frac{1}{T_m^2}E\left(\int_{t_0^k}^{t_0^{k+1}} \frac{F-\Psi'(\tilde{Q}^{(k)}(s))}{\gamma}\,ds\int_{t_0^k}^{t_0^{k+1}} \frac{F-\Psi'(\tilde{Q}^{(k)}(s'))}{\gamma}\,ds'\right)\\
&= \frac{1}{\gamma^2T_m^2}\int_{t_0^k}^{t_0^{k+1}}\int_{t_0^k}^{t_0^{k+1}} E\left((F-\Psi'(\tilde{Q}^{(k)}(s)))(F-\Psi'(\tilde{Q}^{(k)}(s')))\right)ds\,ds'\\
&= \frac{2}{\gamma^2T_m^2}\int_0^{T_m}\int_{s'}^{T_m}\int_{\mathbb{T}}\int_{\mathbb{T}}(F-\Psi'(Q))(F-\Psi'(Q'))P(Q',s'|Q_0,0)P(Q,s-s'|Q',0)\,dQ\,dQ'\,ds\,ds' \quad Q_0=\tilde{Q}^{(k)}(0)\\
&= J_1+J_2+J_3+J_4,
\end{aligned}$$

where

$$J_1 = \frac{2}{\gamma^2T_m^2}\int_0^{T_m}\int_{s'}^{T_m}\int_{\mathbb{T}}\int_{\mathbb{T}}(F-\Psi'(Q))(F-\Psi'(Q'))\left(P(Q',s'\mid Q_0,0)-\rho(Q')\right)\left(P(Q,s-s'\mid Q',0)-\rho(Q)\right)dQ\,dQ'\,ds\,ds',$$

$$J_2 = \frac{2}{\gamma^2T_m^2}\int_0^{T_m}\int_{s'}^{T_m}\int_{\mathbb{T}}\int_{\mathbb{T}}(F-\Psi'(Q))(F-\Psi'(Q'))\left(P(Q',s'\mid Q_0,0)-\rho(Q')\right)\rho(Q)\,dQ\,dQ'\,ds\,ds',$$

$$J_3 = \frac{2}{\gamma^2T_m^2}\int_0^{T_m}\int_{s'}^{T_m}\int_{\mathbb{T}}\int_{\mathbb{T}}(F-\Psi'(Q))(F-\Psi'(Q'))\left(P(Q,s-s'\mid Q',0)-\rho(Q)\right)\rho(Q')\,dQ\,dQ'\,ds\,ds',$$

$$J_4 = \frac{2}{\gamma^2T_m^2}\int_0^{T_m}\int_{s'}^{T_m}\int_{\mathbb{T}}\int_{\mathbb{T}}(F-\Psi'(Q))(F-\Psi'(Q'))\rho(Q)\rho(Q')\,dQ\,dQ'\,ds\,ds'.$$

Next we make use of (4.29) to estimate the above:

$$
\begin{aligned}
|J_1| &\le \frac{2\left(F+\|\Psi'\|_{L^\infty}\right)^2}{\gamma^2 T_m^2}\int_0^{T_m}\int_{s'}^{T_m}\left\|P(Q',s'|Q_0,0)-\rho(Q')\right\|_{L^1(\mathbb{T})}\left\|P(Q,s-s'|Q',0)-\rho(Q)\right\|_{L^1(\mathbb{T})}\,ds\,ds'\\
&\le \frac{2\left(F+\|\Psi'\|_{L^\infty}\right)^2}{\gamma^2 T_m^2}\int_0^{T_m}\int_{s'}^{T_m} C_1 e^{-C_2 s'}\cdot C_1 e^{-C_2(s-s')}\,ds\,ds'\\
&\le \frac{2C_1^2\left(F+\|\Psi'\|_{L^\infty}\right)^2}{C_2\gamma^2 T_m}\left(1-e^{-C_2T_m}\right),\\
|J_2| &\le \frac{2V_F^{\tilde{Q},\infty}\left(F+\|\Psi'\|_{L^\infty}\right)}{\gamma T_m^2}\int_0^{T_m}\int_{s'}^{T_m}\left\|P(Q',s'|Q_0,0)-\rho(Q')\right\|_{L^1(\mathbb{T})}\,ds\,ds'\\
&\le \frac{2C_1V_F^{\tilde{Q},\infty}\left(F+\|\Psi'\|_{L^\infty}\right)}{C_2\gamma T_m}\left(1-e^{-C_2T_m}\right),\\
|J_3| &\le \frac{2V_F^{\tilde{Q},\infty}\left(F+\|\Psi'\|_{L^\infty}\right)}{\gamma T_m^2}\int_0^{T_m}\int_{s'}^{T_m}\left\|P(Q,s-s'|Q',0)-\rho(Q)\right\|_{L^1(\mathbb{T})}\,ds\,ds'\\
&\le \frac{2C_1V_F^{\tilde{Q},\infty}\left(F+\|\Psi'\|_{L^\infty}\right)}{C_2\gamma T_m}+\frac{2C_1V_F^{\tilde{Q},\infty}\left(F+\|\Psi'\|_{L^\infty}\right)}{C_2^2\gamma T_m^2}\left(1-e^{-C_2T_m}\right).
\end{aligned}
$$

Direct computation also gives

$$
\begin{aligned}
J_4 &= \frac{1}{\gamma^2T_m^2}\int_0^{T_m}\int_{s'}^{T_m}\int_{\mathbb{T}}\int_{\mathbb{T}}(F-\Psi'(Q))(F-\Psi'(Q'))\rho(Q)\rho(Q')\,dQ\,dQ'\,ds\,ds'\\
&\quad+\frac{1}{\gamma^2T_m^2}\int_0^{T_m}\int_0^{s}\int_{\mathbb{T}}\int_{\mathbb{T}}(F-\Psi'(Q))(F-\Psi'(Q'))\rho(Q)\rho(Q')\,dQ\,dQ'\,ds'\,ds\\
&= \frac{1}{\gamma^2T_m^2}\int_0^{T_m}\int_0^{T_m}\int_{\mathbb{T}}\int_{\mathbb{T}}(F-\Psi'(Q))(F-\Psi'(Q'))\rho(Q)\rho(Q')\,dQ\,dQ'\,ds'\,ds\\
&= \left(\int_{\mathbb{T}}\frac{F-\Psi'(Q)}{\gamma}\rho(Q)\,dQ\right)\left(\int_{\mathbb{T}}\frac{F-\Psi'(Q')}{\gamma}\rho(Q')\,dQ'\right)\\
&= \left(V_F^{\tilde{Q},\infty}\right)^2.
\end{aligned}
$$

All above lead to

$$
E(D_{k,m}^2)=\left(V_F^{\tilde{Q},\infty}\right)^2+o(1)\quad\text{as } m\to 0. \tag{4.34}
$$

Step III (Estimation of $E(D_{k,m})$). By direct computation, we get

$$
\begin{aligned}
E(D_{k,m}) &= E\left(\frac{1}{T_m}\int_{t_0^k}^{t_0^{k+1}}\frac{F-\Psi'(\tilde{Q}^{(k)}(s))}{\gamma}\,ds\right)=\frac{1}{T_m}\int_{t_0^k}^{t_0^{k+1}}E\left(\frac{F-\Psi'(\tilde{Q}^{(k)}(s))}{\gamma}\right)ds\\
&= \frac{1}{T_m}\int_0^{T_m}\int_{\mathbb{T}}\frac{F-\Psi'(Q)}{\gamma}P(Q,s|Q_0,0)\,dQ\,ds\quad (Q_0=\tilde{Q}^{(k)}(t_0^k))\\
&= \frac{1}{T_m}\int_0^{T_m}\int_{\mathbb{T}}\frac{F-\Psi'(Q)}{\gamma}\left(P(Q,s|Q_0,0)-\rho(Q)\right)dQ\,ds+\int_{\mathbb{T}}\frac{F-\Psi'(Q)}{\gamma}\rho(Q)\,dQ.
\end{aligned}
$$

For the approximation error term in $E(D_{k,m})$, we have

$$
\begin{aligned}
\frac{1}{T_m}\int_0^{T_m}\int_{\mathbb{T}}\frac{F-\Psi'(Q)}{\gamma}\left|P(Q,s|Q_0,0)-\rho(Q)\right|dQ\,ds &\le \frac{F+\|\Psi'\|_{L^\infty}}{\gamma T_m}\int_0^{T_m}\int_{\mathbb{T}}\left|P(Q,s|Q_0,0)-\rho(Q)\right|dQ\,ds\\
&\le \frac{F+\|\Psi'\|_{L^\infty}}{\gamma T_m}\int_0^{T_m}\|P(Q,s|Q_0,0)-\rho(Q)\|_{L^1}\,ds\\
&\le \frac{F+\|\Psi'\|_{L^\infty}}{\gamma T_m}\int_0^{T_m}C_1e^{-C_2s}\,ds\\
&= \frac{C_1(F+\|\Psi')\|_{L^\infty}}{C_2\gamma T_m}\left(1-e^{-C_2T_m}\right).
\end{aligned}
$$

The above leads to

$$
\lim_{m\to 0}E(D_{k,m})=V_F^{\tilde{Q},\infty}. \tag{4.35}
$$

Hence

$$\lim_{m\to 0} C_3(m) = \left|E(D_{k,m}^2) - E(D_{k,m})^2\right| = 0.$$

Step IV (Convergence in $L^2(\Omega)$**).** For each n, we have

$$\begin{aligned} E\left(\left(\frac{\sum_{k=1}^{n}\tilde{A}_{k,m}}{n}\right)^2\right) &= \frac{1}{n^2}\sum_{k=1}^{n} E\left(\tilde{A}_{k,m}^2\right) + \frac{1}{n^2}\sum_{j<l} E(2\tilde{A}_{j,m}\tilde{A}_{l,m}) \\ &\le \frac{1}{n^2}\sum_{k=1}^{n} E\left(\tilde{A}_{k,m}^2\right) + \frac{1}{n^2}\sum_{j<l}\left(E(\tilde{A}_{j,m}^2) + E(\tilde{A}_{l,m}^2)\right) \\ &\le \frac{C_3(m) + C_4 T_m^{-\frac{1}{2}} + C_5 T_m^{-1}}{n} + \frac{n(n-1)(C_3(m) + C_4 T_m^{-\frac{1}{2}} + C_5 T_m^{-1})}{n^2} \\ &\le C_3(m) + C_4 T_m^{-\frac{1}{2}} + C_5 T_m^{-1}. \end{aligned} \tag{4.36}$$

By **Steps I, II, III**, (4.31)–(4.36) and $T_m \to \infty$, **Claim** #1 follows. □

Proof of Claim #2. By (4.19) and (4.23), for $m \ll 1$ the error term satisfies

$$E\left(B_{k,m}^2\right) \le \frac{C_6 m e^{\frac{4\|\Psi''\|_{L^\infty}^2}{\gamma^2}\cdot T_m^2}}{T_m},$$

where C_6 is a universal constant independent of k and m. Note that the $\frac{1}{m}$ factor in (4.23) is multiplied by the m^2 in (4.19). Recall that we set $T_m = \sqrt{\alpha|\ln m|}$ at the beginning. Then we obtain

$$E(B_{k,m}^2) \le \frac{C_6 m e^{\frac{4\|\Psi''\|_{L^\infty}^2}{\gamma^2}\cdot\alpha|\ln m|}}{\sqrt{\alpha|\ln m|}} = \frac{C_6 m^{1-\frac{4\|\Psi''\|_{L^\infty}^2}{\gamma^2}\cdot\alpha}}{\sqrt{\alpha|\ln m|}}.$$

We choose α small enough such that $1 - \frac{4\|\Psi''\|_{L^\infty}^2}{\gamma^2}\cdot\alpha > 0$. It leads to that for each n

$$E\left(\frac{\sum_{k=1}^{n} B_{k,m}^2}{n}\right) \le \frac{C_6 m^{1-\frac{4\|\Psi''\|_{L^\infty}^2}{\gamma^2}\cdot\alpha}}{\sqrt{\alpha|\ln m|}} \xrightarrow{m\to 0} 0.$$

Claim #2 follows then. □

5. The under-damped limit

In this section, we study the long time behavior of the Langevin equation (1.4) in the under-damped limit ($\gamma \to 0$). Throughout this section, we use $V_F = V_F^{\tilde{q}}$ to denote the long time average velocity as defined in (4.9).

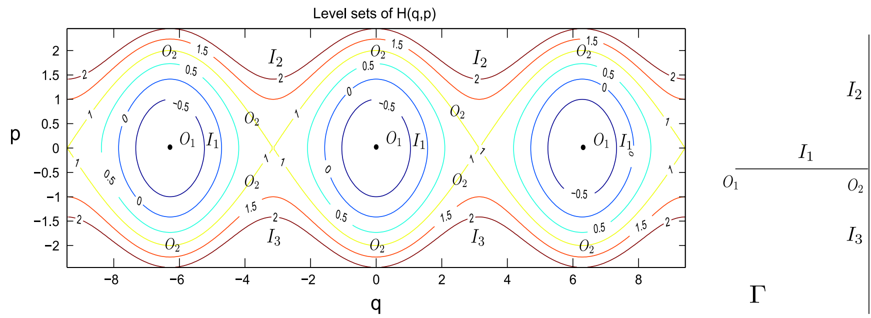
5.1. Bi-stability and three thresholds of the scaled tilt

In the *under-damped limit* ($\gamma \to 0$) of (1.4), inertial effects become significant. We only consider finite value of the *scaled tilt* $F_0 := \frac{F}{\gamma}$. Otherwise, the kinetic energy gain due to decreasing of the potential energy can not be compensated by energy dissipation, leading to unbounded long time average velocity [17, Sec.11.4]. Even for finite F_0, the pinning and de-pinning behavior can be quite complicated due to the fact that both states can co-exist. We call this *bi-stability* phenomenon. It is best depicted in Fig. 4.

For the *deterministic* process obtained by removing the noise term from (1.4), we observe two thresholds F_{01} and F_{03} of F_0 (see Fig. 3): below F_{01} (above F_{03}), only the pinning (running) state exists; between F_{01} and F_{03}, the pinning and running states can both occur – which state the particle ends up with depends on the initial condition [13, Sec. VII.3] – but for fixed F_0, a particle cannot switch between these two states. For $\Psi(q) = -\cos q$, the derivation of $F_{01} = \frac{4}{\pi}$ is carried out by balancing the kinetic energy gain and the energy loss due to friction [17, Sec.11.6.1]. It can also be shown that for general Ψ, $F_{03} = \frac{\|\Psi'\|_{L^\infty}}{\gamma}$ [13, Sec. VII.3].

For the *noisy* process (1.4), by ergodicity (see Section 2.2), V_F exists. Besides F_{01} and F_{03} mentioned above, a new threshold F_{02} emerges, which can be observed (see Fig. 3(b)) in the asymptotics of V_F in the iterated limits by first taking $\gamma \to 0$ (the *under-damped limit*) and then letting $\beta \to \infty$ (the *vanishing noise limit*): between F_{01} and F_{02}, V_F converges to 0; between F_{02} and F_{03}, it converges to a positive number which is the long time average velocity of the running state of the corresponding deterministic (noiseless) process; and near F_{02}, V_F has a steep jump. Note that in this iterated limit, $F_{03} = \frac{\|\Psi'\|_{L^\infty}}{\gamma}$ is not observed in the asymptotics of V_F since it goes to infinity.

The purpose of this section is to make the above description mathematically rigorous. The derivation of F_{01} and F_{02} consists of the following steps. First, in the under-damped limit ($\gamma \to 0$), we make use of the existing machinery invented by Freidlin et al. [15] to



(a) Hamiltonian level sets. (b) Graph Γ.

Fig. 8. Homeomorphism between Hamiltonian level sets and graph Γ. Each point on Γ corresponds to a level curve of H. Note that O_1 is the minimum point of the potential function $\Psi(q) = -\cos q$, O_2 is the heteroclinic orbits between two saddle points of Ψ. Segments I_2 and I_3 correspond to the regions in the phase space with unbounded orbits (with positive and negative p).

reduce the dynamics of a properly time re-scaled system of (1.4) onto a lower dimensional Hamiltonian graph Γ. Second, making use of the invariant measure of (1.4) on the configuration space $\mathscr{C} = \mathbb{T} \times \mathbb{R}$, we derive a formula for V_F in the under-damped limit ($\gamma \to 0$). Third, by Laplace's method we compute the asymptotics of V_F in the vanishing noise limit ($\beta \to \infty$) leading to the precise values of F_{01} and F_{02}.

We now describe the procedure in more detail. We first state the reduction of the dynamics (1.4) to a diffusion on a Hamiltonian graph Γ. Without loss of generality, we consider the following stochastic system (with $m = 1$ and $\Psi(q) = -\cos q$):

$$\dot{\tilde{q}} = \tilde{p}, \qquad \dot{\tilde{p}} = -\gamma \tilde{p} + F - \sin(\tilde{q}) + \sqrt{2\gamma\beta^{-1}}\dot{W} \tag{5.1}$$

which can be written as

$$\dot{\tilde{q}}^\epsilon = \tilde{p}^\epsilon, \qquad \dot{\tilde{p}}^\epsilon = -\sin(\tilde{q}^\epsilon) + \epsilon b(\tilde{q}^\epsilon, \tilde{p}^\epsilon) + \sqrt{\epsilon}\dot{W}, \tag{5.2}$$

where

$$\epsilon = \frac{2\gamma}{\beta} \quad \text{and} \quad b(q, p) = \frac{\beta}{2}(-p + F_0).$$

Note that with fixed β, we have $\epsilon \to 0$ when $\gamma \to 0$.

Introducing the *Hamiltonian function* $H(q, p) = -\cos(q) + \frac{1}{2}p^2$, the deterministic dynamical system is given by:

$$\dot{q} = H_p(q, p), \qquad \dot{p} = -H_q(q, p). \tag{5.3}$$

The value of the Hamiltonian is conserved by the dynamics. On the other hand, for the noisy process, the motion $(\tilde{q}^\epsilon(t), \tilde{p}^\epsilon(t))$ can be roughly decomposed as (i) motion along the trajectories of (5.3), which are connected components of a level curve $\{(q, p) : H(q, p) = H\}$ (see Fig. 8); and (ii) diffusion across them. To study the interactions between these two motions, in particular to identity the limiting description as $\gamma \to 0$, it is convenient to do the time re-scaling $q^\epsilon = \tilde{q}^\epsilon(t/\epsilon)$. Then the dynamics is written as

$$\dot{q}^\epsilon = \frac{1}{\epsilon}H_p(q^\epsilon, p^\epsilon), \qquad \dot{p}^\epsilon = -\frac{1}{\epsilon}H_q(q^\epsilon, p^\epsilon) + b(q^\epsilon, p^\epsilon) + \dot{W}. \tag{5.4}$$

With the above, the process $(q^\epsilon(t), p^\epsilon(t))$ becomes a diffusion on $\mathbb{R}^2$ with infinitesimal generator

$$\mathcal{L}^\epsilon f(\underline{q}) = \frac{1}{2}f_{pp}(\underline{q}) + \frac{1}{\epsilon}\bar{\nabla}H(\underline{q}) \cdot \nabla f(\underline{q}) + b(\underline{q})f_p(\underline{q}), \tag{5.5}$$

where $\underline{q} := (q, p) \in \mathbb{R}^2$ and $\bar{\nabla}H(q, p) := (H_p, -H_q)$.

Note that in the under-damped limit ($\gamma \to 0$), we have $\epsilon \to 0$. Then (q^ϵ, p^ϵ) become the *fast variables* whilst $H(q^\epsilon, p^\epsilon)$ becomes the *slow variable* in (5.4). If we identify all points belonging to the same connected component of a level curve $\{(q, p) : H(q, p) = H\}$, we obtain a graph Γ consisting of three edges I_1, I_2, I_3 connected by one interior vertex O_2. There is also an exterior vertex O_1 at the other end of I_1. The shapes of the level curves of H and the graph Γ are shown in Fig. 8. Their connection is described as follows. Edge I_1 is the collection of periodic orbits parametrized by its energy level which is related to the size of the orbit. Vertex O_1 is the "smallest periodic orbit" and in fact is the *stable point* representing the *minimum* of the potential energy. Vertex O_2 is the "biggest periodic orbit" and in fact is a *heteroclinic orbit* joining two adjacent wells of the potential energy. Edges I_2 and I_3 are the collections of running states in the positive and negative directions. These occur when the initial energy level is high enough and are exactly the unbounded orbits in which the particle overcomes the potential energy barriers indefinitely. Again they are parametrized by the energy level. Edges I_1, I_2 and I_3 are joined together at O_2 which also represents the transition or boundary point between the bounded (periodic) and unbounded states.

The above description thus leads to a *homeomorphism* $Y : \mathscr{C} := \mathbb{T} \times \mathbb{R} \mapsto \Gamma$. The graph Γ is commonly known as the *Hamiltonian graph*. For concreteness, this graph is parametrized in the following way:

$$\Gamma = \bigcup_{i=1}^{3} I_i, \quad \text{where} \begin{cases} I_1 = \{(z; 1), -1 < z < 1\}, \\ I_2 = \{(z; 2), 1 < z < \infty\}, \\ I_3 = \{(z; 3), 1 < z < \infty\}; \end{cases} \tag{5.6}$$

$$O_1 = (-1; 1) \quad \text{and} \quad O_2 = (1; 1) = (1; 2) = (1; 3). \tag{5.7}$$

Note that we use ";" to separate the coordinate z and the index i on each edge of the graph Γ.

In [15], M. Freidlin et al. show that the law of the process $Y(q^\epsilon(t), p^\epsilon(t))$ converges weakly as ϵ tends to zero to some diffusion process $z(t)$ on Γ. This limiting process is described by the following two ingredients:

1. *infinitesimal generators* on each edge, $\mathcal{L}_i$, $i = 1, 2, 3$:

$$\mathcal{L}_i v_i(z) = \frac{1}{2T(z)}\left(\frac{d}{dz}S(z)v_i'(z)\right) \pm \frac{\int_{-q_*}^{q_*} b(q, \pm\sqrt{2z+2\cos(q)})\,dq}{\int_{-q_*}^{q_*}(dq/\sqrt{2z+2\cos(q)})} v_i'(z) \tag{5.8}$$

where

$$T(z) = \int_{-q_*}^{q_*} \frac{dq}{\sqrt{2z+2\cos(q)}} \quad \text{and} \quad S(z) = \int_{-q_*}^{q_*} \sqrt{2z+2\cos(q)}\,dq \tag{5.9}$$

with $q_* = \begin{cases}\arccos(-z) & \text{if } z<1,\\ \pi & \text{if } z>1\end{cases}$. In the expressions (5.8) for the $\mathcal{L}_i$'s, "+" is taken if $i = 2$, "−" is taken if $i = 3$ and the mean value of the "+" and "−" expressions is taken if $i = 1$. Note that $T(1) = \infty$ and $S'(z) = T(z)$.

2. *gluing condition* at O_2: the domains of the generator L_i's satisfy

$$2v_1'(1) = v_2'(1) + v_3'(1) \tag{5.10}$$

which works as an "interior boundary" determining the behavior of the process $z(t)$ when it reaches the interior vertex O_2.

The *invariant density* $\rho_i^\Gamma(z)$ for the process $z(t)$ on each edge I_i associated with the generator $\mathcal{L}_i$ is given explicitly as [15, pp. 624, (3.9), (3.10)]:

$$\rho_1^\Gamma(z) = \frac{4CT(z)}{S(1)}e^{-\beta(z-1)}, \quad \text{for } z \in (-1, 1), \tag{5.11}$$

$$\rho_2^\Gamma(z) = \frac{2CT(z)}{S(1)}e^{-\beta((z-1)-F_0g(z))}, \quad \text{for } z \in (1, \infty), \tag{5.12}$$

$$\rho_3^\Gamma(z) = \frac{2CT(z)}{S(1)}e^{-\beta((z-1)+F_0g(z))}, \quad \text{for } z \in (1, \infty) \tag{5.13}$$

where C is the integration constant set to make $\rho_i^\Gamma(z)$ probability densities and

$$g(z) = \int_1^z \frac{2\pi}{S(\zeta)}\,d\zeta, \quad \text{for } z \geq 1. \tag{5.14}$$

The existence of invariant density also implies that the diffusion on Γ is *recurrent*.

The first result of this section is stated below.

Theorem 5.1. *For the stochastic system (5.1), in the under-damped limit ($\gamma \to 0$) described by the diffusion on Γ, we have*

$$V_F = \frac{2\pi C_2}{C_0+C_1}F_0, \tag{5.15}$$

where

$$C_0 = \frac{\sqrt{2\pi}}{2\sqrt{\beta}}\int_{-\pi}^{\pi} e^{\beta\cos q}\,dq, \tag{5.16}$$

$$C_1 = \int_1^\infty T(z)e^{-\beta z}\{\cosh(\beta F_0 g(z)) - 1\}\,dz, \tag{5.17}$$

$$C_2 = \int_1^\infty \frac{2\pi e^{-\beta z}}{S(z)}\cosh(\beta F_0 g(z))\,dz. \tag{5.18}$$

Taking $\beta \to \infty$ in the above formula, i.e. the vanishing noise limit, we have

$$V_F \sim \begin{cases} \left(\dfrac{2\pi}{S(1) - 2\pi F_0} + \dfrac{2\pi}{S(1)+2\pi F_0}\right)\cdot e^{-2\beta}F_0, & F_0 < F_{01};\\ 2\pi \cdot S'(\xi)^{-\frac{1}{2}} \cdot \beta^{\frac{1}{2}} e^{-\beta(\xi+1-F_0g(\xi))}F_0^{\frac{1}{2}}, & F_{01} < F_0 < F_{02};\\ 2\pi \cdot (S'(\xi))^{-1}, & F_{02} \leq F_0, \end{cases} \tag{5.19}$$

where ξ is the unique solution of $S(\xi) = 2\pi F_0$. (For the last case, $F_{02} < F_0$, β appears in the higher-order asymptotics.)

The thresholds are given by

$$F_{01} = \frac{S(1)}{2\pi}\left(=\frac{4}{\pi}\right), \qquad F_{02} = \frac{S(\xi^*)}{2\pi} \approx 3.3576, \tag{5.20}$$

with ξ^* being the unique solution of $\frac{S(\xi^*)}{2\pi} = \frac{1+\xi^*}{g(\xi^*)}$. (Note that $S(1) = 8$.)

We will give some remarks about the above result. Formula (5.15) recovers Risken's work about the asymptotics of V_F in the under-damped limit [17, (11.129), (11.135)], whilst (5.19) has not been derived before. The value of F_{01} and F_{02} is the same as in [17, (11.190), (11.196)]. These formulas and the connection between F_0 and the function S is best explained using the concept of effective potential (see Section 5.2). Some property of the two thresholds is given here:

- the value of F_{01} in (5.20) coincides with that of the deterministic process (recall $F_{01} = \frac{4}{\pi}$ for deterministic process with $\Psi(q) = -\cos q$);
- for the first case of (5.19) ($F_0 < F_{01}$), the convergence rate as $\beta \to \infty$ is $e^{-2\beta}$. For the second case ($F_{01} < F_0 < F_{02}$), to be shown later, we have $0 < \xi + 1 - F_0 g(\xi) < 2$. Hence V_F converges to zero but with a slower rate.
- for $F_0 > F_{02}$, V_F converges to the positive constant $2\pi \cdot (S'(\xi))^{-1} = 2\pi T(\xi)^{-1}$ which is the long time average velocity of the corresponding deterministic equation of (1.4) (see [17, Sec.11.6.1]).

Thus, near F_{02} a steep jump of V_F is observed in its asymptotics in the iterated limits. This behavior is illustrated in Fig. 3(b). The proof of Theorem 5.1 will be presented in Section 5.3.

In order to better describe the phenomenon of the bi-stability and the behavior of the trajectories, we further consider the mean transition times between the pinning and running states. They provide another interpretation of the thresholds indicating the change of relative stability of two states. For the sake of presentation, we will first heuristically call T_f and T_b to be the "mean transition times" from pinning to running states and vice versa. Their precise definitions will be given later in Section 5.4. With that said, we present our second result.

Theorem 5.2. *In the iterated limits (first $\gamma \to 0$ and then $\beta \to \infty$), for $F_0 > F_{01}$, we have the following asymptotics:*

$$T_f \sim \frac{4S(1)}{2\pi F_0 - S(1)} \cdot \beta^{-1} e^{2\beta}, \tag{5.21}$$

$$T_b \sim \frac{S(\xi)T(\xi)^{\frac{1}{2}}}{(2\pi F_0 - S(1))F_0^{\frac{1}{2}}} \cdot \beta^{-\frac{3}{2}} e^{\beta(1-\xi+F_0 g(\xi))}. \tag{5.22}$$

Furthermore,

$$\frac{T_f}{T_b} \begin{cases} >1, & \text{for } F_{01} < F_0 < F_{02}; \\ <1, & \text{for } F_0 > F_{02}. \end{cases} \tag{5.23}$$

By the asymptotics of T_f and T_b obtained in the above theorem, it can be seen that in the iterated limits, $T_f > T_b$ if $F_0 < F_{02}$ whilst $T_f < F_b$ if $F_0 > F_{02}$. Thus F_{02} can also be considered as a threshold across which the relative ordering between T_f and T_b is switched. This is illustrated by Fig. 5.

Before presenting the proof, we will give some heuristic remarks about the above two Theorems. First, our asymptotic results are obtained by performing the following iterated limits: (i) vanishing dissipation ($\gamma \to 0$) and then (ii) vanishing noise ($\beta \to \infty$). The first procedure allows us to reduce the dynamics onto the lower dimensional Hamiltonian graph Γ while the second facilitates the use of Laplace method leading to precise analytical results. Reversing the order of the limits seems also reasonable, though somewhat unnatural. To illustrate this, consider (5.1) or (5.2). In the original time scale t, letting $\beta \to \infty$ will drive the stochastic dynamics toward its deterministic version which is quite well understood (see for example the work [13] which is also used extensively in the proof of Theorem 4.1). Hence to arrive at more interesting statements, we need to consider longer time scale(s). For this purpose, we explicitly write (5.2) in time scale $O(\beta)$ as

$$\dot{q} = \beta p, \qquad \dot{p} = \beta(-\sin q) + \beta\gamma(-p + F_0) + \sqrt{2\gamma}\dot{W}. \tag{5.24}$$

(Using the time scale $O\left(\frac{\beta}{\gamma}\right)$ will lead to similar consideration.) Letting $\beta \to \infty$ (with γ *fixed*) is to consider on a fast time scale the Hamiltonian dynamics *with dissipation and tilt which come from the term* $\gamma(-p + F_0)$. The long time behaviors can still be captured by the deterministic process perturbed by relatively *weak noise* (since $\gamma \ll \beta$). As a further rescaling, we can decrease the dissipation by letting $\gamma \to 0$ (*simultaneously* as $\beta \to \infty$) so that the conservative part will dominate. We believe that with proper time rescaling, this will lead to similar analysis and results, though the overall procedure might not be as transparent as the approach adopted in the current paper. Essentially, the consideration of long time scales leads to some kind of averaging or homogenization of the underlying inhomogeneous potential.

Second, our results are applicable in the exponentially long time scale $O\left(e^{C\beta}\right)$ for $\beta \gg 1$. This is revealed in the values of T_f and T_b as $\beta \to \infty$. Readers might wonder how informative the value of the long time average velocity V_F is. Complementing this information is the motivation for considering T_f and T_b. They provide more refined properties of the trajectories, illustrating the fluctuations around V_F. In fact, the process on Γ can in principle be recast as a Markov chain with two states: locked and running states. The values of T_f and T_b thus give indication about the transition rates between them. Further quantities related to fluctuations include the distance traveled during the running states, the variances of T_f and T_b and the diffusion coefficient. Central Limit Theorems for long time behaviors of Markov chains can also be considered for our process. In order to keep the current paper within reasonable scope, we do not pursue these ideas.

Third, we find it illustrative to interpret the above results using the concept of effective potential function. This is described in the next section.

5.2. Effective potential and mean transition and return times

The results of Theorem 5.2 enable us to study the *long jump* phenomenon in the under-damped limit [35,36]: in this regime, at low temperatures the surface diffusion of atoms or small clusters proceeds by uncorrelated thermally activated jumps over the barrier from one minimum of the external potential to another. It is related to *bi-stability* in the following way: the *pinning state* corresponds to the state of being trapped in a potential well; the *running state* corresponds to the state of jumping across the barriers between neighboring potential wells; the *return to the pinning or running state* corresponds to being captured again after a long jump or being activated again after being bounded in a potential well for a while. Fig. 4 illustrates the phenomenon of *bi-stability* and *long jumps*.

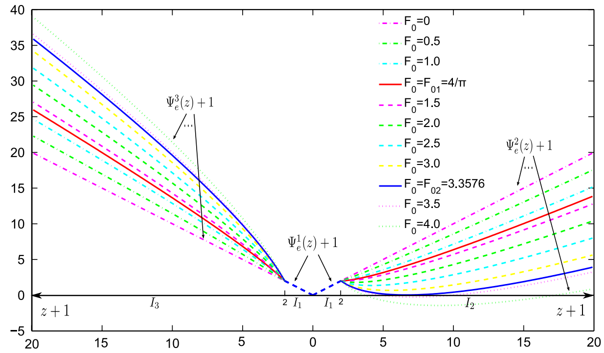


Fig. 9. The graphs of the effective potentials $\Psi_e^i(z)+1$, $i=1,2,3$. Note that the horizontal axis represents $(I_3 \cup I_1) \cup (I_1 \cup I_2)$. It is parametrized using $z+1$.

To best illustrate the diffusion on Γ, we make use of the concept of *effective potential* $\Psi_e^i(z)$ to characterize the energy level. This is introduced by Risken in [17, Ch.11] which is very much related to the invariant densities (5.11)–(5.13) for the diffusion on Γ:

$$\Psi_e^i(z) = \begin{cases} z, & i=1 \text{ and } z \in (-1,1); \\ z+(-1)^{i+1}F_0 g(z), & i=2,3 \text{ and } z \in (1,\infty), \end{cases}$$

where $z(q,p)=H(q,p)$ is the Hamiltonian at (q,p). It is depicted in Fig. 9. It will be shown later that the invariant measure of the limiting motion on the configuration space $\mathscr{C}$ is in the form of a *Gibbs measure*, that is, its density function has the following form:

$$\frac{1}{U_i(\beta)} e^{-\beta \Psi_e^i(z(q,p))}, \quad i=1,2,3$$

where the normalizing constant $U_i(\beta)$ is the *effective partition function*. The exact formulas for the density function will be given in (5.25)–(5.27).

The above concept of effective potential function can be used to give an interpretation of the thresholds F_{01} and F_{02}. Recall the meaning of the Hamiltonian graph Γ. Each point on I_1 represents a bounded (periodic) orbit while each point on I_2 and I_3 represents an unbounded running state. To better illustrate the idea, we plot in Fig. 9 the overall effective potential Ψ_e as a function over $(I_3 \cup I_1) \cup (I_1 \cup I_2)$ so that Ψ_e is the union of the graphs Ψ_e^3, Ψ_e^1, Ψ_e^1 and Ψ_e^2.

Now $O_1=(-1;1)$ is always a local minimum of $\Psi_e(z)$ representing the *stable pinning state* of (5.1). However, depending on the value of F_0, another local minimum will appear on Ψ_e^2. (Note that as F_0 is positive, there is no local minima on the graph of Ψ_e^3.) We can find the minimum on Ψ_e^2 by taking its derivative,

$$\Psi_e^{2'}(z) = 1 - F_0 g'(z) = 1 - \frac{2\pi F_0}{S(z)} = 0 \quad (\text{for } z>1)$$

leading to the solution ξ satisfying $S(\xi)=2\pi F_0$. The smallest value of the function S (for $z \geq 1$) is at $z=1$. This gives the first threshold,

$$F_{01} = \frac{S(1)}{2\pi} = \frac{4}{\pi}.$$

Hence for $F_0 \leq F_{01}$, $O_1=(-1;1)$ is the only minima of Ψ_e whilst for $F_{01}<F_0$, $(\xi;2)$ (on I_2) is another local minimum representing a *stable running state*. Comparing the energy value $\Psi_e^1(-1)=-1$ and $\Psi_e^2(\xi)=\xi-F_0 g(\xi)$, we can determine which is the global minimum. The cross over point is given by $F_0=F_{02}$ and $\xi=\xi^*$ which satisfy

$$\xi^* - F_{02} g(\xi^*) = -1, \quad \text{i.e. } \frac{S(\xi^*)}{2\pi} = F_{02} = \frac{\xi^*+1}{g(\xi^*)}.$$

Hence for $F_{01}<F_0<F_{02}$, $(\xi;2)$ is less stable than $(-1;1)$ while for $F_{02}<F_0$, $(\xi;2)$ becomes more stable. To conclude, F_{02} is exactly the (rescaled) tilt such that the relative stabilities of $(-1;1)$ and $(\xi;2)$ switch.

The switching of the stability property of the pinning and running states also leads to the reversal of the order between the mean transition times T_f and T_b. This is analogous to the Kramer's rate of escape from local minimum: the rate is essentially determined by the energy difference between the local minimum and the barrier to be overcome. In this case, the two barriers to compare are $\Psi_e^1(1)-\Psi_e^1(-1)$ and $\Psi_e^2(1)-\Psi_e^2(\xi)$. However, to go beyond the above heuristic and qualitative description, due to the inhomogeneous diffusion along the Hamiltonian graph Γ, the analytical verification will require some work as seen from the proof of Theorem 5.2.

Now we present the proofs of Theorems 5.1 and 5.2 in the rest of this section.

5.3. *Proof of Theorem 5.1*

First we derive the invariant density $\left\{\rho_i^{\mathscr{C}}(q,p)\right\}_{i=1,2,3}$ on the configuration space $\mathscr{C}=\mathbb{T}\times\mathbb{R}$ induced by the invariant density $\left\{\rho_i^{\Gamma}(z)\right\}_{i=1,2,3}$ on the Hamiltonian graph Γ. For the following computation, we make use of the crucial fact that the function $\rho_i^{\mathscr{C}}(q,p)$

is constant along the trajectories of the unperturbed Hamiltonian system (5.3), which is a consequence of the Liouville Theorem. This can also be seen alternatively by the fact that $\rho_i^\Gamma = Y_\# \rho_i^{\mathscr{C}}$ and Y maps orbits (with constant value of H) to a single point on Γ.

Note that the independent variables on $\mathscr{C}$ and Γ are (q,p) and z respectively. Recall $z(q,p) := -\cos q + \frac{1}{2}p^2$ and let $p(q,z) = \sqrt{2z+2\cos q}$. Then for all $(q,p) \in \mathbb{T}\times\mathbb{R}$,

$$\rho_i^{\mathscr{C}}(q,p)\,dq\,dp = \rho_i^{\mathscr{C}}(q,\operatorname{sgn}(p)\cdot p(q,z))\cdot\left|\frac{\partial(q,p)}{\partial(q,z)}\right| dq\,dz, \quad \text{where } \left|\frac{\partial(q,p)}{\partial(q,z)}\right| = \frac{1}{\sqrt{2z+2\cos q}}.$$

Here $\operatorname{sgn}(p)$ denotes the sign of p. On each Hamiltonian level set with level value z where $\rho_i^{\mathscr{C}}$ is constant, by the correspondence between $\rho_i^{\mathscr{C}}$ and ρ_i^Γ, we obtain the following identities (recall $q_* = \arccos(-z)$ if $z<1$ and $q_* = \pi$ if $z>1$):

$$2\int_{-q_*}^{q^*} \rho_i^{\mathscr{C}}(q,p(q,z))\cdot\left|\frac{\partial(q,p)}{\partial(q,z)}\right| dq = 2T(z(q,p))\rho_i^{\mathscr{C}}(q,p) = \rho_i^\Gamma(z(q,p)), \quad \text{for } i=1,$$

$$\int_{-q_*}^{q^*} \rho_i^{\mathscr{C}}(q,p(q,z))\cdot\left|\frac{\partial(q,p)}{\partial(q,z)}\right| dq = T(z(q,p))\rho_i^{\mathscr{C}}(q,p) = \rho_i^\Gamma(z(q,p)), \quad \text{for } i=2,$$

$$\int_{-q_*}^{q^*} \rho_i^{\mathscr{C}}(q,-p(q,z))\cdot\left|\frac{\partial(q,p)}{\partial(q,z)}\right| dq = T(z(q,p))\rho_i^{\mathscr{C}}(q,-p) = \rho_i^\Gamma(z(q,p)), \quad \text{for } i=3.$$

(The first equality in each identity above is obtained by $T(z) = \int_{-q^*}^{q^*}\frac{dq}{\sqrt{2z+2\cos q}} = \int_{-q^*}^{q^*}\left|\frac{\partial(q,p)}{\partial(q,z)}\right| dq$; the second equality holds since $\rho_i^\Gamma(z)$, the density function over the Hamiltonian level z, is the same as the marginal distribution of the density function with independent variable (q,z) on the configuration space $\mathscr{C}$, integrating out the variable q.) Note that $\rho_i^{\mathscr{C}}(q,p) = \rho_i^{\mathscr{C}}(q,-p)$ for $i=1$ by the symmetry of the trajectory of the unperturbed system (5.3) for $z\in(-1,1)$ (see Fig. 8). The above identities and (5.11)–(5.13) yields:

$$\rho_1^{\mathscr{C}}(q,p) = \frac{2C}{S(1)}e^{-\beta(z(q,p)-1)}, \tag{5.25}$$

$$\rho_2^{\mathscr{C}}(q,p) = \frac{2C}{S(1)}e^{-\beta((z(q,p)-1)-F_0 g(z(q,p)))}, \tag{5.26}$$

$$\rho_3^{\mathscr{C}}(q,p) = \frac{2C}{S(1)}e^{-\beta((z(q,p)-1)+F_0 g(z(q,p)))}, \tag{5.27}$$

for $q\in[-q^*,q^*]$, and $\rho_i^{\mathscr{C}}(q,p) = 0$ for $q\notin[-q^*,q^*]$. The constant C, for normalization purpose, is computed as follows:

$$\begin{aligned}1 &= \int_{-\infty}^{\infty}\int_{-q_*}^{q_*}\sum_{i=1}^{3}\rho_i^{\mathscr{C}}(q,p)\,dq\,dp \\ &= \frac{2Ce^\beta}{S(1)}\sqrt{\frac{2\pi}{\beta}}\int_{-\pi}^{\pi} e^{\beta\cos q}\,dq + \frac{4Ce^\beta}{S(1)}\int_1^\infty T(z)e^{-\beta z}\left(\cosh(\beta F_0 g(z))-1\right)\,dz,\end{aligned}$$

so that

$$C = \frac{e^{-\beta}S(1)}{2}\cdot\left(\sqrt{\frac{2\pi}{\beta}}\int_{-\pi}^{\pi} e^{\beta\cos q}\,dq + 2\int_1^\infty T(z)e^{-\beta z}\left(\cosh(\beta F_0 g(z))-1\right)\,dz\right)^{-1}.$$

By taking the expectation of the velocity variable p with respect to the invariant density $\rho_i^{\mathscr{C}}(q,p)$ on $\mathbb{T}\times\mathbb{R}$:

$$V_F = \int_1^\infty\int_{-q_*}^{q_*}(\rho_2^{\mathscr{C}} - \rho_3^{\mathscr{C}})p\left|\frac{\partial(q,p)}{\partial(q,z)}\right| dq\,dz = \frac{4Ce^\beta}{S(1)}\int_1^\infty\int_{-\pi}^{\pi} e^{-\beta z}\sinh(\beta F_0 g(z))\,dq\,dz.$$

In the above we make use of the fact that $\left|\frac{\partial(q,p)}{\partial(q,z)}\right| = \frac{1}{\sqrt{2z+2\cos q}} = \frac{1}{p}$.

We thus obtain using integration by parts that

$$\begin{aligned}V_F &= \frac{4Ce^\beta}{S(1)}\cdot 2\pi\int_1^\infty\left(-\frac{e^{-\beta z}}{\beta}\right)'\sinh(\beta F_0 g(z))\,dz = \frac{8\pi Ce^\beta}{S(1)}\cdot 2\pi\int_1^\infty\frac{e^{-\beta z}\cosh(\beta F_0 g(z))}{S(z)}\,dz\cdot F_0 \\ &= \frac{2\pi C_2}{C_0+C_1}F_0\end{aligned} \tag{5.28}$$

with C_0, C_1 and C_2 given by (5.16)–(5.18).

Theorem 5.1 is a consequence of the following asymptotics for C_0, C_1 and C_2.

Lemma 5.3. *As $\beta \to \infty$, we have the following asymptotics:*

$$C_0 \sim \pi \cdot \beta^{-1} e^{\beta}; \tag{5.29}$$

$$C_1(F_0) \sim \begin{cases} O\left(\beta^{-1}e^{-\beta}\right), & F_0 < \dfrac{S(1)}{2\pi}, \\ \pi\left(F_0 S'(\xi)\right)^{\frac{1}{2}} \cdot \beta^{-\frac{1}{2}} e^{\beta(F_0 g(\xi)-\xi)}, & F_0 > \dfrac{S(1)}{2\pi}; \end{cases} \tag{5.30}$$

$$C_2(F_0) \sim \begin{cases} \left(\dfrac{\pi}{S(1)-2\pi F_0} + \dfrac{\pi}{S(1)+2\pi F_0}\right) \cdot \beta^{-1} e^{-\beta}, & F_0 < \dfrac{S(1)}{2\pi}; \\ \pi\left(F_0 S'(\xi)\right)^{-\frac{1}{2}} \cdot \beta^{-\frac{1}{2}} e^{\beta(F_0 g(\xi)-\xi)}, & F_0 > \dfrac{S(1)}{2\pi}. \end{cases} \tag{5.31}$$

Proof of (5.29). Rewrite C_0 as

$$C_0 = \frac{\sqrt{2\pi}}{2\sqrt{\beta}} \int_{-\pi}^{\pi} e^{\beta h(q)}\, dq, \quad \text{where } h(q) = \cos q.$$

Note that $h(q)$ attains its maximum at 0. By Laplace's method, we get as $\beta \to \infty$

$$C_0 = \frac{\sqrt{2\pi}}{2\sqrt{\beta}} \left[\left(\frac{-2\pi}{\beta h''(0)}\right)^{\frac{1}{2}} e^{\beta h(0)} + e^{\beta h(0)}\, O\left(\beta^{-\frac{3}{2}}\right)\right] = \frac{\pi}{\beta} \cdot e^{\beta} + \frac{\sqrt{2\pi}}{2\sqrt{\beta}} e^{\beta}\, O\left(\beta^{-\frac{3}{2}}\right)$$
$$\sim \pi \cdot \beta^{-1} e^{\beta}. \quad \square$$

Proof of (5.30). Let

$$C_1(F_0) = \int_1^{\infty} T(z) e^{-\beta z} \left(\cosh\left(\beta F_0 g(E)\right) - 1\right)\, dz = C_{1,0} + C_{1,1} + C_{1,2},$$

where

$$C_{1,0} = \frac{1}{2}\int_1^{\infty} T(z) e^{-\beta z} e^{\beta F_0 g(z)}\, dz, \qquad C_{1,1} = \frac{1}{2}\int_1^{\infty} T(z) e^{-\beta z} e^{-\beta F_0 g(z)}\, dz, \qquad C_{1,2} = -\int_1^{\infty} T(z) e^{-\beta z}\, dz.$$

By integration by parts and Laplace's method (see [31, formula (2.37)]), we obtain as $\beta \to \infty$,

$$C_{1,2} = S(1)e^{-\beta} - \beta \int_1^{\infty} e^{-\beta z} S(z)\, dz$$
$$\sim S(1)e^{-\beta} - \beta\left(\left(-\frac{S(1)}{\beta(-1)}\right) e^{-\beta} + e^{-\beta}\, O\left(\beta^{-2}\right)\right) = O\left(\beta^{-1}e^{-\beta}\right).$$

(Note that we have used the fact that $S'(z) = T(z)$.)

Similarly we get as $\beta \to \infty$,

$$C_{1,1} = -\frac{S(1)}{2}e^{-\beta} + \frac{\beta}{2}\int_1^{\infty} \left(S(z) + 2\pi F_0\right) e^{-\beta(z - F_0 g(z))}\, dz$$
$$\sim -\frac{S(1)}{2}e^{-\beta} + \frac{\beta}{2}\left(\left(-\frac{S(1)+2\pi F_0}{\beta\left(-1-\frac{2\pi F_0}{S(1)}\right)}\right) e^{-\beta} + e^{-\beta}\, O\left(\beta^{-2}\right)\right) = O\left(\beta^{-1}e^{-\beta}\right).$$

Observe that $C_{1,0}$ is of the form $\frac{1}{2}\int_1^{\infty} G(z) e^{\beta H(z)}\, dz$, where $G(z) = T(z)$ and $H(z) = -z + F_0 g(z)$. Note also that $H'(z) = \frac{2\pi F_0}{S(z)} - 1$ and $H'(z)$ is monotonically decreasing as $S(z)$ is monotonically increasing. There are the following two cases:

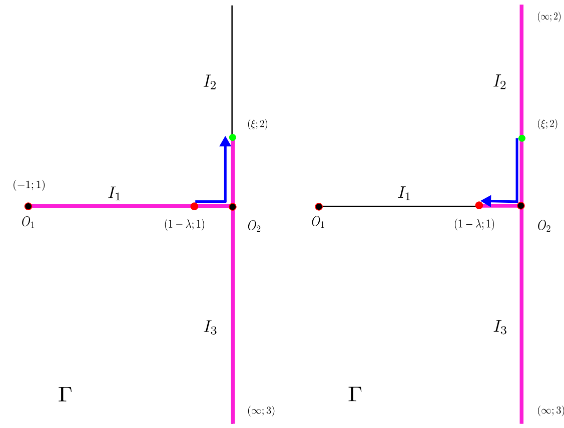
1. If $F_0 < \frac{S(1)}{2\pi}$, then $H'(1) < 0$ and $H'(z) < H'(1) < 0$ for $z > 1$. Thus $H(z)$ reaches its maximum at 1. Then by integration by parts and Laplace's method again, we get as $\beta \to \infty$,

$$C_{1,0} \sim O\left(\beta^{-1}e^{-\beta}\right).$$

2. If $F_0 > \frac{S(1)}{2\pi}$, then $H'(1) > 0$ and $H''(z) < 0$ as $H'(z)$ is monotonically decreasing. Thus $H(z)$ achieves its maximum at ξ with $1 < \xi < \infty$ and $H'(\xi) = 0$. Hence ξ is the solution of $S(\xi) = 2\pi F_0$. Then (see [31, formula 2.34]) we obtain as $\beta \to \infty$,

$$C_{1,0}(F_0) \sim \frac{S'(\xi)}{2} \cdot \left(\frac{-2\pi}{\beta\left(-\frac{2\pi F_0 S'(\xi)}{S^2(\xi)}\right)}\right)^{\frac{1}{2}} e^{\beta(F_0 g(\xi)-\xi)} + \frac{1}{2} e^{\beta(F_0 g(\xi)-\xi)}\, O\left(\beta^{-\frac{3}{2}}\right)$$
$$\sim \pi\left(F_0 S'(\xi)\right)^{\frac{1}{2}} \cdot \beta^{-\frac{1}{2}} e^{\beta(F_0 g(\xi)-\xi)}.$$

The assertion then follows. $\square$



(a) G_1 for the computation of T_f. (b) G_2 for the computation of T_b.

Fig. 10. The domains G_1 and G_2 used in the definition of T_f and T_b.

Proof of (5.31). The computation is analogous to that of (5.30). □

Proof of Theorem 5.1. Formula (5.19) and (5.20) are simple consequences of the above asymptotics. We will only concentrate on the latter. By (5.30) and (5.31), across $F_{01} = \frac{S(1)}{2\pi}$, C_1 and C_2 have different asymptotic expansions. To determine F_{02}, we assume $F_0 > F_{01}$. Note that for $F_0 < \frac{S(\xi^*)}{2\pi}$, $F_0 g(\xi) - \xi < 1$ whilst for $F_0 > \frac{S(\xi^*)}{2\pi}$, $F_0 g(\xi) - \xi > 1$. Then in the former case the denominator in (5.28) is dominated by $C_0 = O\left(\beta^{-1}e^{\beta}\right)$, while in the latter case the dominant term is $C_1 = O\left(\beta^{-\frac{1}{2}}e^{\beta(F_0 g(\xi)-\xi)}\right)$. This shows $F_{02} = \frac{S(\xi^*)}{2\pi}$. □

5.4. *Proof of Theorem 5.2*

We first give a precise definition of T_f and T_b. They are defined as the *mean first exit time* from appropriate domains of the Hamiltonian graph Γ. Given $G \subset \Gamma$, we define:

$$\tau^G := \inf\{t : z(t) \notin G\} \quad \text{and} \quad v_i^G(z_0) := E_{(z_0;i)}\tau^G$$

where $z(\cdot)$ is the diffusion process on Γ corresponding to the infinitesimal generator $\mathcal{L}_i$'s (5.8) with initial data $(z_0; i)$. In principle, we like to define T_f as the time it takes the particle to diffuse from $O_1 = (-1; 1)$ (the most stable pinning state) to $(\xi; 2)$ (the most stable running state) and vice versa for T_b. However, O_1 is inaccessible with probability 1 [15]. Hence we will choose the point $(1-\lambda; 1)$ instead of O_1 and then let $\lambda \to 0^+$.

Precisely, let G_1 be the subset of Γ bounded by $z_1^{G_1} \to -1$, $z_2^{G_1} = \xi$, and $z_3^{G_1} \to \infty$ on I_1, I_2 and I_3 respectively (see Fig. 10(a)). Note that both $(-1; 1)$ and $(\infty; 3)$ are inaccessible with probability 1. Then we define T_f as:

$$T_f := \lim_{\lambda\to 0} \lim_{z_1^{G_1}\to -1, z_3^{G_1}\to\infty} v_1^{G_1}(1-\lambda). \tag{5.32}$$

Similarly, let G_2 be the subset of Γ bounded by $z_1^{G_2} = 1-\lambda$, $z_2^{G_2} \to \infty$ and $z_3^{G_2} \to \infty$ on I_1, I_2 and I_3 respectively (see Fig. 10(b)). In this case both $(\infty; 2)$ and $(\infty; 3)$ are inaccessible with probability 1. We define T_b as follows:

$$T_b := \lim_{\lambda\to 0} \lim_{z_2^{G_2}, z_3^{G_2}\to\infty} v_2^{G_2}(\xi). \tag{5.33}$$

See Fig. 10 for an illustration of G_1 and G_2. Even though the above definitions of T_f and T_b seems artificial, they are quite intuitive. The fact that our result demonstrate the reversal of their order at exactly F_{02} demonstrates that they indeed capture the essential behavior.

With the above definition of G_1 and G_2 (below we let $G = G_1$ or $G = G_2$), the function $v_i^G(z) := E_{(z;i)}\tau^G$ is the solution of the boundary value problem

$$\mathcal{L}_i v_i^G(z) = -1, \quad (z; i) \in G \setminus \{z_1^G, z_2^G, z_3^G\}, \tag{5.34}$$

$$v_i^G(z_i^G) = 0, \quad i = 1, 2, 3, \tag{5.35}$$

$$v_1^G(1) = v_2^G(1) = v_3^G(1), \tag{5.36}$$

$$2(v_1^G)'(1) = (v_2^G)'(1) + (v_3^G)'(1), \quad \text{(gluing condition)}. \tag{5.37}$$

This linear system can be solved as follows [15, pp. 625]:

$$v_i^G(z) = c_i^G \Delta s_i^G(z) + \tilde{v}_i^G(z), \quad i = 1, 2, 3, \quad (z; i) \in G \setminus \{O_2\}, \tag{5.38}$$

where

$$c_i^G = \frac{1}{\Delta s_i^G(1)} \left(\sum_{j=1}^{3} \frac{\alpha_j}{\Delta s_j^G(1)}\right)^{-1} \left(\sum_{j=1}^{3} \frac{\alpha_j \tilde{v}_j^G(1)}{\Delta s_j^G(1)} + \rho(G)\right) - \frac{\tilde{v}_i^G(1)}{\Delta s_i^G(1)}, \tag{5.39}$$

and

$$\rho(G)=\sum_{i=1}^{3}\alpha_i\left|\int_1^{z_i^G} dm_i\right|, \qquad \text{(the invariant measure on } G\text{)}$$

$$\Delta s_i^G(z)=8\left|\int_z^{z_i^G} S(y)^{-1}e^{-K_i(y)}\,dy\right|, \qquad \text{(difference of the scale function at } z \text{ and } z_i^G\text{)}$$

$$\tilde{v}_i^G(z)=8\cdot\frac{2}{S(1)}\int_z^{z_i^G}\int_y^z S(x)^{-1}e^{-K_i(x)}T(y)e^{K_i(y)}\,dx\,dy,\quad i=1,2,3, \qquad \text{(a solution of (5.34))}$$

$$K_1(z)=-\beta(z-1),\quad -1<z<1, \qquad \text{(exponent of the invariant density on } I_1\text{)}$$

$$K_i(z)=-\beta\left((z-1)-(-1)^iF_0g(z)\right),\quad i=2,3,\quad z>1, \qquad \text{(exponent of the invariant density on } I_2 \text{ and } I_3\text{)}$$

$$\alpha_1=2,\quad \alpha_2=\alpha_3=1, \qquad \text{(gluing condition constants)}$$

with $m_i(\,dz)$ being the *speed measure* of the limiting diffusion $z(t)$ on I_i with densities:

$$m_1'(z)=\frac{2T(z)}{S(1)}e^{-\beta(z-1)}, \tag{5.40}$$

$$m_2'(z)=\frac{2T(z)}{S(1)}e^{-\beta((z-1)-F_0g(z))}, \tag{5.41}$$

$$m_3'(z)=\frac{2T(z)}{S(1)}e^{-\beta((z-1)+F_0g(z))}. \tag{5.42}$$

Next we give the proofs of (5.21)–(5.23). In each proof, we suppress the notational dependence on the domain G_1 or G_2. Furthermore, we let $G=G_1$ in the proof of (5.21) and $G=G_2$ in the proof of (5.22).

Proof of (5.21) — asymptotics of T_f. By definition we have

$$\begin{aligned}
T_f &= \lim_{\lambda\to 0}\lim_{z_1\to -1, z_3\to\infty} v_1(1-\lambda)\\
&= \lim_{\lambda\to 0}\lim_{z_1\to -1, z_3\to\infty} (c_1\Delta s_1(1-\lambda)+\tilde{v}_1(1-\lambda))\\
&= \lim_{\lambda\to 0}\lim_{z_1\to -1, z_3\to\infty}\left(\left(\frac{2}{\Delta s_1(1)}+\frac{1}{\Delta s_2(1)}+\frac{1}{\Delta s_3(1)}\right)^{-1}\cdot\left(\frac{2\tilde{v}_1(1)}{\Delta s_1(1)}+\frac{\tilde{v}_2(1)}{\Delta s_2(1)}+\frac{\tilde{v}_3(1)}{\Delta s_3(1)}+\rho(G)\right)\right.\\
&\qquad\left.\times\frac{\Delta s_1(1-\lambda)}{\Delta s_1(1)}-\tilde{v}_1(1)\cdot\frac{\Delta s_1(1-\lambda)}{\Delta s_1(1)}+\tilde{v}_1(1-\lambda)\right)\\
&= \lim_{\lambda\to 0}\lim_{z_1\to -1, z_3\to\infty} (\tilde{v}_2(1)+\rho(G)\Delta s_2(1)-\tilde{v}_1(1)+\tilde{v}_1(1-\lambda))\\
&= \tilde{v}_2(1)+\Delta s_2(1)\lim_{z_1\to -1, z_3\to\infty}\rho(G).
\end{aligned} \tag{5.43}$$

In the above computation we have used the following limits

$$\lim_{z_1\to -1}\Delta s_1(1)=\infty,\qquad \lim_{z_3\to\infty}\Delta s_3(1)=\infty,\qquad \lim_{z_1\to -1}\frac{\Delta s_1(1-\lambda)}{\Delta s_1(1)}=1,\qquad \lim_{\lambda\to 0}\tilde{v}_1(1-\lambda)=\tilde{v}_1(1).$$

We then obtain the following expression for T_f:

$$\begin{aligned}
T_f &= 8\cdot\frac{2}{S(1)}\int_1^{\xi}\int_y^1 S(x)^{-1}e^{-K_2(x)}T(y)e^{K_2(y)}\,dx\,dy\\
&\quad +8\cdot\left|\int_1^{\xi}S(y)^{-1}e^{-K_2(y)}\,dy\right|\cdot\left(2\left|\int_1^{-1}m_1'(z)\,dz\right|+\left|\int_1^{\xi}m_2'(z)\,dz\right|+\left|\int_1^{\infty}m_3'(z)\,dz\right|\right).
\end{aligned} \tag{5.44}$$

Let

$$I_1(\beta)=\int_1^{\xi}\int_1^{y}\frac{T(y)}{S(x)}e^{\beta((x-y)+F_0g(y)-F_0g(x))}\,dx\,dy,$$

$$I_2(\beta)=\int_1^{\xi}\frac{e^{\beta((y-1)-F_0g(y))}}{S(y)}\,dy,$$

$$C_3(\beta)=\frac{2}{S(1)}\int_{-1}^{1}T(z)e^{-\beta(z-1)}\,dz,$$

$$C_4(\beta)=\frac{2}{S(1)}\int_1^{\xi}T(z)e^{-\beta((z-1)-F_0g(z))}\,dz,$$

$$C_5(\beta)=\frac{2}{S(1)}\int_1^{\infty}T(z)e^{-\beta((z-1)+F_0g(z))}\,dz.$$

Then T_f is given by (the fact that $S(1) = 8$ is used to simply the expression)

$$T_f = -2I_1 + 8I_2(2C_3 + C_4 + C_5).$$

The asymptotics of T_f is a consequence of the following result.

Lemma 5.4. *As $\beta \to \infty$, we have*

$$I_1(\beta) \sim \frac{1}{2} \cdot \frac{S(\xi)T(\xi)^{\frac{1}{2}}}{(2\pi F_0 - S(1))F_0^{\frac{1}{2}}} \cdot \beta^{-\frac{3}{2}} e^{\beta(1-\xi+F_0 g(\xi))}, \tag{5.45}$$

$$I_2(\beta) \sim \frac{1}{2\pi F_0 - S(1)} \cdot \beta^{-1} + O\left(\beta^{-2}\right), \tag{5.46}$$

$$C_3(\beta) \sim 2e^{2\beta} + e^{2\beta}\, O\left(\beta^{-1}\right), \tag{5.47}$$

$$C_4(\beta) \sim \frac{1}{S(1)} S(\xi)T(\xi)^{\frac{1}{2}} \beta^{-\frac{1}{2}} F_0^{-\frac{1}{2}} e^{\beta(1-\xi+F_0 g(\xi))} + \frac{2}{S(1)} e^{\beta(1-\xi+F_0 g(\xi))}\, O\left(\beta^{-1}\right), \tag{5.48}$$

$$C_5(\beta) \sim O\left(\beta^{-1}\right). \tag{5.49}$$

Proof of (5.45). Rewrite I_1 as

$$I_1(\beta) = \iint_D k(x,y) e^{-\beta f(x,y)}\, dx\, dy,$$

where

$$f(x,y) = y - x + F_0 g(x) - F_0 g(y), \qquad k(x,y) = \frac{T(y)}{S(x)}, \qquad D = \{(x,y) | y \in [1,\xi], x \in [1,y]\}.$$

Direct computation gives

$$f_x(x,y) = -1 + F_0 \cdot \frac{2\pi}{S(x)}, \qquad f_y(x,y) = 1 - F_0 \cdot \frac{2\pi}{S(y)},$$

$$f_{xx}(x,y) = -\frac{2\pi F_0 T(x)}{(S(x))^2}, \qquad f_{xy}(x,y) = f_{yx}(x,y) = 0, \qquad f_{yy}(x,y) = \frac{2\pi F_0 T(y)}{(S(y))^2},$$

$$H(f)(x,y) := \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{vmatrix} = -\frac{4\pi^2 F_0^2 T(x)T(y)}{(S(x)S(y))^2}.$$

Then (ξ, ξ) is the unique critical point of $f(x,y)$, which is a saddle point since

$$H(f)(x,y)\Big|_{(\xi,\xi)} = -\frac{4\pi^2 F_0^2 (T(\xi))^2}{(S(\xi))^4} < 0.$$

We next check all corner points of D. Observe that $f(1,1) = f(\xi,\xi) = 0$. And for $1 \le z \le \xi$, $S(z) \le 2\pi F_0$ by monotonicity of $S(\cdot)$ and $S(\xi) = 2\pi F_0$. Thus

$$f(1,\xi) = \xi - 1 - F_0(g(\xi) - g(1)) = \xi - 1 - F_0 \int_1^\xi \frac{2\pi}{S(z)}\, dz < \xi - 1 - \int_1^\xi dz = 0.$$

So $f(x,y)$ attains its minimum at $(1,\xi)$ on D. Moreover, by

$$f_x(1,\xi) = -1 + \frac{2\pi F_0}{S(1)} > 0, \qquad f_y(1,\xi) = 1 - \frac{2\pi F_0}{S(\xi)} = 0, \qquad f_{yy}(1,\xi) = \frac{2\pi F_0 T(\xi)}{(S(\xi))^2} > 0,$$

the level curve $f(x,y) = C$ is tangent to the boundary of D at $(1,\xi)$. We next follow the method presented in [32, Sec. VIII.7] to derive the asymptotics of $I_1(\beta)$.

We write

$$f(x,y) - f(1,\xi) = f_x(1,\xi)(x-1)[1 + P(x-1, y-\xi)] + \frac{f_{yy}(1,\xi)}{2}(y-\xi)^2[1 + Q(x-1, y-\xi)],$$

where $P(x,y)$ and $Q(x,y)$ are power series in x and y satisfying $P(0,0) = Q(0,0) = 0$. Let

$$u = (x-1)[1 + P(x-1, y-\xi)], \qquad v = (y-\xi)[1 + Q(x-1, y-\xi)]^{\frac{1}{2}},$$

then

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|_{(1,\xi)} = 1, \qquad f(x,y) - f(1,\xi) = f_x(1,\xi)u + \frac{f_{yy}(1,\xi)}{2} v^2 = F(u,v).$$

Let $K(u,v)=k(x(u,v),y(u,v))|\partial(x,y)/\partial(u,v)|$ and D' be the image of D under this transformation. We consider the Taylor expansion of K at $(0,0)$: $K(u,v)=\sum_{i,j=0}K_{ij}u^iv^j$, where K_{ij} denote the Taylor series coefficients. We have the coefficient of the leading term $K_{00}=k(1,\xi)=\frac{T(\xi)}{S(1)}$. Then

$$I_1(\beta)=e^{-\beta f(1,\xi)}\iint_{D'}K(u,v)e^{-\beta F(u,v)}\,du\,dv.$$

By the method of resolution of multiple integrals, the double integral $I_1(\beta)$ can be written as

$$I_1(\beta)=e^{-\beta f(1,\xi)}\int_0^M h(t)e^{-\beta t}\,dt, \tag{5.50}$$

where M denotes the maximum of F in D' (the minimum of F is zero clearly), and

$$h(t)=\int_{F(u,v)=t}\frac{K(u,v)}{\sqrt{F_u^2+F_v^2}}\,d\sigma,$$

with σ being the arc length of the curve $F(u,v)=t$.

In order to compute $h(t)$, we then transform (x,y) to (ζ,η) by

$$u=\frac{\zeta}{f_x(1,\xi)}\cos^2\eta,\qquad v=\left(\frac{2\zeta}{f_{yy}(1,\xi)}\right)^{\frac12}\sin\eta.$$

Thus $F(u,v)=\zeta$. By direct computation,

$$\left|\frac{\partial(u,v)}{\partial(\zeta,\eta)}\right|=\begin{vmatrix}\frac{\partial u}{\partial\zeta} & \frac{\partial u}{\partial\eta}\\ \frac{\partial v}{\partial\zeta} & \frac{\partial v}{\partial\eta}\end{vmatrix}=\frac{1}{f_x(1,\xi)}\left(\frac{2\zeta}{f_{yy}(1,\xi)}\right)^{\frac12}\cos\eta.$$

Let $\Phi(\zeta,\eta)=K(u,v)\left|\frac{\partial(u,v)}{\partial(\zeta,\eta)}\right|$. Then

$$\Phi(\zeta,\eta)=\sum_{i,j}K_{ij}u^iv^j\frac{1}{f_x(1,\xi)}\left(\frac{2\zeta}{f_{yy}(1,\xi)}\right)^{\frac12}\cos\eta=\sum_{i,j}\Phi_{ij}\zeta^{i+\frac{(j+1)}{2}}(\cos\eta)^{2i+1}(\sin\eta)^j,$$

with $\Phi_{ij}=\frac{K_{ij}}{f_x(1,\xi)^{i+1}\left(\frac{f_{yy}(1,\xi)}{2}\right)^{\frac{j+1}{2}}}$. And $h(t)=\int_{-\frac{\pi}{2}}^0\Phi(t,\eta)\,d\eta$ (note that $v\le 0$ in D', thus $\eta\in[-\pi/2,0]$). Hence

$$\begin{aligned}h(t)&=\sum_{i,j}\Phi_{ij}t^{i+\frac{j+1}{2}}\int_{-\frac{\pi}{2}}^0(\cos\eta)^{2i+1}(\sin\eta)^j\,d\eta=\frac12\sum_{i,j}\Phi_{ij}t^{i+\frac{j+1}{2}}(-1)^jB\left(i+1,\frac{j+1}{2}\right)\\&=\frac12\sum_{i,j}(-1)^j\Phi_{ij}t^{i+\frac{j+1}{2}}\cdot\frac{\Gamma(i+1)\Gamma(\frac{j+1}{2})}{\Gamma(i+\frac{j+3}{2})}=\frac12\sum_{i,j}(-1)^j\Phi_{ij}t^{i+\frac{j+1}{2}}\cdot\frac{i!\,\Gamma(\frac{j+1}{2})}{\Gamma(i+\frac{j+3}{2})}.\end{aligned}\tag{5.51}$$

Plugging (5.51) into (5.50), we get as $\beta\to\infty$

$$\begin{aligned}I_1(\beta)&\sim e^{-\beta f(1,\xi)}\frac12\Phi_{00}\frac{\Gamma\left(\frac12\right)}{\Gamma\left(\frac32\right)}\int_0^M t^{\frac12}e^{-\beta t}\,dt\sim\frac12\sqrt{\pi}\Phi_{00}e^{-\beta f(1,\xi)}\beta^{-\frac32}\\&\sim\frac12\sqrt{\pi}\frac{T(\xi)}{S(1)f_x(1,\xi)\left(\frac{f_{yy}(1,\xi)}{2}\right)^{\frac12}}e^{-\beta f(1,\xi)}\beta^{-\frac32}\sim\frac12\cdot\frac{S(\xi)T(\xi)^{\frac12}}{(2\pi F_0-S(1))F_0^{\frac12}}\cdot e^{-\beta f(1,\xi)}\beta^{-\frac32},\end{aligned}$$

by Watson's Lemma [31, Sec. 2.1,pp. 24]. □

Proof of (5.46). Write

$$I_2(\beta)=\int_1^\xi\frac{1}{S(y)}e^{\beta h(y)}\,dy,$$

where $h(y)=y-1-F_0g(y)$. Note that $h'(y)=1-\frac{2\pi F_0}{S(y)}\le 0$ for $y\in[1,\xi]$. This is a direct result of the monotonicity property of $h'(y)$: $h''(y)=\frac{2\pi F_0T(y)}{(S(y))^2}>0$, and the fact that $h'(\xi)=1-\frac{2\pi F_0}{S(\xi)}=0$ by definition of ξ. So $h(y)$ attains its maximum at 1. By [31, pp. 35, (2.38)], as $\beta\to\infty$

$$I_2(\beta)\sim\left(\frac{-\frac{1}{S(1)}}{\beta h'(1)}\right)e^{\beta h(1)}+e^{\beta h(1)}O\left(\beta^{-2}\right)\sim\frac{1}{2\pi F_0-S(1)}\cdot\beta^{-1}+O\left(\beta^{-2}\right).\quad\square$$

Proof of (5.47). By integration by parts and Laplace's method, we get as $\beta \to \infty$

$$\begin{aligned} C_3(\beta) &= \frac{2}{S(1)}\left(S(z)e^{-\beta(z-1)}\Big|_{-1}^{1} - \int_{-1}^{1} S(z)(-\beta)e^{\beta(1-z)}\,dz \right) \\ &= \frac{2}{S(1)}\left(S(1) + \beta \int_{-1}^{1} S(z)e^{\beta(1-z)}\,dz \right) \\ &\sim \frac{2}{S(1)}\left(S(1) + \beta\left(\frac{-S(1)}{\beta(-1)}e^{2\beta} + e^{2\beta}\,O\left(\beta^{-2}\right)\right)\right) \\ &\sim 2e^{2\beta} + e^{2\beta}\,O\left(\beta^{-1}\right). \quad \square \end{aligned}$$

Proof of (5.48). Write

$$C_4(\beta) = \frac{2}{S(1)}e^{\beta}\int_1^{\xi} T(z)e^{\beta h(z)}\,dz,$$

where $h(z) = -z + F_0 g(z)$. Observe that $h'(z) = \frac{2\pi F_0}{S(z)} - 1$, $h'(1) > 0$ when $F_0 > \frac{S(1)}{2\pi}$, as well as $h'(\xi) = 0$ and $h''(z) = -\frac{2\pi F_0 T(z)}{(S(z))^2} < 0$. Hence $h(z)$ attains its maximum at ξ. By [31, pp. 33, (2.31)], we obtain as $\beta \to \infty$

$$\begin{aligned} \int_1^{\xi} T(z)e^{\beta(-z+F_0 g(z))}\,dz &\sim T(\xi)\left(\frac{-\pi}{2\beta\cdot\frac{-2\pi F_0 T(\xi)}{(S(\xi))^2}} \right)^{\frac{1}{2}} e^{\beta(-\xi+F_0 g(\xi))} + e^{\beta(-\xi+F_0 g(\xi))}\,O\left(\beta^{-1}\right) \\ &\sim \frac{1}{2}S(\xi)T(\xi)^{\frac{1}{2}}F_0^{-\frac{1}{2}}\beta^{-\frac{1}{2}}e^{\beta(-\xi+F_0 g(\xi))} + e^{\beta(-\xi+F_0 g(\xi))}\,O\left(\beta^{-1}\right). \end{aligned}$$

The conclusion follows then. $\square$

Proof of (5.49). Let $h(z) = -z - F_0 g(z)$. Write

$$\begin{aligned} C_5(\beta) &= \frac{2}{S(1)}e^{\beta}\int_1^{\infty} T(z)e^{\beta h(z)}\,dz = \frac{2}{S(1)}e^{\beta}\left(S(z)e^{\beta h(z)}\Big|_1^{\infty} - \int_1^{\infty} S(z)\beta h'(z)e^{\beta h(z)}\,dz \right) \\ &= \frac{2}{S(1)}e^{\beta}\left(-S(1)e^{-\beta} - \int_1^{\infty} S(z)\beta h'(z)e^{\beta h(z)}\,dz \right). \end{aligned}$$

By $h'(z) = -1 - \frac{2\pi F_0}{S(z)} < 0$, $h(z)$ attains its maximum at 1. By [31, pp. 38, (2.38)], we get as $\beta \to \infty$

$$\begin{aligned} C_5(\beta) &\sim \frac{2}{S(1)}e^{\beta}\left(-S(1)e^{-\beta} - \beta\left(\frac{-S(1)h'(1)}{\beta h'(1)} \right) e^{-\beta} + \beta e^{-\beta}\,O\left(\beta^{-2}\right) \right) \\ &\sim \frac{2}{S(1)}e^{\beta}\,O\left(e^{-\beta}\beta^{-1}\right) \sim O\left(\beta^{-1}\right). \quad \square \end{aligned}$$

The proof of (5.21) is now complete.

Proof of (5.22) — asymptotics of T_b. By definition we have

$$\begin{aligned} T_b &= \lim_{\lambda\to 0}\lim_{z_2,z_3\to\infty} v_2(\xi) \\ &= \lim_{\lambda\to 0}\lim_{z_2,z_3\to\infty} \left(c_2\Delta s_2(\xi) + \tilde{v}_2(\xi)\right) \\ &= \lim_{\lambda\to 0}\lim_{z_2,z_3\to\infty}\left(\left(\frac{2}{\Delta s_1(1)} + \frac{1}{\Delta s_2(1)} + \frac{1}{\Delta s_3(1)} \right)^{-1} \cdot \left(\frac{2\tilde{v}_1(1)}{\Delta s_1(1)} + \frac{\tilde{v}_2(1)}{\Delta s_2(1)} + \frac{\tilde{v}_3(1)}{\Delta s_3(1)} + \rho(G) \right) \cdot \frac{\Delta s_2(\xi)}{\Delta s_2(1)} \right. \\ &\quad \left. + \tilde{v}_2(\xi) - \tilde{v}_2(1)\cdot\frac{\Delta s_2(\xi)}{\Delta s_2(1)} \right) \\ &= \lim_{\lambda\to 0}\lim_{z_2,z_3\to\infty}\left(\tilde{v}_1(1) + \frac{\rho(G)\Delta s_1(1)}{2} + \tilde{v}_2(\xi) - \tilde{v}_2(1) \right). \end{aligned}$$

We have used the following limits

$$\lim_{z_2\to\infty}\Delta s_2(1) = \infty, \qquad \lim_{z_3\to\infty}\Delta s_3(1) = \infty, \qquad \lim_{z_2\to\infty}\frac{\Delta s_2(\xi)}{\Delta s_2(1)} = 1.$$

Then by $\lim_{\lambda\to 0}\tilde{v}_1(1) = 0$, $\lim_{z_2\to\infty}|\tilde{v}_2(\xi) - \tilde{v}_2(1)| < \infty$, and $\lim_{\lambda\to 0}\Delta s_1(1) = 0$, we get

$$T_b = \lim_{z_2\to\infty}\left(\tilde{v}_2(\xi) - \tilde{v}_2(1)\right).$$

By definition of $\tilde{v}_2(z)$ the formula of T_b is given by:

$$T_b = \int_1^{\xi}\int_x^{\infty} S(x)^{-1}e^{-K_2(x)}T(y)e^{K_2(y)}\,dy\,dx. \tag{5.52}$$

Note that T_b is in the form of $\iint_{D''} k(x,y)e^{-\beta f(x,y)}\,dx\,dy$, where $D'' = \{(x,y)|x \in [1,\xi], y \in [x,\infty)\}$ and $f(x,y)$, $k(x,y)$ are the same as defined in I_1 in the proof of (5.45). Direct computation shows that the unique critical point (ξ,ξ) of $f(x,y)$ is a saddle point. Then we check the value of $f(x,y)$ along the boundary of D''. On the bottom boundary, we find that $f(1,1) = f(\xi,\xi) = 0$. On the left and right boundary, i.e. half lines $x = 1$ and $x = \xi$ above $y = 1$, $f_y(x,y)$ hits zero at $y = \xi$. Note also that $f_{yy}(x,y) = \frac{2\pi F_0 T(y)}{(S(y))^2} > 0$. All above lead to that on $x = 1$ and $x = \xi$, $f(x,y)$ attains its local minimum at $y = \xi$. It is easy to verify that $f(1,\xi) < f(\xi,\xi) = 0$. Hence the asymptotics of T_b is determined by the value of $f(x,y)$ near $(1,\xi)$, where the level curve $f(x,y) = C$ is tangent to $x = 1$. The remaining computation is analogous to that of (5.45). Note that the integration domain for η in the first equality of (5.51) in this case is $[-\frac{\pi}{2},\frac{\pi}{2}]$ (y can be greater than ξ). It follows that

$$T_b \sim 2I_1(\beta), \quad \text{as } \beta \to \infty.$$

The asymptotics for T_b follows then.

Proof of (5.23). By (5.21) and (5.22) the leading term of the asymptotics of T_b is $O\left(\beta^{-\frac{3}{2}}e^{\beta(1-\xi+F_0 g(\xi))}\right)$, whilst the leading term of the asymptotics of T_f is $O\left(\beta^{-1}e^{2\beta}\right)$. There are two cases: (i) for $F_{01} < F_0 < F_{02}$, $1-\xi+F_0 g(\xi) < 2$; (ii) for $F_0 > F_{02}$, $1-\xi+F_0 g(\xi) > 2$. Consequently, the ratio $\frac{T_f}{T_b} > 1$ for $F_{01} < F_0 < F_{02}$ as $O\left(\beta^{-1}e^{2\beta}\right) \gg O\left(\beta^{-\frac{3}{2}}e^{\beta(1-\xi+F_0 g(\xi))}\right)$; whereas, $\frac{T_f}{T_b} < 1$ for $F_0 > F_{02}$ since $O\left(\beta^{-1}e^{2\beta}\right) \ll O\left(\beta^{-\frac{3}{2}}e^{\beta(1-\xi+F_0 g(\xi))}\right)$.

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Appendix. Proof of (3.5)

At $F = 1$, f attains its global minimum at $(\frac{\pi}{2},\frac{\pi}{2})$. The asymptotics of $J(\beta)$ as $\beta \to \infty$ depends on the integral I_δ in (3.7). But since this critical point of f is degenerate, we further split the integration region D_δ into three parts:

$$D_\delta^1 = \left\{(q',q)\Big|0 \le q \le \frac{\pi}{2} - \epsilon_1, q-\delta \le q' \le q\right\},$$
$$D_\delta^2 = \left\{(q',q)\Big|\frac{\pi}{2} + \epsilon_2 \le q \le 2\pi, q-\delta \le q' \le q\right\},$$
$$D_\delta^3 = \left\{(q',q)\Big|\frac{\pi}{2} - \epsilon_1 \le q \le \frac{\pi}{2} + \epsilon_2, q-\delta \le q' \le q\right\}$$

where ϵ_1 and ϵ_2 are some arbitrarily chosen small numbers independent of β (see Fig. 6(a)). We define

$$I_\delta^i(\beta) = \iint_{D_\delta^i} e^{-\beta f(q',q)}\,dq'\,dq.$$

It is easy to get as $\beta \to \infty$ that:

$$I_\delta^1(\beta) \sim \beta^{-1}\int_0^{\frac{\pi}{2}-\epsilon_1} \frac{1}{1-\sin q}\,dq = \beta^{-1}\left(\cot\left(\frac{\epsilon_1}{2}\right) - 1\right),$$
$$I_\delta^2(\beta) \sim \beta^{-1}\int_{\frac{\pi}{2}+\epsilon_2}^{2\pi} \frac{1}{1-\sin q}\,dq = \beta^{-1}\left(1 + \cot\left(\frac{\epsilon_2}{2}\right)\right).$$

Thus $I_\delta^1(\beta)$ and $I_\delta^2(\beta)$ are of $O\left(\beta^{-1}\right)$. To compute $I_\delta^3(\beta)$, note first that

$$f(q',q) = \frac{f_{30}}{6}\left(q' - \frac{\pi}{2}\right)^3 + \frac{f_{03}}{6}\left(q - \frac{\pi}{2}\right)^3 + O\left(\left|(q',q) - \left(\frac{\pi}{2},\frac{\pi}{2}\right)\right|^4\right)$$

where

$$f_{30} := f_{q'q'q'}\left(\frac{\pi}{2},\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1, \qquad f_{03} := f_{qqq}\left(\frac{\pi}{2},\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

The above form also reflects the degeneracy of the minimum at $(\frac{\pi}{2},\frac{\pi}{2})$.

We next write

$$f(q',q) = \frac{f_{30}}{6}\left(q' - \frac{\pi}{2}\right)^3\left(1 + P\left(q' - \frac{\pi}{2}, q - \frac{\pi}{2}\right)\right) + \frac{f_{03}}{6}\left(q - \frac{\pi}{2}\right)^3\left(1 + Q\left(q' - \frac{\pi}{2}, q - \frac{\pi}{2}\right)\right)$$

where $P(q',q)$ and $Q(q',q)$ are power series in q' and q, with $P(0,0) = Q(0,0) = 0$. We will use the following variables

$$u = \left(q' - \frac{\pi}{2}\right)\left(1 + P\left(q' - \frac{\pi}{2}, q - \frac{\pi}{2}\right)\right)^{\frac{1}{3}}, \qquad v = \left(q - \frac{\pi}{2}\right)\left(1 + Q\left(q' - \frac{\pi}{2}, q - \frac{\pi}{2}\right)\right)^{\frac{1}{3}}$$

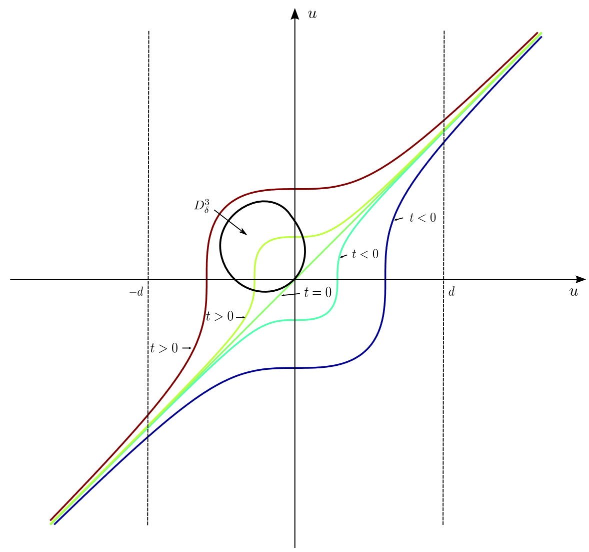

Fig. 11. Level sets of $F(u, v) = t$ and the region D_δ^3.

as independent variables. For this, we introduce

$$F(u, v) \;\; \big(:= f(q', q)\big) = \frac{f_{30}}{6}u^3 + \frac{f_{03}}{6}v^3 \quad \text{and} \quad G(u, v) = \left|\frac{\partial(q', q)}{\partial(u, v)}\right|.$$

Note that $G(\frac{\pi}{2}, \frac{\pi}{2}) = 1$.

Now let m and M be the infimum and supremum of $f(q', q)$ in D_δ^3. As easily seen, $m = 0$ and $M > 0$. Now we apply the method of resolution of multiple integrals (see [32, V.13, Thm. 9, pp. 280] and [32, VIII.10, pp. 463]) to get

$$I_\delta^3(\beta) = \int_m^M k(t)e^{-\beta t}\, dt, \qquad k(t) = \int_{F(u,v)=t} \frac{G(u, v)}{|\nabla F|}\, d\sigma,$$

σ being the arc length of the curve $F(u, v) = t$. We shall evaluate $k(t)$ along the curves $F(u, v) = \frac{f_{30}}{6}u^3 + \frac{f_{03}}{6}v^3 = t$, which are bounded between the vertical lines $u = -d$ and $u = d$ (see Fig. 11). Set

$$\xi = \frac{f_{30}}{6}u^3 + \frac{f_{03}}{6}v^3, \qquad \eta = u, \qquad \Theta(\xi, \eta) = G(u, v)\left|\frac{\partial(u, v)}{\partial(\xi, \eta)}\right|.$$

The line integral $k(t)$ is reduced to $k(t) = \int_{\xi=t} \Theta(\xi, \eta)\, d\eta$. By computation

$$\left|\frac{\partial(u, v)}{\partial(\xi, \eta)}\right| = -2f_{03}^{-\frac{1}{3}}(6\xi - f_{30}\eta^3)^{-\frac{2}{3}}.$$

Assume $G(u, v)$ has the Maclaurin series $G(u, v) = \sum G_{ij}u^i v^j$, with $G_{00} = 1$. We choose the positive value of $\left|\frac{\partial(u,v)}{\partial(\xi,\eta)}\right|$. Then

$$\Theta(\xi, \eta) = -G(u, v)\left|\frac{\partial(u, v)}{\partial(\xi, \eta)}\right| = \sum 2G_{ij}(f_{03})^{-\frac{j+1}{3}}(6\xi - f_{30}\eta^3)^{\frac{j-2}{3}}\eta^i = \sum \Theta_{ij}\eta^i(6\xi - f_{30}\eta^3)^{\frac{j-2}{3}}$$

with $\Theta_{ij} = \frac{2G_{ij}}{f_{03}^{\frac{j+1}{3}}}$. Thus, $k(t) = \sum \Theta_{ln}\int_{-d}^{d}\eta^l(6t - f_{30}\eta^3)^{\frac{n-2}{3}}\, d\eta$.

We only compute the dominating term which is the first expansion term with $l = n = 0$. Let $t > 0$ (this is always true since $m = 0$, and we can disregard the case $t = 0$). Note that d is independent of t. Hence, we may write

$$\int_{-d}^{d}(6t - f_{30}\eta^3)^{-\frac{2}{3}}\, d\eta = \int_{-\infty}^{0}(6t - f_{30}\eta^3)^{-\frac{2}{3}}\, d\eta + \int_0^{\infty}(6t - f_{30}\eta^3)^{-\frac{2}{3}}\, d\eta + \varphi(t),$$

where $\varphi(t)$ is a C^∞ function of t near $t = 0$ and, therefore, do not contribute to the asymptotic expansion of the Laplace integral I_δ^3. Now let $\eta = (-6t/f_{30})^{\frac{1}{3}}(\tan\alpha)^{\frac{2}{3}}$, then

$$\begin{aligned}\int_0^\infty (6t - f_{30}\eta^3)^{-\frac{2}{3}}\, d\eta &= \int_0^{\frac{\pi}{2}} \frac{2}{3}(6t)^{-\frac{2}{3}}(\sec\alpha)^{-\frac{4}{3}}\left(\frac{-6t}{f_{30}}\right)^{\frac{1}{3}}(\tan\alpha)^{-\frac{1}{3}}\sec^2\alpha\, d\alpha \\ &= -\frac{2}{3}6^{-\frac{1}{3}}t^{-\frac{1}{3}}f_{30}^{-\frac{1}{3}}\int_0^{\frac{\pi}{2}}(\sin\alpha)^{-\frac{1}{3}}(\cos\alpha)^{-\frac{1}{3}}\, d\alpha \\ &= -\frac{1}{3}6^{-\frac{1}{3}}t^{-\frac{1}{3}}f_{30}^{-\frac{1}{3}}B\left(\frac{1}{3},\frac{1}{3}\right)\end{aligned}$$

where we have used the identity

$$\int_0^{\frac{\pi}{2}}(\sin x)^{\mu-1}(\cos x)^{\nu-1}\, dx = \frac{1}{2}B\left(\frac{\mu}{2},\frac{\nu}{2}\right),\ \mathrm{Re}(\mu) > 0, \mathrm{Re}(\nu) > 0$$

(see [27, pp. 369, 3.62(5)]). Next, let $\eta = (6t/f_{30})^{\frac{1}{3}}(\sin\alpha)^{\frac{2}{3}}$ on $(\eta^*, 0)$ with $\eta^* = (6t/f_{30})^{\frac{1}{3}}$. Then we obtain

$$\begin{aligned}\int_{\eta^*}^0 (6t - f_{30}\eta^3)^{-\frac{2}{3}}\, d\eta &= -\frac{2}{3}6^{-\frac{1}{3}}t^{-\frac{1}{3}}f_{30}^{-\frac{1}{3}}\int_0^{\frac{\pi}{2}}(\sin\alpha)^{-\frac{1}{3}}(\cos\alpha)^{-\frac{1}{3}}\, d\alpha \\ &= -\frac{1}{3}6^{-\frac{1}{3}}t^{-\frac{1}{3}}f_{30}^{-\frac{1}{3}}B\left(\frac{1}{3},\frac{1}{3}\right).\end{aligned}$$

Let $\eta = (6t/f_{30})^{\frac{1}{3}}(\sec\alpha)^{\frac{2}{3}}$ on $(-\infty, \eta^*)$, then

$$\begin{aligned}\int_{-\infty}^{\eta^*} (6t - f_{30}\eta^3)^{-\frac{2}{3}}\, d\eta &= -\frac{2}{3}6^{-\frac{1}{3}}t^{-\frac{1}{3}}f_{30}^{-\frac{1}{3}}\int_0^{\frac{\pi}{2}}(\sin\alpha)^{-\frac{1}{3}}(\cos\alpha)^{-\frac{1}{3}}\, d\alpha \\ &= -\frac{1}{3}6^{-\frac{1}{3}}t^{-\frac{1}{3}}f_{30}^{-\frac{1}{3}}B\left(\frac{1}{3},\frac{1}{3}\right).\end{aligned}$$

Finally, we have as $\beta \to \infty$ that

$$\begin{aligned}I_\delta^3(\beta) &= \int_m^M k(t)e^{-\beta t}\, dt \sim \int_m^M \left(\Theta_{00}\int_{-d}^d (6t - f_{30}\eta^3)^{-\frac{2}{3}}\, d\eta\right)e^{-\beta t}\, dt \\ &\sim \int_m^M \left(\Theta_{00}\int_{-\infty}^\infty (6t - f_{30}\eta^3)^{-\frac{2}{3}}\, d\eta\right)e^{-\beta t}\, dt \\ &\sim -\frac{\Theta_{00}}{3}6^{-\frac{1}{3}}f_{30}^{-\frac{1}{3}}\left(3B\left(\frac{1}{3},\frac{1}{3}\right)\right)\int_m^M t^{-\frac{1}{3}}e^{-\beta t}\, dt \\ &\sim 2\cdot 6^{-\frac{1}{3}}B\left(\frac{1}{3},\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\beta^{-\frac{2}{3}},\end{aligned}$$

by Watson's Lemma [31, Sec. 2.1, pp. 24]. Hence $I_\delta^3(\beta) \gg I_\delta^1(\beta),\ I_\delta^2(\beta)$. The conclusion then follows. The constant C in (3.5) is given explicitly as

$$C = \frac{2\pi}{2\cdot 6^{-\frac{1}{3}}B\left(\frac{1}{3},\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}. \tag{A.1}$$

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