J. Nonlinear Sci. Vol. 8: pp. 491–579 (1998)

Journal of Nonlinear Science

Existence of Dendritic Crystal Growth with Stochastic Perturbations

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Received April 7, 1997; revised October 30, 1997; accepted November 3, 1997
Communicated by Robert Kohn

Summary. We prove the first mathematical existence result for a model of dendritic crystal growth with thermal fluctuations. The incorporation of noise is widely believed to be important in solidification processes. Our result produces an evolving crystal shape and a temperature field satisfying the Gibbs-Thomson condition at the crystal interface and a heat equation with a driving force in the form of a spatially correlated white noise. We work in the regime of infinite mobility, using a sharp interface model with a smooth and elliptic anisotropic surface energy. Our approach permits the crystal to undergo topological changes.

A time discretization scheme is used to approximate the evolution. We combine techniques from geometric measure theory and stochastic calculus to handle the singular geometries and take advantage of the cancellation properties of the white noise.

Key words. dendritic crystal growth, Gibbs-Theorem condition, Stochastic heat equation, geometric measure theory, stochastic analysis

MSC numbers. 80A22, 60H15, 58E50

1. Introduction

We prove the mathematical existence for dendritic crystal growth in a model that incorporates stochastic perturbations. This is an example of a curvature-driven evolution. Such processes appear frequently in the modeling of various physical phenomena—for example, solidification processes, fluid flows, and bacteria growths. Crystal growth is one of the most vivid examples of pattern formation. It can be described by relatively

* This paper is based on part of the author's doctoral dissertation [Yip].

simple equations, and yet it demonstrates rich and complex structures. The nonlinearities and singularities involved in such evolutions pose many difficult questions, including the existence and regularity of their mathematical solutions.

The effects of noise in such systems have been investigated at length by the physics community through both experiments and theory. See for example [Kar] in the case of crystal growth. It is believed that certain forms of fluctuations must be present in order to generate the observed patterns. Due to the nonlinearity of the evolution, a tiny amount of noise can be magnified tremendously to produce a macroscopic effect. Many physicists and material scientists are interested in knowing the selection mechanisms determining the final pattern. There are still many open questions concerning the formulation of the models with noise, the origins of the noise, and its relevance for the overall dynamics of the evolutions.

The study of such processes has also been taken up from a mathematical point of view. The Stefan problem is a simple model which does not involve any surface tension. A more refined model incorporates curvature information. It gives a more stable interface and allows the phenomenon of undercooling. Local in time classical solutions for this refined model were proved in [CR] and [FR]. [AW] and [Luc] used variational approaches to give general weak solutions in the infinite mobility framework. [Son] used the phase field method to give a varifold solution in the finite mobility case. Numerical works have also been done by (not mentioning the huge physical literatures) [Alr] and [RT] using variational approaches, and by [Kob] and [WMS] using the phase field approach. All these works do not consider the effects of noise except that, in simulations, noise very often is added artificially in order to produce realistic pictures.

In this paper, we introduce **thermal fluctuations** into a model of crystal growth and prove an existence result for a solution which combines the effects of surface tension—the Gibbs-Thomson condition—and stochastic heat diffusion. We hope this can be a step toward a more physical formulation. Using the idea of [AW], we produce an evolving crystal shape with sharp interface. We work in the regime of infinite mobility. The surface energy we use is anisotropic, smooth, and elliptic. The crystal is allowed to undergo topological changes. The whole Theorem is stated in Section 2.2

1.1. Model for Crystal Growth and Stochastic Noise

Our existence theorem is for a model of dendritic crystal growth. This process can be formulated in a wider context of **curvature driven flows**, which are defined as follows. Consider a time-varying domain $K(t)$ in the space R^n. Its boundary $\partial K(t)$—a hypersurface (physically also known as the **interface**)—evolves according to the following form:

$$\text{Norml Velocity}|_{p\in\partial K} = F\left(\text{Curvature}|_{p\in\partial K}, \text{Bulk forces}\right),$$

where F is some prescribed function. The bulk forces are quantities defined on the ambient space. They can be the temperature field, concentration of solutes, impurities, nutrients, and so forth. The use of such a formulation in various physical systems can be found in [KKL] and [Lan].

In the case of crystal growth, the above heuristic equation becomes

$$v_{\partial K}(p) = M(n_{\partial K}(p))\left(h_\Phi(p) + C(p,t)\right), \qquad p \in \partial K, \tag{1.1}$$

where $v_{\partial K}(p)$ = (inward) normal velocity at $p \in \partial K$; $n_{\partial K}(p)$ = (outward) normal to ∂K; M = mobility function; Φ = surface energy integrand; $h_\Phi(p)$ = (Φ-weighted)-mean curvature[1]; C = undercooling; t = time.

The **mobility function** M assigns a value to each normal direction. It describes the response of the interface $K(t)$ in terms of the attachment kinetics—the ease with which atoms attach to (or detach from) the interface $\partial K(t)$ or the crystal lattice.

There are two driving forces. The first one is the **Φ-weighted mean curvature**, h_Φ, which captures the reduction of the Φ **surface energy** of ∂K—$\Phi(\partial K)$. (Surface energy arises whenever there is an interface separating two phases or material composites. In our case, they are the solid and liquid phases.) The system will evolve in such a way that $\Phi(\partial K)$ (together with some other bulk quantities) decreases. The relationship between Φ and h_Φ is given in Section 2.1.7.

The other driving force is the **undercooling** C which is a function of the temperature value T. It describes how much T is lower than the melting point T_* of the material.[2] C can be approximated as $C(p,t) = T(p,t) - T_*(p,t)$ when T is close to T_*. Negative (positive) values of C gives a growing (shrinking) tendency of the crystal. These two forces are competing against each other. Their relative effects govern whether the crystal is growing or shrinking.

Another important ingredient of our model is the **diffusion of latent heat**, which controls the rate of growth. Recall that freezing of the liquid phase releases latent heat. If this heat is not diffused away, it will warm up the interface and then slow down the growth. Thus, in order to have a proper growth model, (1.1) is coupled with the following diffusion equation:

$$dQ = \mathrm{div}(\Sigma_K \nabla T)\, dt, \tag{1.2}$$

where the heat distribution Q is related to the temperature field T through the specific heat capacity: $Q = c_K T$. Σ_K is the diffusivity matrix.

The derivations of (1.1) and (1.2) using thermodynamics can be found in [Gur] and [Gur2]. The concept of (weighted) mean curvature from the materials science point of view is described nicely in [Tay]. Several mathematical methods of tackling phase transition problems are outlined in [TCH].

The motivations for the study of this phenomenon include applications to materials science. The control of the interfacial structures is an important issue in the manufacturing of alloys and semiconductor materials. The description of various physical processes involved in solidifications can be found in [Cha] and [Woo]. In addition, even the questions of how to model and predict the rich patterns pose many fascinating mathematical problems.

The above model is also an example of **diffusion controlled growths**. The qualitative picture in such phenomena is that simple shapes such as planar and circular interfaces are notoriously unstable ([MS], [MS2]). They tend to evolve into a regime of intensive sidebranching activities, but this is later stabilized by the surface energy effect. It is the

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[1] The sign convention is that h_Φ is positive for a sphere.

[2] In reality, the melting point of a material depends on the curvature of the interface. This is called the Gibbs-Thomson Curvature Effect, which is one of the main ingredients in this paper. T_* means the melting point of a planar interface.

interplay between the interfacial kinetics, surface tension, and diffusion that produces the intricate dendritic patterns observed in the solidification processes. The seemingly regular and self-similar patterns seen for example in snowflakes is believed to be a consequence of the **anisotropy** of Φ.

There is extensive physical literature concerning various aspects of these patterns. The quantities they study include tip radii of the dendrites, the spacings between them, and their growth velocities. Several nonlinear (deterministic) models have been proposed (see the accounts in [KKL], [Lan], [Lan2]). However, in some recent works, incorporation of noise is one of the main considerations ([Lan3], [PL], [WL]). It is widely believed that fluctuations are important in initiating the onset of morphological instabilities. The noise is then selectively amplified by the nonlinearity of the process to produce macroscopic patterns. However, the magnitude of the noise needed to simulate the experimental results seems to vary a lot depending on the models used. It is also not quite clear in which stages of growth the effects of noise are most prominent.

Here we take the point of view that thermal fluctuations are natural sources of perturbations. They are always present. They can come from external heat sources, chemical reactions, impurities, etc. In this paper, they are all put together into one stochastic driving force that is white in time but correlated in space. We demonstrate the possibility of a mathematical framework to incorporate such effects and produce an existence result.

Several questions remain open. What are the statistics of our solutions and their long-time behaviors? The answers can give a test of our formulation in comparison with the experimental results. In the framework of this paper, we are in effect considering macroscopic perturbations. Can they be treated as the accumulative effects of microscopic fluctuations? How can we relate quantitatively the noise we use and the physical parameters? Another approach to introduce perturbations is to consider random initial data. How can we compare the overall effects of initial randomness and the fluctuations during the evolutions? We do not know whether our approach gives a unique solution. How can this be formulated? We believe these are important directions for further work.

In this paper, we study the following simplified version of (1.1).

Infinite Mobility ($M \equiv \infty$) and the Gibbs-Thomson Condition. This formally reduces the interfacial dynamics (1.1) to

$$h_\Phi(p) = -C(p,t),$$

i.e., the mean curvature[3] balances with the negative of the undercooling. This is called the **Gibbs-Thomson Curvature Condition**. We further assume that

$$C(p,t) = H(T(p,t)). \tag{1.3}$$

H is called the **Gibbs-Thomson Relation**, which is a decreasing function of the temperature value—the lower the undercooling, the higher the corresponding curvature. It is this effect that provides the barrier to nucleations and gives the possibilities of **undercooled liquid** and **superheated solid**.

[3] From now on, we will drop the word "Φ-weighted." It is understood that mean curvature is always with respect to some surface energy integrand Φ.

Note that the above is an **equilibrium** condition. However, the crystal shape and temperature field are constantly changing. Due to the infinite mobility, the crystal interface can respond as fast as possible to compensate any deviations from the equilibrium.

As far as we know, there has not been any rigorous justification of why such a simplification is considered in much of the literature. Numerically, the inclusion of the curvature term h_Φ is necessary to give a well-posed problem, but the mobility's being finite is not important for such a purpose [Alr]. A finite mobility regime is simulated in [RT], [Kob], and [WMS].

However, the hypothesis of finite mobility is necessary to produce a crystal evolution that is **continuous in time**. In [AW], which considers (1.3), there is an actual example of a discontinuous evolution of the crystal shape. Mathematically, existence results for the motion law (1.1) are harder to establish than (1.3). We need a stronger regularity property for the crystal interface. [Son] gives a solution for (1.1), but the interface is of a varifold nature. A modified version of the approach used in this paper can indeed produce a continuous crystal evolution, but we do not know how to formulate and prove (1.1). The relationship between the solutions for the finite and infinite mobility cases remains to be resolved.

Models for Thermal Fluctuations. We add a stochastic driving force to the diffusion equation to imitate thermal fluctuations. Loosely speaking, we will consider the following stochastic heat equation:

$$dQ = \mathrm{div}(\Sigma_K \nabla T)\, dt + c_K\, f(T)\, dW_t, \tag{1.4}$$

where W_t is a **spatially correlated infinite dimensional Brownian motion** and dW_t is the Ito's differential. We show the existence of an evolution satisfying (1.3) and (1.4). The spatial correlation of W_t is important in our approach to give good regularity properties of the temperature fields so that (1.3) can be shown to be true.

We compare our formulation with those in the physics literature in which *space-time white noises* are considered very often. In [Kar], Langevin noises were introduced to both (1.3) and (1.4). They showed that this was necessary to be thermodynamically consistent with the equilibrium fluctuations of the bulk phases and the interface. The equations therein relevant to our paper are noted as follows:

$$\left\{\begin{array}{rcl} \frac{\partial}{\partial t} T_\nu &=& D_\nu \mathrm{div} \nabla T_\nu - \nabla \cdot \boldsymbol{q}_\nu, \qquad \nu = l, s, \\ L v_n &=& n \cdot [c_s D_s \nabla T_s - c_L D_l \nabla T_l] + n \cdot [c_l \boldsymbol{q}_l - c_l \boldsymbol{q}_s], \\ v_n &=& M(h_\Phi + C + \eta), \end{array}\right. \tag{1.5}$$

where T is the temperature; D_ν is the diffusivity; ν denotes the liquid (l) or solid (s) phases; v_n is the normal velocity; L is the latent heat and c_ν the specific heat capacities; M is the mobility; C is the undercooling; $\boldsymbol{q}$ and η are space-time white noises.

Note that the noises considered there are much more singular than the one used in this paper. However, (1.5) works in the finite mobility regime,[4] which has more regularizing effects than the infinite mobility case as demonstrated in [Str]. Thus it is conceivable that (1.5) can support more singular noises than our equations. It will be interesting to investigate mathematically the "optimal roughness" of the noises permitted in various regimes and to study the regularity properties of the interfaces.

Other Models. At another extreme, we completely ignore the undercooling and heat diffusion. Then (1.1) becomes

$$v_{\partial K}(p) = M(n)h_{\Phi}(p). \tag{1.6}$$

This is called **motion by mean curvature** (if we further assume that $M \equiv 1$). It is a fascinating geometric evolution in its own right. Stochastic perturbations added to (1.6) have been considered in [Yip].

Another interesting question is when h_{Φ} corresponds to a nonconvex surface energy. In this case, (1.6) is then backward parabolic in some directions. The interface might produce infinitesimal wrinkles. Such and related issues are discussed in [Gur2].

1.2. Mathematical Approach

The most difficult aspect of solving (1.3) and (1.4) is the singular behavior of the equations. Singularities may form even from smooth initial data. Topological changes of the crystal shapes may happen during the evolution. Upon the addition of white noise—time derivative of Brownian motion—we need to make sure that the effects of noise are canceled locally in time. The approach employed here can tackle the above difficulties quite efficiently. We combine the machinery of geometric measure theory and stochastic calculus. The function spaces we use can handle the singular geometries, and they have nice compactness and regularity results at our disposal. Furthermore, we have a meaningful formulation of (1.3) even when the curvature is unbounded.

The idea is to write the whole evolution as a gradient flow with respect to an energy functional. Under such a flow, the energy is always decreasing. We minimize a related functional using a time-stepping scheme to approximate the flow. The novel parts in this scheme are the choice of the inner product for the function spaces and the proof of the convergence to a limit.

In the deterministic setting, such an approach was used in [AW], [ATW], [Luc], and [LS] to solve (1.2), (1.3), and (1.6). Here we follow the idea in [AW]. The difficulty in the stochastic case is to control the energy globally in time and then prove the tightness of the probability measures. A technical device used is the smoothing of the noise term to preserve the regularity property of the functions inherited from the minimization steps. The spatial correlation of W_t is crucial in the argument. Ito's Formula and martingale inequality are the main tools from stochastic calculus.

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[4] In the infinite mobility limit, (1.5) essentially becomes $h_{\Phi} = -C + \eta$ and $dQ = \mathrm{div}(\Sigma \nabla T)\,dt + \nabla \cdot q_t$, where Q is the heat distribution.

In order to carry out the above scheme, we take the following analog from the stochastic differential equation:

$$dX_t = A(X_t)\,dt + B(X_t)\,dW_t, \tag{1.7}$$

where X_t denotes some heuristic state variable, $A(X_t)$ is a driving force describing the unperturbed motion law, and $B(X_t)$ is some operator acting on the white noise term dW_t.[5] (1.7) is interpreted as an integral form $X_t = X_0 + \int_0^t A(X_s)\,ds + \int_0^t B(X_s)\,dW_s$. In this paper, we employ a time discretization procedure. The integral can then be approximated as:

$$X_n = X_0 + \sum_{i=1}^{n} A(X_i)\,\triangle t_i + \sum_{i=1}^{n} B(X_i)\,\triangle W_i, \tag{1.8}$$

where $\triangle W_i = W(t_{i+1}) - W(t_i)$. Within each discrete time interval, we minimize some energy functional to approximate $A(X_i)\triangle t_i$, and then we solve a stochastic heat equation to imitate the effects of $B(X_i)\triangle W_i$. These discrete evolutions will be shown to converge.

2. Statement of Result and Outline of Proof

Our goal is to prove the existence of an evolution process of crystal shape and heat distribution satisfying (1.3) and (1.4). First we define the terminologies and notations involved. Further concepts of varifolds and probability theory will be given in the appendix.

The domain we are working in is an n-dimensional torus $\mathcal{O}$ so that we do not need to worry about boundary conditions. $|\mathcal{O}| = \mathcal{L}^n(\mathcal{O}) = \rho^n$ denotes the volume of $\mathcal{O}$. (ρ is the side length of $\mathcal{O}$.)

2.1. Definition of Function Spaces

2.1.1. Crystal Position ($\mathcal{K}$). These are described by subsets of $\mathcal{O}$ with **finite perimeter**. K is called such a set if

$$|\partial K| = \sup\left\{\int_K \operatorname{div} g\, d\mathcal{L}^n\colon\ g \in C_0^1(\mathcal{O}, R^n),\ \|g\|_\infty \le 1\right\} < \infty. \tag{2.1}$$

$|\partial K|$ is called the **perimeter** of K. $\mathcal{K}$ is metrized by the L^1 norm,

$$\|K - L\|_{L^1} = \int_{x\in\mathcal{O}} |K(x) - L(x)|\, d\mathcal{L}^n x = \mathcal{L}^n(K \triangle L). \tag{2.2}$$

By abuse of notation, K can mean either the set K or its characteristic function χ_K.

The main properties we need for this kind of set are **compactness** under L^1 of the collection $\{K \in \mathcal{K}\colon\ |\partial K| \le M < \infty\}$ and the existence of a well-defined notion of normal and boundary—**approximate normal** (n_K) and **reduced boundary** ($\partial^* K$). These concepts are elaborated nicely in [EG] and [Giu]. Each $K \in \mathcal{K}$ also can be considered as an n-dimensional integral current in the context of geometric measure theory [Fed].

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[5] Here we are considering Ito's differential.

2.1.2. Surface Energy (Φ). A **surface energy integrand** Φ is a function from S^{n-1} to R_+. It is usually extended to a map from R^n to R_+ by positive homogeneity of degree 1: $\Phi(\lambda v) = \lambda\Phi(v)$ $\left(\lambda \geq 0, v \in S^{n-1}\right)$.

The Φ **surface energy** of $K \in \mathcal{K}$ is defined as

$$\Phi(\partial K) = \int_{\partial K} \Phi(n_K)\, d\mathcal{H}^{n-1}, \tag{2.3}$$

where n_K is the outward normal vector of ∂K.[6]

In this paper, we assume that Φ is **smooth** and **elliptic**, i.e., it is twice differentiable and has positive second derivative when restricted to any straight line not passing through the origin.[7]

2.1.3. Heat Distribution ($\mathcal{Q}$). A heat distribution is any positive L^2 function Q defined on $\mathcal{O}$. They are metrized by the **modified Monge-Kantorovich Norm**,

$$\|P - Q\|_\sim = \|P - Q\|_* + \left|\bar{P} - \bar{Q}\right|, \tag{2.4}$$

where $\|P - Q\|_*$ is the number

$$\sup\left\{\int_{\mathcal{O}} \varphi(x)(P(x) - Q(x))\, d\mathcal{L}^n x\colon\ \mathrm{Lip}(\varphi) \leq 1\ \text{ and }\ \int_{\mathcal{O}} \varphi\, d\mathcal{L}^n = 0\right\}, \tag{2.5}$$

and $\bar{P} = \dfrac{1}{|\mathcal{O}|}\displaystyle\int_{\mathcal{O}} P\, d\mathcal{L}^n$ is the spatial average of P.

Such a norm is used in many mass transport problems [Rav]. An important property of $\mathcal{Q}$ is that $\{\|Q\|_{L^2} \leq M < \infty\}$ is compact in the $\|\cdot\|_\sim$ topology ([AW] Appendix B).

2.1.4. Temperature Field ($\mathcal{T}$). A temperature field is any positive L^2 function T on $\mathcal{O}$. This space is metrized by the L^2 norm,

$$\|T_1 - T_2\|_{L^2} = \left(\int_{x\in\mathcal{O}} |T_1(x) - T_2(x)|^2\, d\mathcal{L}^n x\right)^{1/2}. \tag{2.6}$$

We denote $U = 1/T$.

2.1.5. Specific Heat Capacity (c) and Diffusivity Matrix (Σ). Specific heat capacity is a number which relates the heat content and temperature of a material. In general, this number depends on the phase.

Let $K \in \mathcal{K}$ be a crystal. $\{x \in K\}$ is called the **solid phase** and $\{x \notin K\}$ the **liquid phase**. The **specific heat capacity** is a piecewise constant function defined on $\mathcal{O}$,

$$c_K = c_s\chi_K + c_l(1 - \chi_K), \tag{2.7}$$

[6] In this paper, ∂K always denotes the reduced boundary of K.

[7] This condition is not necessary in every result. However, in order to prove the existence of minimizers involving Φ, it is at least required to be a convex function.

where c_s, c_l are constants of the material. Usually $c_s < c_l$. Heat content and temperature are related by

$$Q = c_K T. \tag{2.8}$$

Another quantity we need is the **diffusivity matrix** for heat diffusion,

$$\Sigma_K \equiv \Sigma_s \chi_K + \Sigma_l (1 - \chi_K). \tag{2.9}$$

Σ_s and Σ_l are positive symmetric matrices with Σ_l usually taken to be a multiple of the identity.

Note that both c_K and Σ are **discontinuous** functions of the spatial variable.

2.1.6. State Space for Time Evolution (S). As we are studying evolution processes, time will be incorporated into our state space. Precisely, we define

$$S = L^1([0,1], \mathcal{K}) \times L^2([0,1], L^2(\mathcal{O})) \times L^2([0,1], L^2(\mathcal{O})) \times C([0,1], \mathcal{Q}). \tag{2.10}$$

Each element $X \in S$ consists of time-varying crystal position, temperature field, reciprocal temperature field, and heat distribution: $(K(t), T(t), U(t), Q(t))_{t\in[0,1]}$. The metric for S is given by

$$\|X_1 - X_2\|_S = \int_0^1 \|K_1(t) - K_2(t)\|_{L^1}\, dt + \int_0^1 \|T_1(t) - T_2(t)\|_{L^2}^2\, dt + \int_0^1 \|U_1(t) - U_2(t)\|_{L^2}^2\, dt + \sup_{t\in[0,1]} \|Q_1(t) - Q_2(t)\|_{\sim}. \tag{2.11}$$

2.1.7. The Gibbs-Thomson Condition and First Variation. The Gibbs-Thomson condition (1.3) is the equilibrium relationship between the curvature of the crystal boundary and the temperature value. In order to formulate this condition in the case when the crystal boundary is not smooth enough to define its curvature, we make use of the concept of **first variation of surface energy**.

Given the previous definitions of surface energy and integrand (Section 2.1.2), we describe how the (mean) curvature is related to the changes of the surface energy when the set is deformed by vector fields.

- Given a C^1 vector field g: $R^n \longrightarrow R^n$, let $G_s(x) \equiv x + sg(x)$ for $x \in R^n$. Then $G_{(\cdot)}$ is a one-parameter family of diffeomorphisms for small values of s and $G_0(\cdot)$ is the identity map from R^n to R^n. Note that $g(x) = \left.\frac{\partial}{\partial s} G(s,x)\right|_{s=0}$.
- The **first variation** of K with respect to g is defined to be[8] $\left.\frac{d}{ds} \Phi\left(G_{s\sharp}\partial K\right)\right|_{s=0}$. Sometimes it is also denoted by $\langle \partial K,\ g\rangle$.

[8] In the following, the $\sharp$ means the "push forward" or the diffeomorphic image of ∂K under the action of G_s.

- $\vec{H}_\Phi\colon \partial K \longrightarrow R^n$ is called the Φ **first variation vector field** of ∂K if, for all g,

$$\frac{d}{ds}\Phi\left(G_{s\sharp}\partial K\right)\Big|_{s=0} = \int_{x\in\partial K} \langle \vec{H}_\Phi(x), g(x)\rangle \, d\mathcal{H}^{n-1}x. \tag{2.12}$$

- If further, $\vec{H}_\Phi(x) = h_\Phi(x) n_K(x)$, where n_K is the outward normal of ∂K at x, then h_Φ is called the Φ **(weighted) mean curvature**[9] of ∂K. Rewriting the above, we have

$$\frac{d}{ds}\Phi\left(G_{s\sharp}\partial K\right)\Big|_{s=0} = \int_{x\in\partial K} h_\Phi(x)\langle n_K(x), g(x)\rangle \, d\mathcal{H}^{n-1}x. \tag{2.13}$$

Using this point of view, h_Φ is seen to be the **rate of change of the Φ-surface energy of ∂K per volume swept out by deformations.** This definition coincides with the classical one when the boundary is smooth.

2.1.8. Gibbs-Thomson Condition. There is a prescribed function called the **Gibbs-Thomson Relation**, $H\colon R_+ \longrightarrow R$, which relates the temperature and curvature values in equilibrium. (By abuse of notation, we use the same H as the first variation vector field, but without the arrow.) H is **smooth** and **decreasing**. It has the growth rates $H(a) \sim O(a^{-2})$ as $a \longrightarrow 0^+$ and $H(a) \sim O(a^2)$ as $a \longrightarrow +\infty$. (Its exact form will be given in Section 3.1.)

$K \in \mathcal{K}$ and $T \in \mathcal{T}$ are said to satisfy the **Gibbs-Thomson condition** *if, for arbitrary C^1 vector field g,*

$$\frac{d}{ds}\Phi\left(G_{s\sharp}\partial K\right)\Big|_{s=0} = \int_K \mathrm{div}(H(T(x))g(x))\, d\mathcal{L}^n x. \tag{2.14}$$

Clearly this formulation corresponds to the classical sense when everything is smooth.

2.2. Statement of Result

The main result gives a precise meaning by which the following statements are true:

$$dQ = \mathrm{div}(\Sigma_K \nabla T)\, dt + c_K f(T)\, dW_t, \tag{2.15}$$

$$h_\Phi = H(T), \tag{2.16}$$

where

- W_t is an infinite dimensional Brownian motion[10] taking values in $L^2(\mathcal{O})$.
- f is a given function of the temperature value ($f\colon R_+ \longrightarrow R_+$) with growth rates $f(a) \sim O(a^4)$ as $a \longrightarrow 0^+$; $f(a) \sim O(a^{-1})$ as $a \longrightarrow +\infty$. It is used to damp the white noise term dW_t in extreme temperature ranges.

[9] For historical reason, the Φ**-mean curvature vector** is taken to be the **negative** of the Φ-first variation vector.

[10] Such and related concepts about the stochastic integral will be given in the Appendix.

We have achieved the following:

There is a probability space $(\Omega, \mathcal{F}, P)$ *equipped with a filtration* $\{\mathcal{F}_t\}_{t\in[0,1]}$ *and a Brownian motion* $\{W_t,\ \mathcal{F}_t;\ 0 \le t < \infty\}$ *in* $L^2(\mathcal{O})$ *with its covariance operator given by a symmetric kernel* $\Lambda(\cdot,\cdot) \in L^\infty(\mathcal{O}\times\mathcal{O})$ *such that starting from some admissible initial condition*[11] K_0 *and* Q_0*, there is an almost surely positive stopping time* τ *and a predictable stochastic process* $X(t) = (K(t), T(t), Q(t))_{t\in[0,1]}$ *defined on* $(\Omega, \mathcal{F}, P)$ *taking values in the crystal shapes, temperature fields, and heat distributions satisfying the following properties:*

1. $E\left\{\sup_{t\in[0,1]} \Phi(\partial K(t))^m + \|T(t)\|_{L^2}^m + \|U(t)\|_{L^2}^m\right\} < C_m < \infty$ *and*
   $E\left(\int_0^1 \|\nabla T(t)\|_{L^2}^2 + \|\nabla U(t)\|_{L^2}^2\, dt\right)^m < C_m < \infty,$
   where $U = T^{-1}$ *and* m *is any positive integer.*
2. *The heat distribution* $Q(\cdot)$ *with* $Q(0) = Q_0$ *is evolving continuously in time in the modified Monge-Kantorovich norm* $\|\cdot\|_\sim$*. It satisfies the estimate,*
   $$E\,\|Q(t) - Q(s)\|_\sim^{2m} \le C_m\,|t-s|^m \qquad \text{for all } 0 \le s < t \le 1.$$
3. $Q(t) = c_{K(t)}T(t)$ *for* $d\mathcal{L}^1 \times dP$ *a.s. on* $\{(t,\omega)\colon\ t < \tau(\omega)\}$.
4. $(K(t), T(t), Q(t))_{t\in[0,1]}$ *solves the stochastic heat equation (2.15) in the following sense:*
   For all $\varphi \in C^\infty(\mathcal{O})$,
   $$(Q(t\wedge\tau),\ \varphi) = (Q_0,\ \varphi) - \int_0^{t\wedge\tau} \langle \Sigma_K \nabla T,\ \nabla\varphi\rangle\, dr + \int_0^{t\wedge\tau} (c_K\, f(T)dW_r,\ \varphi)\,,$$
   where $(\,\cdot\,,\,\cdot\,)$ *denotes the* L^2 *inner product on* $L^2(\mathcal{O})$*. Here* f *acts as a multiplicative operator on* $L^2(\mathcal{O})$*.*
5. *For* $d\mathcal{L}^1 \times dP$ *a.s on* $\{(t,\omega)\colon\ t < \tau(\omega)\}$*, the Gibbs-Thomson condition (2.16) holds, i.e.:*
   Given any C^1 *vector field* g *on* $\mathcal{O}$*,*
   $$\frac{d}{ds}\Phi\left(G_{s\sharp}\partial K(t)\right)\Big|_{s=0} = \int_{x\in K(t)} \operatorname{div}(H(T(t,x))g(x))\, d\mathcal{L}^n x.$$
   The left-hand side is the first variation of the Φ *energy of* ∂K *when* ∂K *is deformed by the one-parameter family of diffeomorphisms (for small s)* $G_s(x) = x + sg(x)$*.*
6. *For* $d\mathcal{L}^1 \times dP$ *a.s on* $\{(t,\omega)\colon\ t < \tau(\omega)\}$*, the following regularity statements are true:*
   $n = 2$**:** $\partial K(t)$ *is a one-dimensional differentiable submanifold of* $\mathcal{O}$ *without boundary and for any* C^1 *vector field* g *on* $\mathcal{O}$*,*
   $$\frac{d}{ds}\Phi\left(G_{s\sharp}\partial K(t)\right)\Big|_{s=0} = \int_{x\in\partial K(t)} H(T(x,t))\left\langle n_{K(t)},\ g(x)\right\rangle d\mathcal{H}^1 x,$$

[11] See the next section.

where n_K is the outward normal of K and $\mathcal{H}^1$ is the Hausdorff one-dimensional measure on ∂K.

$n = 3$**:** $\partial K(t)$ *is the homeomorphic image in $\mathcal{O}$ of a compact two-dimensional manifold without boundary.*

The whole theorem is split up into Theorems 6.4.5 (Energy Estimates and Heat Continuity), 7.2.1 (Limiting Heat Equation), 8.0.1 (Gibbs-Thomson condition), and 8.0.2 (Regularity of the Crystal Boundary).

2.2.1. Admissible Condition and Stopping Time. The admissible condition and the stopping time are introduced artificially by our method of solution. Concisely, the admissible condition says that the whole domain $\mathcal{O}$ is not completely frozen or melted. The stopping time gives the interval of time such that this condition is ensured to be true. Beyond this, the $\mathcal{O}$ *might* oscillate without control in between the states of being completely frozen or melted, and the compactness argument on which our proof relies heavily will fail. The definition of the admissible condition will be given in Section 6.2.2.

However, by the suggestion of one of the referees of this paper, if we consider the **Dirichlet boundary condition** for the temperature field, $T_{\partial\mathcal{O}} = f > 0$, the stopping time τ in the above main theorem probably could be eliminated. We believe that the extra estimates coming from the boundary of $\mathcal{O}$ can be handled in very much the same way as in the present paper. See also the remark in Section 6.1.1.

2.3. Outline of Proof

We will solve (2.15) and (2.16) using a time-stepping approximation scheme. There are three main ingredients in our method:

- Variational minimizations are used to approximate (or restore) the Gibbs-Thomson condition (2.16). The singular nature of the curvature condition is handled automatically by the techniques of geometric measure theory.
- Global energy estimates are derived using stochastic calculus (especially Ito's Formula and martingale inequalities) to take advantage of the statistical cancellations of the Brownian increments. This leads to tightness of the approximating probability measures.
- Approximating crystals are shown to converge in the varifold sense, which is much stronger than just the L^1 convergence. This allows us to show that the Gibbs-Thomson condition is true in the limit.

Now we give a symbolic outline of the whole scheme.

2.3.1. Generation of Discrete Approximation. Let $\{X(t)\}_{t\in[0,1]}$ denote the evolving state variable.[12] For each X, we associate it with some kind of energy functional $\mathcal{E}(X)$.

[12] $X = (K, T, U, Q)$ with $U = T^{-1}$.

Let N be a positive integer and $\triangle t = 1/N$ be the discretization interval. Also let $X^N_{i^-}$ ($X^N_{i^+}$) be the state variable at $t_i^- = (i\triangle t)^-$ ($t_i^+ = (i\triangle t)^+$) for $0 \leq i \leq N$. We will produce $\left\{X^N(t)\right\}_{t\in[0,1]}$,

$$X^N_{0^-} \longrightarrow X^N_{0^+} \longrightarrow X^N_{1^-} \longrightarrow X^N_{1^+} \longrightarrow \cdots \longrightarrow X^N_{N-1^-} \longrightarrow X^N_{N-1^+} \longrightarrow X^N_{N^-},$$

in the following manner:

- $X^N_{i^-} \longrightarrow X^N_{i^+}$: $X^N_{i^+}$ is chosen to be a **minimizer** for the following "heuristic functional":

$$\mathcal{E}(X) + \frac{1}{\triangle t}\left\| X - X^N_{i^-}\right\|, \tag{2.17}$$

  where $\|\cdot\|$ is some metric for X. It acts as a **penalty function** so that $X^N_{i^+}$ will not differ too much from $X^N_{i^-}$. The form of $\|\cdot\|$ also serves the purpose of giving the right motion law. In the present case, the choice is to make the minimizers satisfy the Gibbs-Thomson condition. As a by-product, the temperature fields enjoy some a priori regularity properties.
- $X^N_{i^+} \longrightarrow X^N_t$ $(t_i \leq t < t_{i+1})$: The heat distribution is diffused by the following stochastic heat equation for a **duration of** $\triangle t$ **with initial condition** $X^N_{i^+}$:

$$dQ = \operatorname{div}(\Sigma_{K^N_{i^+}} \nabla T)\, dt + c_{K^N_{i^+}} f(T)\, dW_t. \tag{2.18}$$

  Note that the **crystal is kept fixed to be** $K^N_{i^+}$ **during this procedure.** Galerkin's scheme is used to solve the above equation. In the actual implementation, we **smooth out** the W in (2.18). This will be explained later in Section 2.3.3.

2.3.2. Derivation of Energy Estimates. The next step is to derive the energy bound $E\left\{\sup_{\lambda\in[0,1]}\mathcal{E}(X^N(\lambda))\right\} \leq C$. It is carried out formally as

$$\begin{aligned}\mathcal{E}(X^N_{i^+}) &\leq \mathcal{E}(X^N_{i^-}) \text{ (by definition of minimization)},\\ \mathcal{E}(X^N_{i+1^-}) &= \mathcal{E}(X^N_{i^+}) + A^N(X^N_{i^+})\triangle t + B^N(X^N_{i^+})\triangle W_i \text{ (by Ito's Formula)},\end{aligned}$$

where A^N and B^N are some controllable operators. Thus, $\mathcal{E}$ can be estimated as

$$\mathcal{E}(X^N(t)) \leq \mathcal{E}(X^N(0)) + \int_0^t A^N(X^N(s))\,ds + \int_0^t B^N(X^N(s))\,dW_s.$$

A Martingale Inequality gives the asserted energy bound. The tightness of the probability measures follow from the compactness property of the function spaces. We can then extract a converging subsequence.

2.3.3. Properties of Limit Evolution. Our goal is to show (2.15) and (2.16). The validity of the stochastic heat equation can be proved by standard procedures using martingale formulation.

However, the Gibbs-Thomson condition is the heart of the matter. In the present case with stochastic perturbation, extra care must be taken to obtain good regularity properties of the temperature fields during the heat flow process. An intricate step is the smoothing of the white noise term in (2.18). This is used to preserve the regularity of the temperature fields inherited from the minimization procedure.

The main ingredient here is to show that the approximating crystals converge in **varifold sense** to the limiting ones. Briefly speaking, we will prove the following:

$$\left\|K^N - K\right\|_{L^1} \longrightarrow 0 \qquad \text{and} \qquad \Phi(\partial K^N) \longrightarrow \Phi(\partial K).$$

The key idea is to exploit the fact that the K^N are minimizers of the functional (2.17).

3. The Minimization Step

In this section, we describe the minimization procedure and its associated estimates. During this step, both the crystal and heat distribution will be changed. The purpose is to restore the Gibbs-Thomson condition. The a priori regularity estimates for the minimizers turn out to be very important to prove the properties of the limiting evolution.

First we define the **energy of the system** as $\mathcal{E}$,

$$\begin{aligned} \mathcal{E}(K, Q) &= \Phi(\partial K) + \textstyle\int_{\mathcal{O}} c_K F(c_K^{-1} Q)\, d\mathcal{L}^n, \qquad && (K \in \mathcal{K}, Q \in \mathcal{Q}), \\ &= \Phi(\partial K) + \textstyle\int_{\mathcal{O}} c_K F(T)\, d\mathcal{L}^n, && (T = c_K^{-1} Q). \end{aligned} \tag{3.1}$$

We recall that Φ is a **smooth** and **elliptic** integrand. F is the bulk energy functional. Its form and relationship with H—the Gibbs-Thomson relation—will be given in the next section.

Let α be any fixed positive number less than 1/44. **Given** $P \in \mathcal{Q}$, we **minimize**

$$\mathcal{E}(K, Q) + \frac{1}{\triangle t^\alpha} \|Q - P\|_\sim , \tag{3.2}$$

among all $K \in \mathcal{K}$ and $Q \in \mathcal{Q}$ such that $\overline{Q} = \overline{P}$, i.e., $\displaystyle\int_{\mathcal{O}} Q\, d\mathcal{L}^n = \int_{\mathcal{O}} P\, d\mathcal{L}^n$. We call any minimizer (K, Q) of the above functional a **minimizer for** $(\mathcal{E}, \triangle t, P)$.

In the actual application, P will be Q_{i^-}, the heat distribution at t_i^-, and the minimizer will become K_{i^+} and Q_{i^+}, the crystal position and heat distribution at t_i^+.

As pointed out in [AW], one of the novel features of this scheme is the use of the Monge-Kantorovich Distance. Its role in (3.2) is to allow a certain degree of heat transport so as to facilitate crystal changes. Due to this freedom, the minimizers satisfy the Gibbs-Thomson condition exactly. The condition on α is such that the effect of this term is not felt when taking the limit $\triangle t \longrightarrow 0$.

3.1. F, H, and their Estimates

Here we follow [AW] Appendix A. F is a **smooth positive uniformly convex function** defined on R_+. Its form can be described as follows ($U = 1/T$):

$$\begin{aligned} F(T) &= \begin{cases} U^2 & \text{if} \quad 0 < T \le a_*, \\ T^2 & \text{if} \quad b_* \le T < \infty, \end{cases} \\ F(T_*) &= 0 \text{ for some } T_* \in [a_*, b_*], \\ F''(T) &\ge c > 0 \text{ for all } T > 0, \end{aligned} \tag{3.3}$$

where a_* and b_* are fixed positive numbers and T_* is interpreted as the melting point of the planar interface of the material.

The Gibbs-Thomson relation (H) is derived from F as

$$H(T) = (c_l - c_s)(F(T) - TF'(T)). \tag{3.4}$$

It is easy to check that H is a smooth decreasing function such that $H(T_*) = 0$.

The following are simple consequences of the above representations:

$$\begin{aligned} |F(T_1) - F(T_2)| &\leq C\left(\left|U_1^2 - U_2^2\right| + \left|T_1^2 - T_2^2\right|\right) \\ &= C\,|T_1 - T_2|\left(U_1U_2^2 + U_1^2U_2 + T_1 + T_2\right), \qquad (3.5) \\ |H(T_1) - H(T_2)| &\leq C\left(\left|U_1^2 - U_2^2\right| + \left|T_1^2 - T_2^2\right|\right) \\ &= C\,|T_1 - T_2|\left(U_1U_2^2 + U_1^2U_2 + T_1 + T_2\right), \qquad (3.6) \end{aligned}$$

where C is a fixed constant. Furthermore, there are bounded Lipschitz functions J and L from R_+ to R such that

$$H(T) = C_1(3U^2 - T^2 + J(T)), \qquad H'(T) = C_2(-6U^3 - 2T + L(T)). \tag{3.7}$$

3.2. The Existence and Properties of Minimizers

The following results are from [AW] Chapter 4.

3.2.1. Theorem (Existence of Minimizer). *For all $P \in \mathcal{Q}$, $\triangle t > 0$, there exists a minimizer (K, Q) for $(\mathcal{E}, \triangle t, P)$, i.e., for all $L \in \mathcal{K}$ and $R \in \mathcal{Q}$ with $\overline{R} = \overline{P}$,*

$$\begin{aligned} &\Phi(\partial K) + \int_{\mathcal{O}} c_K F(c_K^{-1}Q)\,d\mathcal{L}^n + \frac{1}{\triangle t^\alpha}\,\|Q - P\|_\sim \\ &\quad\leq \Phi(\partial L) + \int_{\mathcal{O}} c_L F(c_L^{-1}R)\,d\mathcal{L}^n + \frac{1}{\triangle t^\alpha}\,\|R - P\|_\sim . \end{aligned}$$

3.2.2. Theorem (Regularity of the Temperature Field). *Let $T = c_K^{-1}Q$ where (K, Q) is a minimizer for $(\mathcal{E}, \triangle t, P)$. Then, upon redefining T on a set of measure zero, we have*

1. *T is bounded from below and above. Precisely,*

$$\left[\frac{A}{\mathcal{E}(K, Q) + B}\right]^{1/(3n+2)} \triangle t^{n\alpha/(3n+2)} \leq T(p) \leq C(\overline{Q} + \triangle t^{-\alpha}), \qquad (p \in \mathcal{O}). \tag{3.8}$$

2. *T is Lipschitz with* $\mathrm{Lip}T \leq D\triangle t^{-\alpha}$, *i.e.,*

$$|T(p) - T(q)| \leq D\,|p - q|\,\triangle t^{-\alpha}, \qquad (p, q \in \mathcal{O}). \tag{3.9}$$

(A, B, C, and D are constants depending only on the dimension n and the size of $\mathcal{O}$. $\overline{Q}$ is the spatial average of Q.)

3.2.3. Theorem (Validity of the Gibbs-Thomson Condition for K and T). *The Gibbs-Thomson condition holds for any minimizer (K, Q) of $(\mathcal{E}, \triangle t, P)$, i.e., for all $g \in C_0^1(R^n, R^n)$, $G_s(x) = x + sg(x)$, we have ($T = c_K^{-1} Q$)*

$$\frac{d}{ds}\Phi\left(G_{s\sharp}K\right)\Big|_{s=0} = \int_{x\in\partial K} H(T(x))\,\langle n_K(x),\ g(x)\rangle\ d\mathcal{H}^{n-1}x = \int_K \operatorname{div}(H(T)g)\,d\mathcal{L}^n. \tag{3.10}$$

For completeness, we also mention some regularity properties of the minimizing crystals.

3.2.4. Theorem (Regularity of the Minimizing Crystal). *Suppose (K, Q) is a minimizer for $(\mathcal{E}, \triangle t, P)$. Then,*

1. *There exist positive numbers δ and μ such that*
$$\mathcal{H}^{n-1}(\partial K \cap B^n(p, r)) \geq \mu r^{n-1},$$
for each point p in the support of $\partial[\![K]\!]$ and $0 < r < \delta$.
2. *The support of $\partial[\![K]\!]$ (which equals the closure of ∂K) has finite $\mathcal{H}^{n-1}$ measure and hence zero $\mathcal{L}^n$ measure.*
3. *There exist positive numbers δ and C together with a function $\omega(r) = Cr$ defined for $0 < r < \delta$ such that $\partial[\![K]\!]$ is (Φ, ω, δ)-minimal in the sense of Bomberi [Bom, Definition 1, p. 101].*
4. *When Φ is an even integrand ($\Phi(v) = \Phi(-v)$), there exist positive numbers γ and δ such that the support of $\partial[\![K]\!]$ is (γ, δ) restricted with respect to the empty set in the sense of Almgren [Alm II.1, p. 53].*
5. *When Φ is an even integrand, there exist positive numbers δ and C together with a function $\omega(r) = Cr$ defined for $0 < r < \delta$ such that the support of $\partial[\![K]\!]$ is (Φ, ω, δ)-minimal with respect to the empty set in the sense of Almgren [Alm III.1 p. 75].*
6. *When Φ is smooth and elliptic, then except for a possibly compact singular set of zero $\mathcal{H}^{n-1}$ measure, the support of $\partial[\![K]\!]$ is a two times Hölder continuously differentiable $(n-1)$-dimensional submanifold of $\mathcal{O}$.*
7. *When Φ is smooth and elliptic and $n = 2$ or 3, the support S of $\partial[\![K]\!]$ is a two times Hölder continuously differentiable submanifold of $\mathcal{O}$ (with no singular set). Furthermore, at every point p of S, the weighted mean curvature of S (with respect to the exterior normal of K) exists in the classical sense and equals $H(T(p))$.*

3.3. Measurable Selection

If we want to write the whole evolution as a stochastic integral (Section 5.2), we need to make sure that the evolving crystal positions, heat distributions, and temperature fields are **adapted** to some underlying filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of a probability space $(\Omega, \mathcal{F}, P)$.

Let the state at t_i^- be denoted by $X_i^- = \left(K_i^-, T_i^-, U_i^-, Q_i^-\right)$, which is a $\mathcal{F}_{t_i}$-measurable random variable ($\mathcal{F}_{t_i} \subset \mathcal{F}$), i.e.,

$$\omega \longrightarrow X_i^-(\omega) = \left(K_i^-(\omega), T_i^-(\omega), U_i^-(\omega), Q_i^-(\omega)\right) :\ (\Omega, \mathcal{F}_{t_i}) \longrightarrow (S, \mathcal{B}) \tag{3.11}$$

is measurable. Upon minimization, we get a new state $X_i^+ = \left(K_i^+, T_i^+, U_i^+, Q_i^+\right)$ at t_i^+. In general, the minimizers are not unique. We want to make a **measurable choice** of the X_i^+ so that it is also a $\mathcal{F}_{t_i}$-measurable random variable, i.e.,

$$\omega \longrightarrow X_i^+(\omega) = \left(K_i^+(\omega), T_i^+(\omega), U_i^+(\omega), Q_i^+(\omega)\right) : (\Omega, \mathcal{F}_{t_i}) \longrightarrow (S, \mathcal{B}). \quad (3.12)$$

is a measurable map.

To achieve this, it suffices to demonstrate the existence of a **Borel map** between X_i^- and X_i^+:

$$X_i^- \in (S, \mathcal{B}) \longrightarrow X_i^+ \in (S, \mathcal{B}). \quad (3.13)$$

We omit the proof here. The techniques are from [SV] Chapter 12. For details, see [Yip].

4. Heat Equation and Estimates (Fixed Crystal)

In this section, we carry out the heat flow process to diffuse the latent heat with the addition of stochastic noise. We solve (1.4) with the crystal kept fixed. In the next section, we will derive estimates for the overall solution. This is crucial in proving the compactness property of the evolution process, which is the crux of the matter in the stochastic version of the theorem.

An intricate step in this procedure is the **smoothing of the stochastic noise** in order to preserve the regularity of the heat distributions inherited by the minimization steps (Theorem 3.2.2). The spatial correlation of the noise is essential to achieve this purpose. The regularity estimates will be important in the proof of the Gibbs-Thomson condition in Section 8.

We assume the basic notions of stochastic calculus. They are summarized in the appendix.

4.1. Heat Equation for Fixed Crystal with Smoothing

In between the minimizations, we flow the heat for a duration of $\triangle t$, keeping the crystal fixed. Let K be a **given fixed crystal**. We will solve the following equation:

$$\begin{cases} dQ(t) = \operatorname{div}(\Sigma_K \nabla T(t))\, dt + c_K f(T(t))\, dW_t, \\ Q(t) = c_K T(t). \end{cases} \quad (4.1)$$

Rewriting the first equation, we have

$$dT(t) = \frac{1}{c_K} \operatorname{div}(\Sigma_K \nabla T(t))\, dt + f(T(t))\, dW_t. \quad (4.2)$$

4.1.1. Form of W_t and f.

- $\{W_t,\ \mathcal{F}_t;\ 0 \le t < \infty\}$ is an **infinite dimensional Brownian motion** taking values in $L^2(\mathcal{O})$ with **covariance operator** Λ given by a **symmetric kernel** (still denoted by Λ) belonging to $L^\infty(\mathcal{O} \times \mathcal{O})$. The filtration $\{\mathcal{F}_t\}_{t\ge 0}$ is fixed in this section. Unless otherwise stated, all adaptedness refers to this filtration.

- $f\colon R_+ \longrightarrow R_+$ is a smooth positive function of temperature value which acts as a **multiplicative operator on** L^2 **functions**. We require that $f(a) \longrightarrow 0$ as $a \longrightarrow 0^+$ and $+\infty$ in such a rate that $F'(a) f(a)$ and $F''(a) f(a)$ are uniformly bounded in a.[13] The purpose of this is to damp the white noise term dW_t in extreme temperature regions.

4.1.2. Smoothed Version of the Heat Equation. As mentioned earlier, in the **discrete scheme**, we are actually solving a smoothed version of (4.2),

$$dT(t) = \frac{1}{c_K}\text{div}\,(\Sigma_K \nabla T(t))\; dt + \; f_\delta(T(t))\, dW_t^\epsilon. \tag{4.3}$$

- δ and ϵ are small positive numbers depending on $\triangle t$ and they tend to zero as $\triangle t \longrightarrow 0$.
- f_δ is basically the same function as f but with $f(a) = 0$ for a very large and small. $f_\delta \longrightarrow f$ in C^∞ norm as $\delta \longrightarrow 0$. Its exact form is specified in (4.14). Of course, we still impose that $F'(a) f_\delta'(a)$ and $F''(a) f_\delta'(a)$ are uniformly bounded in δ and a.
- $W^\epsilon = \phi_\epsilon * W_t$ where ϕ_ϵ is a standard symmetric smoothing function tending to the delta function as $\epsilon \longrightarrow 0$. In this case, the covariance operator of W_t^ϵ is given by (see Section B.5.2)

$$\begin{aligned}\Lambda^\epsilon(x,y) \;&=\; (\phi_\epsilon \Lambda \phi_\epsilon)\,(x,y) = \int_{(w,z)\in\mathcal{O}\times\mathcal{O}} \Lambda(w,z)\phi_\epsilon(x-w)\phi_\epsilon(y-z)\, d\mathcal{L}^n w\, d\mathcal{L}^n z\\ &\in\; L^\infty(\mathcal{O}\times\mathcal{O}).\end{aligned} \tag{4.4}$$

4.1.3. Definition of Solution for (4.3). *$T\colon [0,1] \longrightarrow L^2(\mathcal{O})$ is called a solution of (4.3) with initial data T_0 if T is predictable, P a.s. belongs to*[14]

$$C([0,1], L^2(\mathcal{O})) \bigcap L^2([0,1], H^1(\mathcal{O}))$$

and for all $\varphi \in C^\infty(\mathcal{O})$, $t \in [0,1]$, the following identity is satisfied:

$$\begin{aligned}&\int_{x\in\mathcal{O}} c_K(x)T(t,x)\varphi(x)\, d\mathcal{L}^n x\\ &= \int_{x\in\mathcal{O}} c_K(x)T_0(x)\varphi(x)\, d\mathcal{L}^n x - \int_0^t \int_{x\in\mathcal{O}} \langle \Sigma_K(x)\nabla T(s,x),\; \nabla\varphi(x)\rangle\; d\mathcal{L}^n x\, ds\\ &\quad + \int_0^t \int_{x\in\mathcal{O}} \varphi(x)c_K(x)\, f_\delta(T(s,x))\, dW^\epsilon(s,x)\, d\mathcal{L}^n x.\end{aligned} \tag{4.5}$$

Then we set $Q(t) = c_K T(t)$.

In the actual scheme, we solve (4.3) during the intervals $[t_0, t_1), [t_1, t_2), \ldots$. The initial condition T_0 will then be $T_0^+, T_1^+, \ldots$—the temperature fields right after minimizations.

The first result we need is as follows.

---

[13] F is the function defined in the energy functional $\mathcal{E}$ in (3.1).

[14] $H^1(\mathcal{O})$ is the space of L^2 functions on $\mathcal{O}$ with L^2 spatial derivatives.

4.1.4. Theorem (Existence and Uniqueness). *There is a unique solution $T(\cdot)$ for (4.3) in the sense of Definition 4.1.3 satisfying*

$$E\left\{\sup_{t\in[0,1]}\|T(t)\|_{L^2}^2+\int_0^1\|\nabla T(t)\|_{L^2}^2\,dt\right\}<\infty. \tag{4.6}$$

The Theorem is proved by **Galerkin's Scheme** and **Picard's Iteration**. The whole procedure is very similar to the one carried out in [KR] and [Par]. The fact that we have discontinuous coefficients c_K and Σ can be remedied by using a weighted L^2 norm. For completeness, we give the proof in Appendix C.1. Now we concentrate on the effects of the smoothing and the corresponding estimates.

For the use of later sections, we set forth some notations.

4.1.5. Function Spaces and Operators.

- Let H be the Hilbert space of L^2 functions on $\mathcal{O}$ with inner product,

$$(u,\ v)_H=\int_{x\in\mathcal{O}}c_K(x)u(x)v(x)\,d\mathcal{L}^nx. \tag{4.7}$$

- Let V be the Hilbert space of $H^1(\mathcal{O})$ functions with inner product,

$$[u,\ v]_V\ =\int_{x\in\mathcal{O}}\langle\Sigma_K\nabla u,\ \nabla v\rangle\ d\mathcal{L}^nx+\int_{x\in\mathcal{O}}c_K(x)u(x)v(x)\,d\mathcal{L}^nx. \tag{4.8}$$

- Denote V^* and H^* to be the dual of V and H. Then, (V, H, V^*) forms a **Gelfand Triple**. (See page 568.)
- Let A: $V\longrightarrow V^*$ be defined as

$$\langle Au,\ v\rangle=\int_{\mathcal{O}}\langle\Sigma_K\nabla u,\ \nabla v\rangle\ d\mathcal{L}^n=\left\langle\Sigma_K^{1/2}\nabla u,\ \Sigma_K^{1/2}\nabla v\right\rangle_{L^2}. \tag{4.9}$$

If $Au\in H,\ \ v\in V$, then $\langle Au,\ v\rangle=(Au,\ v)_H$. Formally, this means that

$$Au=-\frac{1}{c_K}\operatorname{div}(\Sigma_K\nabla u)\,.$$

We have the following properties for A:
Boundedness: $\|Au\|_{V^*}\le\|u\|_V$.
Positivity: $0\le\langle Au,\ u\rangle$ and $\|u\|_V^2=\langle Au,\ u\rangle+\|u\|_H^2$.
Self-Adjointness: For all $u, v\in V$, $\langle Au,\ v\rangle=\langle u,\ Av\rangle$.

- Let B: $H\longrightarrow L(H)$ be the multiplicative operator,

$$(B(u)h)(x)=f(u(x))h(x),\qquad (u,h\in H). \tag{4.10}$$

Let $\|B(u)\|_\Lambda^2=\mathrm{Tr}\,[B(u)\Lambda B(u)^*]$. Then by Section B.5.1, we have

1. (by the **boundedness** of f)

$$\|B(u)\|_\Lambda^2=\int_{x\in\mathcal{O}}\Lambda(x,x)\,f(u(x))^2\,d\mathcal{L}^nx\le C. \tag{4.11}$$

2. (by the **Lipschitz** property of f)

$$\|B(u)-B(v)\|_\Lambda^2\le C\,\|u-v\|_H^2\,. \tag{4.12}$$

4.2. Lower and Upper Bounds for the Temperature Field

We investigate the role of δ in (4.3).

From Theorem 3.2.2, we know that right after minimization, the temperature field T is bounded from below and above,

$$\left[\frac{A}{\mathcal{E}(K, Q) + B}\right]^{1/(3n+2)} \triangle t^{n\alpha/(3n+2)} \leq T(p) \leq C(\overline{Q} + \triangle t^{-\alpha}), \qquad (p \in \mathcal{O}). \quad (4.13)$$

Choose $f_\delta\colon R_+ \longrightarrow R$ to be a smooth positive function such that

$$f_\delta(T) = 0 \qquad \text{for } T \leq (A/B)^{\frac{1}{3n+2}} \triangle t^{\frac{n\alpha}{3n+2}} \text{ and } T \geq C(\overline{Q} + \triangle t^{-\alpha}), \qquad (4.14)$$

where A, B, and C are the same as in (4.13).

4.2.1. Theorem. *Let $T(t)$ be a solution for (4.3). Suppose the initial condition T_0 satisfies (4.13), then so does $T(t)$ for $t \geq 0$.*

Proof. The idea of the proof is from [Par] p. 152.

Set $G_{\triangle t}\colon R \longrightarrow R_+$ to be a smooth positive convex function such that

$$G_{\triangle t}(T) = 0 \qquad \text{for } T \in \left[\left(\frac{A}{\mathcal{E}(K, Q) + B}\right)^{\frac{1}{3n+2}} \triangle t^{\frac{n\alpha}{3n+2}},\ \ C(\overline{Q} + \triangle t^{-\alpha})\right],$$

and $G_{\triangle t}(\cdot)$ tends to a linear function as $T \longrightarrow \pm\infty$.

Then by the same method as in the derivation of Theorem 5.1.1, we get

$$\int_{\mathcal{O}} c_K G_{\triangle t}(T(t))\, d\mathcal{L}^n \leq \int_{\mathcal{O}} c_K G_{\triangle t}(T_0)\, d\mathcal{L}^n + \int_0^t \int_{\mathcal{O}} c_K G'_{\triangle t}(T)\, f_{\triangle t}(T)\, dW_s^\epsilon\, d\mathcal{L}^n + \frac{1}{2}\int_0^t \mathrm{Tr}\left[c_K G''_{\triangle t}(T)\, f_{\triangle t}(T) \Lambda^\epsilon f_{\triangle t}(T)\right] ds.$$

However,

- $\int_{\mathcal{O}} c_K G_{\triangle t}(T_0)\, d\mathcal{L}^n = 0$ as T_0 satisfies (4.13);
- $\int_0^t \int_{\mathcal{O}} c_K G'_{\triangle t}(T)\, f_{\triangle t}(T)\, dW_s^\epsilon\, d\mathcal{L}^n = 0$ as $G'_{\triangle t}(\cdot)\, f_{\triangle t}(\cdot) = 0$;
- $\frac{1}{2}\int_0^t \mathrm{Tr}\left[c_K G''_{\triangle t}(T)\, f_{\triangle t}(T) \Lambda^\epsilon f_{\triangle t}(T)\right] ds = 0$ as $G''_{\triangle t}(\cdot)\, f_{\triangle t}(\cdot) = 0$.

Hence, $\int_{\mathcal{O}} c_K G_{\triangle t}(T(t))\, d\mathcal{L}^n = 0$, i.e., (4.13) is preserved. □

To proceed further for the use of Section 8.3.2, define

$$m_i = \inf_{(t,x)} \{T(t, x), T_0(x)\}, \qquad M_i = \sup_{(t,x)} \{T(t, x), T_0(x)\}, \qquad t \geq 0, \qquad x \in \mathcal{O}. \quad (4.15)$$

Then, from the above proposition, we have

$$m_i \geq \left(\frac{A}{\mathcal{E}(K,Q)+B}\right)^{\frac{1}{3n+2}} \triangle t^{\frac{n\alpha}{3n+2}} \qquad \text{and} \qquad M_i \leq C(\overline{Q} + \triangle t^{-\alpha}),$$

and so,

$$m_i^{-24} \leq \left(\frac{\mathcal{E}(K,Q)+B}{A}\right)^{\frac{24}{3n+2}} \triangle t^{\frac{-24n\alpha}{3n+2}} \qquad \text{and} \qquad M_i^8 \leq C(\overline{Q}^8 + \triangle t^{-8\alpha}).$$

Using the energy estimates Theorem 5.2.2, we get the following.

4.2.2. Corollary.

$$E\left(m_i^{-24}\right) \leq C\triangle t^{\frac{-24n\alpha}{3n+2}} \qquad \textit{and} \qquad E\left(M_i^8\right) \leq C\triangle t^{-8\alpha}. \tag{4.16}$$

4.3. Temperature Gradient Bound

Now we take into account the effect of ϵ in (4.3).

Remark. As mentioned earlier, the equation we are solving is (4.3) where ϵ is a small number depending on $\triangle t$. Such a smoothing of the noise will help preserve the gradient bound of the temperature field under the heat flow. In this section and eventually, we take ϵ to be $\triangle t^{\gamma}$ where γ is a very small positive number.

Our goal is the following:

4.3.1. Theorem (Temperature Gradient Bound). *If $\{T(t)\}_{t\geq 0}$ solves (4.3), and suppose $\|\nabla T_0\|_{L^2} \leq C\triangle t^{-\alpha}$, then*

$$E\left\{\sup_{t\in[0,\triangle t]} \|\nabla T(t)\|_{L^2}^8\right\} \leq \frac{C}{\epsilon^{8n+8}\triangle t^{8\alpha}}. \tag{4.17}$$

Note that by Theorem 3.2.2, the temperature fields right after minimizations satisfy the hypothesis in the above proposition concerning the gradient bound.

The method is by Galerkin's Scheme and Picard's Iteration, which are used in the proof of Theorem 4.1.4 (Appendix C.1). But this time, the computations are much more involved. The crucial fact is that the covariance operator of W^{ϵ} is given by a smooth kernel Λ^{ϵ}.

In the following, the δ in f_δ will be suppressed.

4.3.2. Strategy of Proving Theorem 4.3.1. The following two sections will lead to the desired result:

Section 4.3.9: *Let $S(t)$ be a time-varying temperature field adapted to $\mathcal{F}_t$, and let $T(t)$ be the unique solution of the following:*

$$dT(t) = \frac{1}{c_K}\operatorname{div}\left(\Sigma_K^{1/2}\nabla T(t)\right) dt + f(S(t))\, dW_t^{\epsilon}, \tag{4.18}$$

in the sense of (C.2). Suppose $\|\nabla T_0\|_{L^2} \le C\triangle t^{-\alpha}$, *then for* $0 \le t \le \triangle t$,

$$E\left\{\sup_{\lambda\in[0,t]} \|\nabla T(\lambda)\|_{L^2}^8\right\} \le \frac{C}{\epsilon^{8n+8}\triangle t^{8\alpha}} + \int_0^t E\left\{\sup_{\lambda\in[0,r]} \|\nabla S(\lambda)\|_{L^2}^8\right\} dr. \tag{4.19}$$

Section 4.3.10: *Set* $S(t) = T^{n-1}(t)$, $T(t) = T^n(t)$ *in (4.18) where* $\{T^n(t)\}_{n\ge1}$ *are the solutions in Picard's Iteration (Appendix C.1.5) in the process of solving (4.3). We will iteratively make use of (4.19) to achieve (4.17).*

Now we start to prove (4.19). We will show that during the process of solving (4.18) using Galerkin's scheme, each approximated solution T_n satisfies (4.19) and hence so does T (by the lower-semicontinuity of the gradient norm under uniform or weak convergence).

4.3.3. Special Basis for H**.** In addition to the notations in Section 4.1.5, we further define H_0 to be the subspace of H consisting of elements u such that $\int_{\mathcal{O}} c_K u(x)\, d\mathcal{L}^n x = 0$ and V_0 to be a subspace of H_0 with inner product

$$\langle u,\ v\rangle_{V_0} = \int_{\mathcal{O}} \left\langle \Sigma_K^{1/2}\nabla u,\ \Sigma_K^{1/2}\nabla v\right\rangle d\mathcal{L}^n.$$

By the Poincaré Inequality and Rellich's Lemma, V_0 is **compactly embedded** in H_0. Now, $Au = -\frac{1}{c_K}\mathrm{div}\,(\Sigma_K \nabla u)$ is a postive elliptic operator. From [Eva] Section 6.5, we can find a sequence $\{u_i\}_{i\ge1}$ with the following properties:

1. $\{u_i\}_{i\ge1}$ forms a complete sequence of eigenvectors of A, i.e., $Au_i = \lambda_i u_i$.
2. $\{u_i\}_{i\ge1}$ forms an orthonormal basis for H_0, and it is also an orthogonal basis for V_0, i.e.,
$$(u,\ v)_{H_0} = \delta_{ij} \qquad \text{and} \qquad \langle u,\ v\rangle_{V_0} = \delta_{ij}\,\|u_i\|_{V_0}^2. \tag{4.20}$$
3. For $u \in V_0$, if we write $u = \sum_i c_i u_i$ as vectors in H_0, then the series also converges in V_0.
4. Let Π_n be the orthogonal projection onto $\{u_1, u_2, \ldots, u_n\}$ in V_0. Then, for $u, v \in V_0$,

$$\langle u,\ u_i\rangle_{V_0} = (u,\ u_i)_{H_0}\,\langle u_i,\ u_i\rangle_{V_0}, \tag{4.21}$$

$$\begin{aligned}\langle \Pi_n u,\ \Pi_n v\rangle_{V_0} &= \sum_{j=1}^n \langle u,\ u_i\rangle_{V_0}\,(v,\ u_i)_{H_0} \\ &= \sum_{j=1}^n (u,\ u_i)_{H_0}\,(v,\ u_i)_{H_0}\,\langle u_i,\ u_i\rangle_{V_0}. \end{aligned} \tag{4.22}$$

We then adjoint the constant function $u_0 = \left(\int_{\mathcal{O}} c_K(x)\, d\mathcal{L}^n x\right)^{-1}$ to $\{u_i\}_{i\ge1}$ to form a complete O.N.B. for H.

4.3.4. Finite Dimensional Approximation for ∇T**.** Let $T_n(t) = \sum_{i=1}^n c_n^i(t)u_i$. We know from Section C.1.1 that

$$dT_n(t) = -\Pi_n A T_n\, dt + \Pi_n f(S(t))\, dW_t$$

has a solution with

$$dc_n^i(t) = -\sum_{j=1}^{n} c_n^j(t)\left\langle \Sigma_K^{1/2}\nabla u_j,\ \Sigma_K^{1/2}\nabla u_i\right\rangle dt + \left(f(S(t))\,dW_t^{\epsilon},\ c_K u_i\right)_{L^2}.$$

Since $\nabla T_n(t) = \sum_{i=1}^{n} c_n^i(t)\nabla u_i$, we have

$$\left\langle \Sigma_K^{1/2}\nabla T_n,\ \Sigma_K^{1/2}\nabla T_n\right\rangle = \sum_{ij}^{n} c_n^i(t)c_n^j(t)\left\langle \Sigma_K^{1/2}\nabla u_i,\ \Sigma_K^{1/2}\nabla u_j\right\rangle.$$

Hence,

$$\begin{aligned} & d\left\langle \Sigma_K^{1/2}\nabla T_n(t),\ \Sigma_K^{1/2}\nabla T_n(t)\right\rangle \\ &= 2\sum_{ij}^{n} c_n^i(t)\,dc_n^j(t)\left\langle \Sigma_K^{1/2}\nabla u_i,\ \Sigma_K^{1/2}\nabla u_j\right\rangle + \sum_{ij}^{n} d\left\langle c_n^i(t),\ c_n^j(t)\right\rangle\left\langle \Sigma_K^{1/2}\nabla u_i,\ \Sigma_K^{1/2}\nabla u_j\right\rangle \\ &= 2\left\langle \Sigma_K^{1/2}\nabla T_n(t),\ d\Sigma_K^{1/2}\nabla T_n(t)\right\rangle + d\,\mathrm{Tr}\left\langle\!\left\langle \Sigma_K^{1/2}\nabla T_n(t)\right\rangle\!\right\rangle. \end{aligned} \tag{4.23}$$

We proceed to investigate each term of the above.

4.3.5. Computation of $2\langle \Sigma_K^{1/2}\nabla T_n(t),\ d\Sigma_K^{1/2}\nabla T_n(t)\rangle$.

This term can be written as

$$\begin{aligned} & 2\sum_{ij}^{n} c_n^i(t)\,dc_n^j(t)\left\langle \Sigma_K^{1/2}\nabla u_i,\ \Sigma_K^{1/2}\nabla u_j\right\rangle \\ &= 2\sum_{ij}^{n} c_n^i(t)\left\{-\sum_{k}^{n} c_n^k(t)\left\langle \Sigma_K^{1/2}\nabla u_k,\ \Sigma_K^{1/2}\nabla u_j\right\rangle dt + \left(f(S(t))\,dW_t^{\epsilon},\ c_K u_j\right)\right\} \\ &\quad \times\left\langle \Sigma_K^{1/2}\nabla u_i,\ \Sigma_K^{1/2}\nabla u_j\right\rangle \\ &= -2\sum_{ijk}^{n} c_n^i(t)c_n^k(t)\left\langle \Sigma_K^{1/2}\nabla u_j,\ \Sigma_K^{1/2}\nabla u_k\right\rangle\left\langle \Sigma_K^{1/2}\nabla u_j,\ \Sigma_K^{1/2}\nabla u_i\right\rangle dt \\ &\quad +2\sum_{ij}^{n} c_n^i(t)\left(f(S(t))\,dW_t^{\epsilon},\ c_K u_j\right)\left\langle \Sigma_K^{1/2}\nabla u_i,\ \Sigma_K^{1/2}\nabla u_j\right\rangle \\ &= -2\sum_{j}^{n}\left\langle \Sigma_K^{1/2}\nabla T_n(t),\ \Sigma_K^{1/2}\nabla u_j\right\rangle^2 dt \\ &\quad +\sum_{j}^{n}\left(f(S(t))\,dW_t^{\epsilon},\ c_K u_j\right)\left\langle \Sigma_K^{1/2}\nabla T_n(t),\ \Sigma_K^{1/2}\nabla u_j\right\rangle \qquad (4.24) \\ &= -2\sum_{j}^{n}\left\langle \Sigma_K^{1/2}\nabla T_n(t),\ \Sigma_K^{1/2}\nabla u_j\right\rangle^2 dt \\ &\quad +\left(\Sigma_K^{1/2}\nabla T_n(t),\ \Sigma_K^{1/2}\nabla\left(f(S(t))\,dW_t^{\epsilon}\right)\right) \qquad (4.25) \end{aligned}$$

$$= -2\sum_{j}^{n}\left\langle \Sigma_K^{1/2}\nabla T_n(t),\ \Sigma_K^{1/2}\nabla u_j\right\rangle^2\ dt \tag{4.26}$$

$$+\left(f'(S(t))\left(\Sigma_K^{1/2}\nabla S(t)\right)\ dW_t^{\epsilon} + f(S(t))\,d\left(\Sigma_K^{1/2}\nabla W_t^{\epsilon}\right),\ \Sigma_K^{1/2}\nabla T_n(t)\right). \tag{4.27}$$

Note that, from (4.24) to (4.25), we have made use of the special property (4.22) of the basis $\{u_i\}_{i\geq 1}$.

For (4.26), it is a negative term.

For (4.27), we write it as $dM_t^{(1)} + dM_t^{(2)}$ where

$$dM_t^{(1)} = \left(f'(S(t))\left(\Sigma_K^{1/2}\nabla S(t)\right)\ dW_t^{\epsilon},\ \Sigma_K^{1/2}\nabla T_n(t)\right), \tag{4.28}$$

$$dM_t^{(2)} = \left(f(S(t))\,d\left(\Sigma_K^{1/2}\nabla W_t^{\epsilon}\right),\ \Sigma_K^{1/2}\nabla T_n(t)\right). \tag{4.29}$$

Note that $M_t^{(1)}$ and $M_t^{(2)}$ are martingales.

From (4.23) and (4.27), we have

$$\begin{aligned}
\left\|\Sigma_K^{1/2}\nabla T_n(t)\right\|^2 &\leq \left\|\Sigma_K^{1/2}\nabla T_n(0)\right\|^2 + M_t^{(1)} + M_t^{(2)} + \int_0^t d\,\mathrm{Tr}\left\langle\!\left\langle \Sigma_K^{1/2}\nabla T_n(t)\right\rangle\!\right\rangle\\
\Longrightarrow\ \sup_{r\in[0,t]}\left\|\Sigma_K^{1/2}\nabla T_n(r)\right\|^8 &\leq C\left\{\left\|\Sigma_K^{1/2}\nabla T_n(0)\right\|^8 + \sup_{r\in[0,t]}\left|M_r^{(1)}\right|^4 + \sup_{r\in[0,t]}\left|M_r^{(2)}\right|^4\right.\\
&\qquad\left. + \left|\int_0^t d\,\mathrm{Tr}\left\langle\!\left\langle \Sigma_K^{1/2}\nabla T_n(r)\right\rangle\!\right\rangle\right|^4\right\}.
\end{aligned} \tag{4.30}$$

Using Burkholder's Inequality, $E\left\{\sup_{\lambda\in[0,t]}\left\|\Sigma_K^{1/2}\nabla T_n(\lambda)\right\|^8\right\}$ can be bounded by

$$C\left\{E\left\|\Sigma_K^{1/2}\nabla T_n(0)\right\|^8 + E\left\langle M_t^{(1)}\right\rangle^2 + E\left\langle M_t^{(2)}\right\rangle^2 + E\left|\int_0^t d\,\mathrm{Tr}\left\langle\!\left\langle \Sigma_K^{1/2}\nabla T_n(r)\right\rangle\!\right\rangle\right|^4\right\}. \tag{4.31}$$

We will treat each term separately.

4.3.6. Computation of $d\left\langle M_t^{(1)}\right\rangle$.

Recall that

$$\begin{aligned}
dM_t^{(1)} &= \left(f'(S(t))\left(\Sigma_K^{1/2}\nabla S(t)\right) dW_t^{\epsilon},\ \Sigma_K^{1/2}\nabla T_n(t)\right)\\
&= \left(f'(S(t))\,dW_t^{\epsilon},\ \left\langle \Sigma_K^{1/2}\nabla S(t),\ \Sigma_K^{1/2}\nabla T_n(t)\right\rangle\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&d\left\langle M_t^{(1)}\right\rangle\\
&= \left\langle\left[f'(S(t))\Lambda^{\epsilon} f'(S(t))\right]\left\langle \Sigma_K^{1/2}\nabla S(t),\ \Sigma_K^{1/2}\nabla T_n(t)\right\rangle, \left\langle \Sigma_K^{1/2}\nabla S(t),\ \Sigma_K^{1/2}\nabla T_n(t)\right\rangle\right\rangle dt\\
&= \iint_{(x,y)\in\mathcal{O}\times\mathcal{O}} f'(S(t,x))\Lambda^{\epsilon}(x,y)\,f'(S(t,y))\\
&\quad\times\left\langle \Sigma_K^{1/2}\nabla S(t,y),\ \Sigma_K^{1/2}\nabla T_n(t,y)\right\rangle\left\langle \Sigma_K^{1/2}\nabla S(t,x),\ \Sigma_K^{1/2}\nabla T_n(t,x)\right\rangle\,d\mathcal{L}^n y\,d\mathcal{L}^n x\,dt
\end{aligned}$$

$$
\begin{aligned}
&\leq\ C \iint_{(x,y)} |\nabla S(t,y)|\,|\nabla T_n(t,y)|\,|\nabla S(t,x)|\,|\nabla T_n(t,x)|\,d\mathcal{L}^n x\,d\mathcal{L}^n y\,dt \\
&\leq\ C\,\|\nabla S(t)\|_{L^2}^2\,\|\nabla T_n(t)\|_{L^2}^2\ dt.
\end{aligned}
$$

(In the process, we have made use of (B.37).)

Thus,

$$
\left\langle M_t^{(1)} \right\rangle \leq C \int_0^t \|\nabla S(r)\|_{L^2}^2\,\|\nabla T_n(r)\|_{L^2}^2\ dr. \tag{4.32}
$$

4.3.7. Computation of $d\left\langle M_t^{(2)}\right\rangle$**.** For simplicity, we will omit the $\Sigma_K^{1/2}$ factor, which merely introduces a bounded transform of the function space with a bounded inverse. In this case,

$$
d\left\langle M_t^{(2)}\right\rangle = \left\langle f(S(t))\,d\nabla W_t^{\epsilon},\ \nabla T_n(t)\right\rangle = \sum_p \left\langle f(S(t))\,d\partial_p W_t^{\epsilon},\ \partial_p T_n(t)\right\rangle\,dt.
$$

Thus,

$$
\begin{aligned}
d\left\langle M_t^{(2)}\right\rangle &= \sum_p \left\langle f(S(t))\left[\partial_p\varphi_\epsilon \Lambda \partial_p\varphi_\epsilon\right] f(S(t))\partial_p T_n(t),\ \partial_p T_n(t)\right\rangle\,dt \\
&= \sum_p \iint_{(x,y)} f(S(t,x))\left[\partial_p\varphi_\epsilon \Lambda \partial_p\varphi_\epsilon\right](x,y) \\
&\qquad \times\ f(S(t,y))\partial_p T_n(t,y)\partial_p T_n(t,x)\,d\mathcal{L}^n x\,d\mathcal{L}^n y\,dt \\
&\leq \frac{C}{\epsilon^{2n+2}} \iint_{(x,y)} |\nabla T_n(t,y)|\,|\nabla T_n(t,x)|\ d\mathcal{L}^n y\,d\mathcal{L}^n x\,dt \qquad \text{(by (B.38))} \\
&\leq \frac{C}{\epsilon^{2n+2}}\,\|\nabla T_n(t)\|_{L^2}^2\,.
\end{aligned}
$$

Hence,

$$
\left\langle M_t^{(2)}\right\rangle = \frac{C}{\epsilon^{2n+2}} \int_0^t \|\nabla T_n(r)\|_{L^2}^2\ dr. \tag{4.33}
$$

4.3.8. Computation of $d\,\mathrm{Tr}\left\langle\!\left\langle \Sigma_K^{1/2}\nabla T_n(t)\right\rangle\!\right\rangle$**.** The above equals

$$
\begin{aligned}
&\sum_{ij}^{n} d\left\langle c_n^i(t),\ c_n^j(t)\right\rangle \left\langle \Sigma_K^{1/2}\nabla u_i,\ \Sigma_K^{1/2}\nabla u_j\right\rangle \\
&= \sum_{ij}\left(f(S(t))\Lambda^{\epsilon} f(S(t))c_K u_i,\ c_K u_j\right)\left\langle \Sigma_K^{1/2}\nabla u_i,\ \Sigma_K^{1/2}\nabla u_j\right\rangle\,dt \\
&= \sum_{i}\left(f(S(t))\Lambda^{\epsilon} f(S(t))c_K u_i,\ c_K u_i\right)\left\langle \Sigma_K^{1/2}\nabla u_i,\ \Sigma_K^{1/2}\nabla u_i\right\rangle\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= \iint_{(x,y)\in\mathcal{O}\times\mathcal{O}} f(S(t,x))\Lambda^\epsilon(x,y) f(S(t,y))c_K(y)u_i(y)c_K(x)u_i(x)\,d\mathcal{L}^n y\, d\mathcal{L}^n x\\
&\quad \times \left\langle \Sigma_K^{1/2}\nabla u_i,\ \Sigma_K^{1/2}\nabla u_i\right\rangle dt\\
&= \int_x c_K(x)u_i(x) f(S(t,x)) \left\{\int_y \Lambda^\epsilon(x,y) f(S(t,y))c_K(y)u_i(y)\,d\mathcal{L}^n y\right\}\\
&\quad \times \left\langle \Sigma_K^{1/2}\nabla u_i,\ \Sigma_K^{1/2}\nabla u_i\right\rangle d\mathcal{L}^n x\, dt\\
&= \int_x c_K(x)u_i(x) f(S(t,x)) \int_y \left\langle \Sigma_K^{1/2}(y)\nabla_y\left(\Lambda^\epsilon(x,y) f(S(t,y))\right),\ \Sigma_K^{1/2}(y)\nabla_y u_i(y)\right\rangle\\
&\quad d\mathcal{L}^n y\, d\mathcal{L}^n x\, dt\\
&= \int_x\int_y \langle \Sigma_K^{1/2}(x)\Sigma_K^{1/2}(y)\nabla_y\nabla_x\left[f(S(t,x))\Lambda^\epsilon(x,y) f(S(t,y))\right]\Sigma_K^{1/2}(y)\nabla_y u_i(y),\\
&\quad \Sigma_K^{1/2}(x)\nabla_x u_i(x)\rangle\, d\mathcal{L}^n y\, d\mathcal{L}^n x \times \left\langle \Sigma_K^{1/2}\nabla u_i,\ \Sigma_K^{1/2}\nabla u_i\right\rangle^{-1} dt. \qquad (4.34)
\end{aligned}
$$

(Note the use of (4.21) in the above computations.)

To continue, we set $\boldsymbol{U}_i = \Sigma_K^{1/2}\nabla u_i / \left\| \Sigma_K^{1/2}\nabla u_i \right\|_{L^2}$. Then, $\left\{\boldsymbol{U}_i = \left(\boldsymbol{U}_i^1, \ldots, \boldsymbol{U}_i^n\right)\right\}_{i\geq 1}$ forms a sequence of **orthonormal** vectors for the **vector-valued** Hilbert space,

$$
L^2_{(n)}(\mathcal{O}) = L^2(\mathcal{O}) \times \cdots \times L^2(\mathcal{O}) \qquad (n\text{-fold product}).
$$

Set $K(t,x,y) = f(S(t,x))\Lambda^\epsilon(x,y) f(S(t,y))$. Then the integrand in (4.34) can be written as

$$
\begin{aligned}
&\Sigma_K^{1/2}(x)\Sigma_K^{1/2}(y)\sum_q \partial_{x_q}\left(\sum_p \partial_{y_p} K(t,x,y)\boldsymbol{U}_i^p(y)\right)\boldsymbol{U}_i^q(x)\\
&\qquad = \Sigma_K^{1/2}(x)\Sigma_K^{1/2}(y)\sum_{pq}\left[\partial_{x_q}\partial_{y_p} K(t,x,y)\right]\boldsymbol{U}_i^p(y)\boldsymbol{U}_i^q(x).
\end{aligned}
$$

Let $\boldsymbol{A}_{pq}(t,x,y) = \Sigma_K^{1/2}(x)\Sigma_K^{1/2}(y)\partial_{x_q}\partial_{y_p} K(t,x,y)$. (4.34) is the same as

$$
\sum_i^n \iint_{(x,y)} \sum_{pq} \boldsymbol{A}_{pq}(t,x,y)\boldsymbol{U}_i^p(y)\boldsymbol{U}_i^q(y)\, d\mathcal{L}^n y\, d\mathcal{L}^n x\, dt.
$$

We can now make use of the formula for vector valued trace class operators as in Proposition B.4.2:

$$
\begin{aligned}
&d\mathrm{Tr}\left\langle\!\left\langle \Sigma_K^{1/2}\nabla T_n(t)\right\rangle\!\right\rangle\\
&\ \leq\ \mathrm{Tr}\,\boldsymbol{A}(t)\,dt\\
&\ =\ \int_x \sum_p \left(\boldsymbol{A}_{pp}(t,x,x)\right) d\mathcal{L}^n x\, dt\\
&\ =\ \int_x \left\{\sum_p \Sigma_K(x)\partial_{x_p}\partial_{y_p}\left[f(S(t,x))\Lambda^\epsilon(x,y) f(S(t,y))\right]\Big|_{x=y}\right\} d\mathcal{L}^n x\, dt.
\end{aligned}
$$

The term in the bracket equals

$$\begin{aligned} & f'(S(t,x))\left(\partial_{x_p} S(t,x)\right)\left(\partial_{y_p}\Lambda^{\epsilon}(x,y)\right) f(S(t,y)) \\ & + \ f(S(t,x))\left(\partial_{x_p}\Lambda^{\epsilon}(x,y)\right) f'(S(t,y))\left(\partial_{y_p} S(t,y)\right) \\ & + \ f(S(t,x))\left(\partial_{x_p}\partial_{y_p}\Lambda^{\epsilon}(x,y)\right) f(S(t,y)) \\ & + \ f'(S(t,x))\left(\partial_{x_p} S(t,x)\right)\Lambda^{\epsilon}(x,y) f'(S(t,y))\left(\partial_{y_p} S(t,y)\right). \end{aligned}$$

Making use of the computations in Section B.5.3 and their simple variants,

$$d\mathrm{Tr}\left\langle\!\left\langle \Sigma_K^{1/2}\nabla T_n(t)\right\rangle\!\right\rangle \le C\left\{\int_x |\nabla S(t,x)|^2 \ d\mathcal{L}^n x + \frac{1}{\epsilon^{n+1}}\int_x |\nabla S(t,x)| \ d\mathcal{L}^n x + \frac{1}{\epsilon^{2n+2}}\right\} dt.$$

Absorbing the middle term, we get

$$\mathrm{Tr}\left\langle\!\left\langle \Sigma_K^{1/2}\nabla T_n(t)\right\rangle\!\right\rangle \le C\left\{\int_0^t \|\nabla S(r)\|_{L^2}^2 \ dr + \frac{t}{\epsilon^{2n+2}}\right\}. \tag{4.35}$$

4.3.9. Combination of Results. Substituting (4.32), (4.33), and (4.35) into (4.31), we get (recall that we only care about $0 \le t \le \triangle t$)

$$\begin{aligned} E\left\{\sup_{r\in[0,t]} \|\nabla T_n(r)\|_{L^2}^8\right\} \ \le \ C\Bigg\{ & E\,\|\nabla T_n(0)\|_{L^2}^8 + \triangle t\int_0^t E\,\|\nabla S(r)\|_{L^2}^4\,\|\nabla T_n(r)\|_{L^2}^4 \ dr \\ & + \frac{\triangle t}{\epsilon^{4n+4}}\int_0^t E\,\|\nabla T_n(r)\|_{L^2}^4 \ dr \\ & + \triangle t^3\int_0^t E\,\|\nabla S(r)\|_{L^2}^8 \ dr + \frac{\triangle t^4}{\epsilon^{8n+8}}\Bigg\}. \end{aligned}$$

Let θ be a small positive number to be specified later. Recall the inequalities $ab \le \frac{1}{\theta}a^2 + \theta b^2$ and $a \le a^2 + 1$. Then the above can be bounded by

$$\begin{aligned} & C\Bigg\{E\,\|\nabla T_n(0)\|_{L^2}^8 + \frac{\triangle t}{\theta}\int_0^t E\,\|\nabla S(r)\|_{L^2}^8 \ dr + \theta\triangle t\int_0^t E\,\|\nabla T_n(r)\|_{L^2}^8 \ dr \\ & \quad + \frac{\triangle t}{\epsilon^{4n+4}}\int_0^t \left(E\,\|\nabla T_n(r)\|_{L^2}^8 + 1\right) dr + \triangle t^3\int_0^t E\,\|\nabla S(r)\|_{L^2}^8 \ dr + \frac{\triangle t^4}{\epsilon^{8n+8}}\Bigg\} \\ & \le \ C\Bigg\{E\,\|\nabla T_n(0)\|_{L^2}^8 + \frac{\triangle t}{\epsilon^{8n+8}} + \left(\frac{\triangle t}{\theta} + \triangle t^3\right)\int_0^t E\,\|\nabla S(r)\|_{L^2}^8 \ dr \\ & \quad + \left(\theta\triangle t + \frac{\triangle t}{\epsilon^{4n+4}}\right)\int_0^t E\,\|\nabla T_n(r)\|_{L^2}^8 \ dr\Bigg\}. \end{aligned} \tag{4.36}$$

Let $D_n(t) = E\left\{\sup_{\lambda\in[0,t]} \|\nabla T_n(\lambda)\|_{L^2}^8\right\}, \ \ D_S(t) = E\left\{\sup_{\lambda\in[0,t]} \|\nabla S(\lambda)\|_{L^2}^8\right\}$. Since we have $E\,\|\nabla T_n(0)\|_{L^2}^8 \ \le \ C\triangle t^{-8\alpha}$ because of the property of the special basis ((3) in Section 4.3.3), we can write (4.36) as

$$D_n(t) \le C\left\{\frac{1}{\epsilon^{8n+8}\triangle t^{8\alpha}} + \left(\frac{\triangle t}{\theta} + \triangle t^3\right)\int_0^t D_S(r)\,dr + \left(\theta\triangle t^2 + \frac{\triangle t^2}{\epsilon^{4n+4}}\right) D_n(t)\right\}.$$

Choose $\theta = 1/(4C\triangle t^2)$ and $\epsilon^{4n+4} \geq 4C\triangle t^2$. (Recall the remark at the beginning of Section 4.3.) Then,

$$D_n(t) \leq \frac{C}{\epsilon^{8n+8}\triangle t^{8\alpha}} + C\int_0^t D_S(r)\,dr.$$

Let $n \longrightarrow \infty$. We have

$$\begin{aligned} E\left\{\sup_{\lambda\in[0,t]} \|\nabla T(\lambda)\|_{L^2}^8\right\} &\leq \liminf_n D_n(t) \\ &\leq \frac{C}{\epsilon^{8n+8}\triangle t^{8\alpha}} + C\int_0^t E\left\{\sup_{\lambda\in[0,r]} \|\nabla S(\lambda)\|_{L^2}^8\right\} dr. \qquad (4.37) \end{aligned}$$

4.3.10. Proof of Theorem 4.3.1. Recall that the solution of (4.3) is obtained by Picard's Iteration (Appendix C.1.5).

Set S in the previous section to be the solution in the $(n-1)$-th iteration, T^{n-1}. Let

$$D^{(n)}(t) = E\left\{\sup_{\lambda\in[0,t]} \left\|\nabla T^n(\lambda)\right\|_{L^2}^8\right\}.$$

Let also $A = \dfrac{C}{\epsilon^{8n+8}\triangle t^{8\alpha}}$, $\quad B = C$. Proceed inductively from (4.37),

$$\begin{aligned} D^{(n)}(t) &\leq A + B\int_0^t D^{(n-1)}(t_1)\,dt_1 \leq A + ABt + B^2\int_0^t\int_0^{t_1} D^{(n-2)}(t_2)\,dt_2\,dt_1 \\ &\;\;\vdots \\ &\leq A + ABt + A\frac{B^2t^2}{2} + \cdots + A\frac{B^nt^n}{n!} + \frac{B^{n+1}t^{n+1}}{(n+1)!}. \end{aligned}$$

(We have used the identity as in (C.15).) Taking the limit $n \longrightarrow \infty$,

$$\limsup_n D^{(n)}(t) \leq A\exp(Bt) \leq \frac{C}{\epsilon^{8n+8}\triangle t^{8\alpha}}.$$

Finally,

$$E\left\{\sup_{r\in[0,\triangle t]} \|\nabla T(\lambda)\|_{L^2}^8\right\} \leq \liminf_n D^{(n)}(t) \leq \frac{C}{\epsilon^{8n+8}\triangle t^{8\alpha}}. \qquad (4.38)$$

That is exactly the statement of Theorem 4.3.1.

4.4. Continuity in Time Estimate for the Temperature Field

4.4.1. Theorem. *If $\{T(t)\}_{t\geq 0}$ solves (4.3) with $\|\nabla T_0\|_{L^2} \leq C\triangle t^{-\alpha}$, then*

$$E\left\{\sup_{t\in[0,\triangle t]} \|T(t) - T_0\|_{L^2}^8\right\} \leq \frac{C\triangle t^2}{\epsilon^{8n+8}}. \qquad (4.39)$$

Remark. What is crucial here is the positive exponent of the $\triangle t$. Its exact value is not important. It can be improved to 3 by estimating in an iterative way.

Proof. The proof makes use of the temperature gradient estimates (4.17).

From the definition of the solution of (4.3),

$$T(t) - T_0 = \int_0^t AT(s)\,ds + \int_0^t f(T(s))\,dW_s^\epsilon.$$

By Ito's Formula,

$$\begin{aligned}
&\|T(t) - T_0\|_H^2 \\
&= 2\int_0^t \langle AT(s),\ T(s) - T_0\rangle\ ds + 2\int_0^t \left\langle f(T(s))\,dW_s^\epsilon,\ T(s) - T_0\right\rangle \\
&\quad + \int_0^t \mathrm{Tr}\left[f(T(s))\Lambda^\epsilon f(T(s))\right]\,ds \\
&= -2\int_0^t \left\langle \Sigma_K^{1/2}\nabla T(s),\ \Sigma_K^{1/2}\nabla T(s)\right\rangle ds + 2\int_0^t \left\langle \Sigma_K^{1/2}\nabla T_0,\ \Sigma_K^{1/2}\nabla T(s)\right\rangle ds \\
&\quad + 2\int_0^t \left\langle f(T(s))\,dW_s^\epsilon,\ T(s) - T_0\right\rangle + \int_0^t \mathrm{Tr}\left[f(T(s))\Lambda^\epsilon f(T(s))\right]\,ds \\
&\Longrightarrow \sup_{\lambda\in[0,t]} \|T(\lambda) - T_0\|_H^2 \\
&\leq C\left\{\int_0^{\triangle t} \|\nabla T_0\|\,\|\nabla T(s)\|\ ds + \sup_{\lambda\in[0,t]}\left|\int_0^\lambda \left\langle f(T(s))\,dW_s^\epsilon,\ T(s) - T_0\right\rangle\right| + \triangle t\right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sup_{\lambda\in[0,t]} \|T(\lambda) - T_0\|_{L^2}^8 &\leq C\triangle t^4 + C\left(\int_0^{\triangle t} \|\nabla T_0\|\,\|\nabla T(s)\|\ ds\right)^4 \\
&\quad + C\sup_{\lambda\in[0,t]}\left|\int_0^\lambda \left\langle f(T(s))\,dW_s^\epsilon,\ T(s) - T_0\right\rangle\right|^4 \\
&\leq C\triangle t^4 + C\triangle t^4\left\{\sup_{t\in[0,\triangle t]} \|\nabla T_0\|^4\,\|\nabla T(s)\|^4\right\} \\
&\quad + C\sup_{\lambda\in[0,t]}\left|\int_0^\lambda \left\langle f(T(s))\,dW_s^\epsilon,\ T(s) - T_0\right\rangle\right|^4 \\
&\leq C\triangle t^4\left\{1 + \sup_{s\in[0,\triangle t]}\left(\|\nabla T_0\|^8 + \|\nabla T(s)\|^8\right)\right\} \\
&\quad + C\sup_{\lambda\in[0,t]}\left|\int_0^\lambda \left\langle f(T(s))\,dW_s^\epsilon,\ T(s) - T_0\right\rangle\right|^4.
\end{aligned}$$

Upon taking expectation, using (4.17) and Burkholder's Inequality,

$$
\begin{aligned}
&E\left\{\sup_{\lambda\in[0,t]}\|T(\lambda)-T_0\|_{L^2}^8\right\}\\
&\le\ C\triangle t^4\left\{\frac{1}{\triangle t^{8\alpha}}+\frac{1}{\triangle t^{8\alpha}\epsilon^{8n+8}}\right\}\\
&\qquad +CE\left(\int_0^t\langle f(T(s))\Lambda^\epsilon f(T(s))(T(s)-T_0),\ T(s)-T_0\rangle\,ds\right)^2\\
&\le\ C\frac{\triangle t^{4-8\alpha}}{\epsilon^{8n+8}}+C\triangle tE\int_0^t\|T(s)-T_0\|_{L^2}^4\,ds\\
&\le\ \left(C\frac{\triangle t^{4-8\alpha}}{\epsilon^{8n+8}}+\triangle t^2\right)+C\triangle tE\int_0^t\|T(s)-T_0\|_{L^2}^8\,ds,\qquad (a\le a^2+1)\\
&\le\ C\frac{\triangle t^2}{\epsilon^{8n+8}}+\triangle t\int_0^t E\left\{\sup_{\lambda\in[0,s]}\|T(\lambda)-T_0\|_{L^2}^8\right\}ds.\qquad \text{(provided } \alpha \text{ is small enough.)}
\end{aligned}
$$

By Gronwall's Inequality, we have the desired result:

$$
E\left\{\sup_{\lambda\in[0,\triangle t]}\|T(\lambda)-T_0\|_{L^2}^8\right\}\le\frac{C\triangle t^2}{\epsilon^{8n+8}}. \tag{4.40}
$$

□

5. Global Energy Estimates

A crucial step in proving the main theorem is the global in time estimation of the energy $\mathcal{E}$. It will be used to prove the compactness results in Section 6.3. Two facts in this aspect are that the energy $\mathcal{E}$ is always **decreasing** after each minimization, and its changes during the heat flow steps can be estimated by means of **stochastic integrals**. The final result then follows from martingale inequalities.

5.1. Ito's Formula for the Bulk Energy (Fixed Crystal)

To achieve our goal, we make use of an extension of Ito's Formula (B.29) for the norm square of a process which takes values in a Hilbert space H. It is this formula that takes into account the **statistical cancellation** property of **white noise** type driving force.

The main result we need is the following Theorem. It was proved in [Par]. In [Yip], a simpler proof is given making use of the **compact embedding** between our function spaces.

5.1.1. Theorem (Ito's Formula for Energy—Fixed Crystal). *Let F be the bulk energy functional as in Section 3.1 and $T(t)$ be the solution of the heat equation (4.2) (or (4.3),* *to be exact*[15]*). Then,*

$$\int_{\mathcal{O}} c_K F(T(t))\, d\mathcal{L}^n + \int_0^t \int_{\mathcal{O}} F''(T(s)) \langle \Sigma_K \nabla T(s),\ \nabla T(s)\rangle\ d\mathcal{L}^n\, ds$$
$$= \int_{\mathcal{O}} c_K F(T(0))\, d\mathcal{L}^n + \int_0^t \int_{\mathcal{O}} c_K F'(T(s))\, f(T(s))\, dW_s\, d\mathcal{L}^n$$
$$+\frac{1}{2} \int_0^t \mathrm{Tr}\left[c_K F''(T(s)\, f(T(s))\Lambda f(T(s)))\right]\, ds. \tag{5.1}$$

Remark. Here, $F''(T(s))$ and $f(T(s))$ are regarded as multiplicative operators on spatial functions (Appendix B.5.1).

5.2. Global Energy Estimates (Varying Crystals)

Now we proceed to estimate the energy globally, taking into consideration the minimization steps. Recall the notation $\left(K_i^-, T_i^-, U_i^-, Q_i^-\right)$ as the state at t_i^- (right before minimization at t_i) and $\left(K_i^+, T_i^+, U_i^+, Q_i^+\right)$ the state at t_i^+ (right after minimization at t_i). Note that $T_i^+ \longrightarrow T_{i+1}^-$ by the heat flow process and $K_{i+1}^- = K_i^+$. By the measurable selection (Section 3.3), $\left(K_i^+, T_i^+, U_i^+, Q_i^+\right)$, is $\mathcal{F}_{t_i}$-measurable. This allows us to use the formulas concerning stochastic integrations.

Let $\mathcal{E}(t) = \mathcal{E}\,(K(t), T(t))$. Since $\mathcal{E}(t_i^+) \le \mathcal{E}(t_i^-)$ by the minimization procedure, we deduce

$$\begin{aligned}
\mathcal{E}(t_n^+) &\le \mathcal{E}(t_n^-) \\
&= \mathcal{E}(t_n^-) - \mathcal{E}(t_{n-1}^+) + \mathcal{E}(t_{n-1}^+) \\
&\ \vdots \\
&\le \mathcal{E}(t_0^+) + \sum_{i=1}^n \mathcal{E}(t_i^-) - \mathcal{E}(t_{i-1}^+) \\
&\le \mathcal{E}(t_0^-) + \sum_{i=1}^n \mathcal{E}(t_i^-) - \mathcal{E}(t_{i-1}^+).
\end{aligned} \tag{5.2}$$

Making use of Theorem 5.1.1, we arrive at the following.

5.2.1. Proposition (Energy Bound—Varying Crystals).

$$\mathcal{E}(t) + \int_0^t \int_{\mathcal{O}} F''(T) \langle \Sigma_K \nabla T,\ \nabla T\rangle\ d\mathcal{L}^n\, ds$$
$$\le\ \mathcal{E}(0) + \int_0^t \int_{\mathcal{O}} c_K F'(T)\, f(T)\, dW_s\, d\mathcal{L}^n + \frac{1}{2}\int_0^t \mathrm{Tr}\left[c_K F''(T)\, f(T)\Lambda f(T)\right]\, ds.$$

This leads to the following global energy estimates.

[15] The smoothing has no effect in the global energy estimate.

5.2.2. Theorem (Global Energy Estimates). *For all positive integers m,*

$$E\left\{\sup_{t\in[0,1]}\Phi(\partial K_t)^m\right\}, \qquad E\left\{\sup_{t\in[0,1]}\|T_t\|_{L^2}^m+\|U_t\|_{L^2}^m\right\} \qquad \textit{and}$$
$$E\left[\left(\int_0^1 \|\nabla T_s\|_{L^2}^2+\|\nabla U_s\|_{L^2}^2\, ds\right)^m\right] < C_m. \tag{5.3}$$

Proof. Let $M_t = \int_0^t\int_{\mathcal{O}} c_K F'(T)\, f(T)\, dW_s\, d\mathcal{L}^n = \int_0^t \left\langle c_K F'(T)\, f(T),\ dW_s\right\rangle$. We then write Proposition 5.2.1 as:

$$\mathcal{E}(t) + \int_0^t\int_{\mathcal{O}} F''(T)\,\langle \Sigma_K\nabla T,\ \nabla T\rangle\, d\mathcal{L}^n\, ds$$
$$\leq \mathcal{E}(0) + M_t + \frac{1}{2}\int_0^t \mathrm{Tr}\left[c_K F''(T)\, f(T)\Lambda f(T)\right] ds. \tag{5.4}$$

By the fact that $F'(T)\, f(T)$ and $F''(T)\, f(T)$ are **bounded**, we have

$$\langle M\rangle_t = \int_0^t \left\langle \Lambda c_K F'(T)\, f(T),\ c_K F'(T)\, f(T)\right\rangle ds \leq Ct,$$
$$\frac{1}{2}\int_0^t \mathrm{Tr}\left[c_K F''(T)\, f(T)\Lambda f(T)\right] ds \leq Ct.$$

Applying Burkholder's Inequality (B.4) to M_t,

$$E\left(\sup_{t\in[0,1]}|M_t|^{2m}\right) \leq C_m E\left[\langle M\rangle_1^m\right], \qquad (m>0).$$

Then, starting from (5.4), upon taking power and expectation, we arrive at

- $E\left\{\sup_{t\in[0,1]}\Phi(\partial K_t)^m\right\} \leq C_m;$
- $E\left\{\sup_{t\in[0,1]}\left(\int_{\mathcal{O}} c_K F(T_t)\, d\mathcal{L}^n\right)^m\right\} \leq C_m;$
- $E\left[\left(\int_0^1\int_{\mathcal{O}} F''(T)\,\langle\Sigma_K\nabla T,\ \nabla T\rangle\, d\mathcal{L}^n\, ds\right)^m\right] \leq C_m.$

By the growth form for F (Section 3.1),

$$T^2+U^2 \leq F(T)+C, \qquad C_1+C_2U^4 \leq F''(T)$$
$$\Longrightarrow\ |\nabla T|^2+|\nabla U|^2 \leq F''(T)\,\langle\Sigma_K\nabla T,\ \nabla T\rangle + C;$$

the asserted result follows. □

5.3. Hölder Continuity Estimate for the Heat Evolution

In this section, we are going to show that the heat evolves "Hölder continuously in time" in the discrete scheme. Precisely, we will establish the following:

5.3.1. Theorem (Hölder Continuity in Time of Heat Evolution). *We can decompose $Q(t)$ as $Q'(t) + R'(t)$ such that, for all positive integers m and for all $0 \le s < t \le 1$,*

$$E \left\| Q'(t) - Q'(s) \right\|_{\sim}^{2m} \le C_m \left| t - s \right|^m, \tag{5.5}$$

$$E \sup_{t\in[0,1]} \left\| R'(t) \right\|_{\sim}^{2m} \le C_m \triangle t^{2m\alpha}. \tag{5.6}$$

Precisely, $Q'(t)$ will describe the evolutions during the heat flow and $R'(t)$ the changes during the minimizations.[16]

Recall that the space of heat distributions is endowed with the **Modified Monge-Kantorovich Norm** (Section 2.1.3),

$$\| Q - P \|_{\sim} = \| Q - P \|_* + \left| \overline{Q - P} \right|.$$

The proof of the Theorem goes by careful estimations of the corresponding terms.

5.3.2. Decomposition for $Q(t)$. Let Q_i^- and Q_i^+ be the heat distributions at time t_i^- (right before the minimization) and t_i^+ (right after). Then Q_i^+ is changed to Q_{i+1}^- by the heat flow process.

Consider (for simplicity, $0 \le t_q \le 1$)

$$\begin{aligned} Q_q^+ &= Q_q^+ - Q_q^- + Q_q^- - Q_{q-1}^+ + Q_{q-1}^+ - Q_{q-1}^- + Q_{q-1}^- - Q_{q-2}^+ \\ &\quad \cdots\cdots + Q_1^+ - Q_1^- + Q_1^- - Q_0^+ + Q_0^+ \\ &= \sum_{i=1}^{q} Q_i^+ - Q_i^- + \sum_{i=1}^{q} Q_i^- - Q_{i-1}^+ + Q_0^+. \end{aligned}$$

Set

$$R'(t) = \sum_{0<t_i\le t} Q_i^+ - Q_i^-, \tag{5.7}$$

$$Q'(t) = Q(t) - R'(t) = \sum_{0<t_i\le t} Q_i^- - Q_{i-1}^+ + Q_0^+. \tag{5.8}$$

We will estimate $R'(t)$ and $Q'(t)$ separately. The theorem will then follow.

5.3.3. Lemma (Estimates for $R'(t)$—Minimization Step). *For all positive integers m,*

$$E \sup_{t\in[0,1]} \left\| R'(t) \right\|_{\sim}^{2m} \le C \triangle t^{2m\alpha}. \tag{5.9}$$

[16] Because of the jumps in the heat distribution during the minimization steps, the heat evolution is not continuous in time, but rather **right continuous with left-hand limit**, the so-called **càdlàg** process.

Proof. The terms in $R'(t)$ measure the changes of the heat during the minimization steps.

By the definition of the minimization procedure, we have

$$\begin{aligned}
&\mathcal{E}(t_i^+) + \frac{1}{\triangle t^\alpha} \left\| Q_i^+ - Q_i^- \right\|_* \leq \mathcal{E}(t_i^-) \\
&\quad \Longrightarrow \quad \left\| Q_i^+ - Q_i^- \right\|_* \leq \triangle t^\alpha \left\{ \mathcal{E}(t_i^-) - \mathcal{E}(t_i^+) \right\}.
\end{aligned}$$

Note that the heat contents of Q_i^+ and Q_i^- are the same. Hence,

$$\begin{aligned}
&\left\| R'(t) \right\|_\sim \\
&\leq \sum_{i=0}^{q} \left\| Q_i^+ - Q_i^- \right\|_* \\
&\leq \triangle t^\alpha \sum_{i=0}^{q} \mathcal{E}(t_i^-) - \mathcal{E}(t_i^+) \\
&\leq \triangle t^\alpha \left\{ \sum_{i=0}^{q} \mathcal{E}(t_i^-) - \mathcal{E}(t_{i-1}^+) + \sum_{i=0}^{q} \mathcal{E}(t_{i-1}^+) - \mathcal{E}(t_{i-1}^+) \right\} \\
&\leq \triangle t^\alpha \left\{ \mathcal{E}(t_0^+) - \mathcal{E}(t_q^+) \right\} + \triangle t^\alpha \left\{ \sum_{i=0}^{q} \int_{t_{i-1}}^{t_i} \int_{\mathcal{O}} c_K F'(T) f(T)\, dW_r\, d\mathcal{L}^n \right. \\
&\qquad \left. + \frac{1}{2} \int_{t_{i-1}}^{t_i} \operatorname{Tr}\left[c_K F''(T) f(T) \Lambda f(T) \right] dr \right\} \\
&\leq \triangle t^\alpha \left\{ \mathcal{E}(t_0^-) + \int_{t_0}^{t_q} \left\langle c_K F'(T) f(T),\, dW_r \right\rangle + \frac{1}{2} \int_{t_p}^{t_q} \operatorname{Tr}\left[c_K F''(T) f(T) \Lambda f(T) \right] dr \right\} \\
&\leq \triangle t^\alpha \left\{ \mathcal{E}(0) + \int_{0}^{t_q} \left\langle c_K F'(T) f(T),\, dW_r \right\rangle + \frac{1}{2} \int_{0}^{t_q} \operatorname{Tr}\left[c_K F''(T) f(T) \Lambda f(T) \right] dr \right\}.
\end{aligned}$$

The lemma follows from taking powers and then invoking Burkholder's Inequality. It is quite similar to the proof of Theorem 5.2.2. □

5.3.4. Lemma (Estimates for $Q'(t) - Q'(s)$—Heat Flow Process). *For all positive integers m and for all* $0 \leq s < t \leq 1$,

$$E \left\| Q'(t) - Q'(s) \right\|_\sim^{2m} \leq C\, |t-s|^m. \tag{5.10}$$

Proof. The terms in $Q'(t) - Q'(s)$ measure the changes of heat caused by the following diffusion equation:

$$\begin{aligned}
dQ &= \operatorname{div}(\Sigma_K \nabla T)\, dt + c_K f(T)\, dW_t; \\
\text{i.e.,} \qquad Q'(t) - Q'(s) &= \int_s^t \operatorname{div}(\Sigma_K \nabla T)\, dr + \int_s^t c_K f(T)\, dW_r.
\end{aligned}$$

Without loss of generality, assume $s = t_p$ and $t = t_q$. Now,

$$\left\| Q'(t) - Q'(s) \right\|_{\sim} = \left| \sum_{i=p+1}^{q} \overline{Q_i^- - Q_{i-1}^+} \right| + \left\| \sum_{i=p+1}^{q} Q_i^- - Q_{i-1}^+ \right\|_* .$$

Step I: Estimation for the Change of the Heat Content $\left| \sum_{i=p+1}^{q} \overline{Q_i^- - Q_{i-1}^+} \right|$**.** Consider (note that $\partial \mathcal{O} = \emptyset$.),

$$\begin{aligned}\int_{\mathcal{O}} \sum_{i=p+1}^{q} \left(Q_i^- - Q_{i-1}^+ \right) d\mathcal{L}^n &= \int_{t_p}^{t_q} \int_{\mathcal{O}} \operatorname{div} \left(\Sigma_K \nabla T \right) d\mathcal{L}^n \, dr + c_K f(T) \, dW_r \, d\mathcal{L}^n \\ &= \int_{t_p}^{t_q} \int_{\mathcal{O}} c_K f(T) \, dW_r \, d\mathcal{L}^n = \int_{t_p}^{t_q} \langle c_K f(T), \, dW_r \rangle .\end{aligned}$$

Hence,

$$\left| \sum_{i=p+1}^{q} \overline{\left(Q_i^- - Q_{i-1}^+ \right)} \right| \leq \left| \int_{t_p}^{t_q} \langle c_K f(T), \, dW_r \rangle \right|. \tag{5.11}$$

By Proposition 5.3.5, we get

$$E \left| \sum_{i=p+1}^{q} \overline{\left(Q_i^- - Q_{i-1}^+ \right)} \right|^{2m} \leq C_m \left| t - s \right|^m . \tag{5.12}$$

Step II: Estimation for the Monge-Kantorovich Norm $\left\| \sum_{i=p+1}^{q} Q_i^- - Q_{i-1}^+ \right\|_*$**.** Let $\operatorname{Lip} \varphi \leq 1$ and $\int_{\mathcal{O}} \varphi \, d\mathcal{L}^n = 0$. Consider:

$$\begin{aligned}&\int_{\mathcal{O}} \sum_{i=p+1}^{q} \left(Q_i^- - Q_{i-1}^+ \right) \varphi \, d\mathcal{L}^n \\ &= \int_{t_p}^{t_q} \int_{\mathcal{O}} \operatorname{div} \left(\Sigma_K \nabla T \right) \varphi \, d\mathcal{L}^n \, dr + \int_{t_p}^{t_q} \int_{\mathcal{O}} \varphi c_K f(T) \, dW_r \, d\mathcal{L}^n \\ &= - \int_{t_p}^{t_q} \int_{\mathcal{O}} \langle \Sigma_K \nabla T, \, \nabla \varphi \rangle \, d\mathcal{L}^n \, dr + \int_{t_p}^{t_q} \langle \varphi, \, c_K f(T) \, dW_r \rangle \\ &\leq C \left(\int_{t_p}^{t_q} \int_{\mathcal{O}} |\nabla \varphi|^2 \, d\mathcal{L}^n \, dr \right)^{1/2} \left(\int_{t_p}^{t_q} \int_{\mathcal{O}} |\nabla T|^2 \, d\mathcal{L}^n \, dr \right)^{1/2} \\ &\quad + \left| \int_{t_p}^{t_q} \langle \varphi, \, c_K f(T) \, dW_r \rangle \right| \\ &\leq C \left| t_q - t_p \right|^{1/2} \left(\int_{t_p}^{t_q} \int_{\mathcal{O}} |\nabla T|^2 \, d\mathcal{L}^n \, dr \right)^{1/2} + \left| \int_{t_p}^{t_q} \langle \varphi, \, c_K f(T) \, dW_r \rangle \right| .\end{aligned} \tag{5.13}$$

The second term of the above can be estimated as

$$\left| \int_{t_p}^{t_q} \langle \varphi,\ c_K\, f(T)\, dW_r \rangle \right| = \left| \left\langle \varphi,\ \int_{t_p}^{t_q} c_K\, f(T)\, dW_r \right\rangle \right| \le \|\varphi\|_{L^2} \left\| \int_{t_p}^{t_q} c_K\, f(T)\, dW_r \right\|_{L^2}.$$

However, since $\operatorname{Lip}\varphi \le 1$ and $\int_{\mathcal{O}} \varphi\, d\mathcal{L}^n = 0$, by Poincaré's Inequality, we have $\|\varphi\|_{L^2} \le C$. Hence, taking powers on both sides of (5.13) leads to

$$\left\| \sum_{i=p+1}^{q} Q_i^- - Q_{i-1}^+ \right\|_* \ \le\ C_m \left| t_q - t_p \right|^m \left(\int_{t_p}^{t_q} \|\nabla T\|_{L^2}^2\ ds \right)^{2m} + C_m \left\| \int_{t_p}^{t_q} c_K\, f(T)\, dW_s \right\|_{L^2}^{2m}. \tag{5.14}$$

The whole lemma then follows by the energy estimates Theorem 5.2.2 and the next result. (Its proof is elementary.) □

5.3.5. Proposition. *Let M_t be a continuous square integrable Hilbert space valued martingale with $d\,\langle\langle M \rangle\rangle_t = \Gamma_t\, dt$ and $\operatorname{Tr}\Gamma_t \le C$ (a deterministic number). Then, for all $m \ge 1$,*

$$E\left(\|M_t - M_s\|^{2m} \right) \le C_m\, |t-s|^m, \qquad 0 \le s \le t \le 1. \tag{5.15}$$

6. Convergent Subsequence

In this section, we are going to show the tightness of probability measures induced by the discrete scheme. The notion of convergence we use is the **weak convergence of probability measures** or **convergence in distribution** (see Appendix B.2).

There are several steps. We need some compactness properties of the function spaces. In addition, we will introduce a **stopped version** of the evolution. (This is an artifact due to the limitation of our compactness results.) Tightness follows from the global energy estimates. Finally, we will make use of an extended version of Skorokhod's Theorem to reformulate weak convergence in terms of **almost sure convergence on the same probability space**. This last part not only makes many later computations more transparent but also allows us to freely **compare the discrete and limiting evolutions**. This is crucial when we want to prove varifold convergence of the crystal positions by exploiting the fact that the discrete evolutions are obtained through minimization procedures.

6.1. Compactness Properties for Crystals, Temperatures, and Heats

The main idea for the compactness of the function spaces comes from the relationship

$$Q = c_K T \qquad \text{or} \qquad T = c_K^{-1} Q.$$

(In practice, they refer to $Q^N = c_{K^N} T^N$, where $N = 1/\triangle t$—the discretization level.) Note that c_K is a **discontinuous** function in the spatial variable. Now $\|T\|_{L^2}$, $\|1/T\|_{L^2}$, $\|\nabla T\|_{L^2}$, and $\|\nabla 1/T\|_{L^2}$ can be controlled by Theorem 5.2.2. Hence, if Q^N converges, so must c_{K^N}. But from Theorem 5.3.1, indeed we have Q^N converging to some limiting heat distribution (up to a subsequence).

First we describe the compactness results in the deterministic case. They are proved in [AW] Chapter 6 and reformulated by an anonymous referee for [AW].

6.1.1. Remark—Nontrivial Crystal Configurations. The compactness criterias developed here rely on the fact that the crystals are **nontrivial**, i.e., the domain is **not totally frozen** or **melted**. In this section, we will assume that there is a (small) positive number γ such that every crystal $K \in \mathcal{K}$ satisfies the condition

$$\gamma\,|\mathcal{O}| < \mathcal{L}^n(K) < (1-\gamma)\,|\mathcal{O}|\,. \tag{6.1}$$

Such a condition is to "force" the crystal to have some boundary inside the domain $\mathcal{O}$. This will then exclude the wild oscillations of the crystal between $K = \emptyset$ and $K = \mathcal{O}$ when we apply the following results.

The next proposition is the starting point of our compactness argument. However, we believe that it is not necessary to restrict to such nontrivial configurations if we impose the Dirichlet boundary condition for the temperature field (instead of the periodic boundary condition as in the present setting). This was suggested by one referee of this paper.

6.1.2. Proposition. [AW, Thm 6.1] *Suppose*

- *K and L are two crystals in $\mathcal{K}$ with $\mathcal{L}^n(K\triangle L) > 0$.*
- *Q is a single heat distribution in $\mathcal{Q}$.*
- *$T = c_K^{-1}Q$ and $S = c_L^{-1}Q$ are the corresponding temperature fields.*

Then, $\|\nabla T\|_{L^2} + \|\nabla 1/T\|_{L^2} + \|\nabla S\|_{L^2} + \|\nabla 1/S\|_{L^2} = \infty$.

6.1.3. Corollary (Close Crystal Positions). *[AW Cor. 6.2] Given any positive numbers ϵ (small) and M (large), there exists $\delta > 0$ such that, if*

- *P and Q are two heat distributions with $\|P - Q\|_\sim \le \delta$,*
- *K and L are two crystals in $\mathcal{K}$ with $\Phi(\partial K)$ and $\Phi(\partial L) \le M$,*
- *$T = c_K^{-1}P$ and $S = c_L^{-1}Q$ are the corresponding temperature fields,*
- *$\|T\|_{L^2} + \|1/T\|_{L^2} + \|S\|_{L^2} + \|1/S\|_{L^2} \le M$,*
- *$\|\nabla T\|_{L^2} + \|\nabla 1/T\|_{L^2} + \|\nabla S\|_{L^2} + \|\nabla 1/S\|_{L^2} \le M$,*

then $\mathcal{L}^n(K\triangle L) < \epsilon$.

6.1.4. Corollary (Close Temperature Fields). *[AW Cor. 6.3] Given any positive numbers ϵ (small) and M (large), there exists a $\delta > 0$ such that, if*

- *P and Q are two heat distributions with $\|P - Q\|_\sim \le \delta$,*
- *K and L are two crystals with $\mathcal{L}^n(K\triangle L) \le \delta$, $\Phi(\partial K)$, and $\Phi(\partial L) \le M$,*

- *$T = c_K^{-1}P$ and $S = c_L^{-1}Q$ are the corresponding temperature fields,*
- $\|T\|_{L^2} + \|1/T\|_{L^2} + \|S\|_{L^2} + \|1/S\|_{L^2} \le M$,
- $\|\nabla T\|_{L^2} + \|\nabla 1/T\|_{L^2} + \|\nabla S\|_{L^2} + \|\nabla 1/S\|_{L^2} \le M$,

then $\|T - S\|_{L^2} + \|1/T - 1/S\|_{L^2} \le \epsilon$.

From the above results, we can formulate **space time convergence**. This was suggested by an anonymous referee for [AW]. Such an approach makes the convergence scheme much more transparent and easier to be handled than the original arguments in [AW].

6.1.5. Theorem (Space Time Compactness for Function Space). *Given a sequence of time-evolving crystal positions, temperature fields, and the corresponding heat distributions $K_i(\cdot)$, $T_i(\cdot)$, $Q(\cdot) = c_{K_i(\cdot)}T_i(\cdot)$ such that, for all i,*

- $\sup_{t\in[0,1]} \Phi(\partial K_i(t)) \le C$,
- $\sup_{t\in[0,1]} \|T_i(t)\|_{L^2} + \|1/T_i(t)\|_{L^2} \le C$,
- $\int_0^1 \|\nabla T_i(t)\|_{L^2} + \|\nabla 1/T_i(t)\|_{L^2}\, dt \le C$,
- *$Q_i(t) \longrightarrow Q(t)$ uniformly in $t \in [0,1]$ in the modified Monge-Kantorovich norm,*

then there exists $K(\cdot)$ and $T(\cdot)$ such that

$$\int_0^1 \|K_i(t) - K(t)\|_{L^1}\, dt \longrightarrow 0, \tag{6.2}$$

$$\int_0^1 \|T_i(t) - T(t)\|_{L^2}^2 + \|1/T_i(t) - 1/T(t)\|_{L^2}^2\, dt \longrightarrow 0, \tag{6.3}$$

and $Q(\cdot) = c_{K(\cdot)}T(\cdot)$.

Proof. For the first statement, let ϵ be an arbitrary positive number. Choose M to be such that $C/M \le \epsilon$. Let δ be the number gotten by Corollary 6.1.3 for such an ϵ and M. Next choose N such that, for all i, $j > N$,

$$\left\| Q_i(t) - Q_j(t) \right\|_\sim \le \delta, \qquad \forall t \in [0,1].$$

Let G_i and G_j be the set of **good times**,

$$\begin{aligned} G_i &= \{t\colon \|\nabla T_i(t)\|_{L^2} + \|\nabla 1/T_i(t)\|_{L^2} \le M\}, \\ G_j &= \left\{t\colon \left\|\nabla T_j(t)\right\|_{L^2} + \left\|\nabla 1/T_j(t)\right\|_{L^2} \le M\right\}. \end{aligned}$$

Then $\mathcal{L}^1([0,1]\backslash G_i) \le C/M$, $\quad \mathcal{L}^1([0,1]\backslash G_j) \le C/M$, and $\left\|K_i(t) - K_j(t)\right\|_{L^1} \le \epsilon$ for all $t \in G_i \bigcap G_j$. Hence,

$$\begin{aligned}&\int_0^1 \left\|K_i(t) - K_j(t)\right\|_{L^1} dt \\ &= \int_{G_i \bigcap G_j} \left\|K_i(t) - K_j(t)\right\|_{L^1} dt + \int_{[0,1]\backslash G_i \bigcap G_j} \left\|K_i(t) - K_j(t)\right\|_{L^1} dt \\ &\leq \epsilon + 2C/M \leq 3\epsilon;\end{aligned}$$

i.e., K_i is Cauchy in $L^1_{loc}(\mathcal{O} \times [0, \infty))$.

For the second statement, let ϵ be any positive number. Choose $M > 0$ such that $C/M \leq \epsilon$. Let δ be the number gotten from Corollary 6.1.4 for ϵ and M.

From the first statement, there is an N such that, for all $i, j > N$,

$$\begin{aligned}&\int_0^1 \left\|K_i(t) - K_j(t)\right\|_{L^1} dt \leq \epsilon\delta, \\ &\left\|Q_i(t) - Q_j(t)\right\|_{\sim} \leq \delta, \qquad \forall t \in [0, 1].\end{aligned}$$

Let H_i and H_j be the set of **good times**,

$$\begin{aligned}H_i &= \left\{t\colon \left\|K_i(t) - K_j(t)\right\|_{L^1} \leq \delta;\ \|\nabla T_i(t)\|^2_{L^2} + \|\nabla 1/T_i(t)\|^2_{L^2} \leq M,\right\} \\ H_j &= \left\{t\colon \left\|K_i(t) - K_j(t)\right\|_{L^1} \leq \delta;\ \left\|\nabla T_j(t)\right\|^2_{L^2} + \left\|\nabla 1/T_j(t)\right\|^2_{L^2} \leq M.\right\}\end{aligned}$$

Then, $\mathcal{L}^1([0, 1]\backslash H_i) \leq \epsilon\delta/\delta + C/M = 2\epsilon$ (similarly, $\mathcal{L}^1([0, 1]\backslash H_j) \leq 2\epsilon$). In addition, we have $\left\|T_i(t) - T_j(t)\right\|_{L^2} \leq \epsilon$ for all $t \in H_i \bigcap H_j$. Hence,

$$\begin{aligned}&\int_0^1 \left\|T_i(t) - T_j(t)\right\|^2_{L^2} dt \\ &= \int_{H_i \bigcap H_j} \left\|T_i(t) - T_j(t)\right\|^2_{L^2} dt + \int_{[0,1]\backslash H_i \bigcap H_j} \left\|1/T_i(t) - 1/T_j(t)\right\|^2_{L^2} dt \\ &\leq \epsilon + C(4\epsilon);\end{aligned}$$

i.e., T_i is Cauchy in $L^2_{loc}(\mathcal{O} \times [0, \infty))$. Similar results hold for $1/T_i$. □

6.2. Nontrivial Crystal Configurations

Now we return to Remark 6.1.1.

The condition of nontrivial crystal occupancies (necessary for the compactness result) might not be preserved for all time due to the white-noise driving force in the heat equation. (In the deterministic case, the energy of the system is always decreasing, so there are ample initial conditions such that this nontriviality of the crystal shape holds for all time.) We show that **this condition is true up to a stopping time.**

Let $\varrho > 0$ be a fixed small number throughout this section.

6.2.1. Proposition (Sufficient Condition for Nontrivial Crystals). *[AW Prop. 3.5] Given any (large) $M > 0$, there is a (small) $\gamma > 0$ such that, for any $L \in \mathcal{K}$ and $P \in \mathcal{Q}$ with*

$$\int_{\mathcal{O}} P^2 + P^{-2}\, d\mathcal{L}^n \;\leq\; M \qquad \textit{and} \tag{6.4}$$

$$\Phi(\partial L) + \int_{\mathcal{O}} c_L F(c_L^{-1} P)\, d\mathcal{L}^n \;\leq\; \min\left\{c_l F(c_l^{-1}\overline{P}), c_s F(c_s^{-1}\overline{P})\right\} |\mathcal{O}| - \varrho, \tag{6.5}$$

where $\overline{P}$ is the spatial average of P, then any $(\mathcal{E}, \triangle t, P)$ minimizer (K, Q) satisfies

$$\gamma\, |\mathcal{O}| \leq \mathcal{L}^n(K) \leq (1-\gamma)\, |\mathcal{O}|\, .$$

Proof. Assume the contrary, i.e., there is an $M > 0$, $\gamma_i \longrightarrow 0$, and $L_i \in \mathcal{K}$, $P_i \in \mathcal{Q}$, such that

$$\int_{\mathcal{O}} P_i^2 + P_i^{-2}\, d\mathcal{L}^n \;\leq\; M, \tag{6.6}$$

$$\Phi(\partial L_i) + \int_{\mathcal{O}} c_{L_i} F(c_{L_i}^{-1} P_i)\, d\mathcal{L}^n \;\leq\; \min\left\{c_l F(c_l^{-1}\overline{P_i}), c_s F(c_s^{-1}\overline{P_i})\right\} |\mathcal{O}| - \varrho, \tag{6.7}$$

and a $(\mathcal{E}, \triangle t, P_i)$ minimizer (K_i, Q_i) satisfies

$$\mathcal{L}^n(K_i) = \gamma_i\, |\mathcal{O}| \qquad \text{or} \qquad \mathcal{L}^n(K_i) = (1-\gamma_i)\, |\mathcal{O}|$$

By the definition of a minimizer, we have

$$\Phi(\partial K_i) + \int_{\mathcal{O}} c_{K_i} F(c_{K_i} Q_i)\, d\mathcal{L}^n + \frac{1}{\triangle t^\alpha} \|Q_i - P_i\|_* \leq \Phi(\partial L_i) + \int_{\mathcal{O}} c_{L_i} F(c_{L_i}^{-1} P_i)\, d\mathcal{L}^n.$$

Due to the uniform convexity of F, the middle term of the L.H.S. of the above is bounded below by ([AW], Prop. 3.1)

$$\int_{\mathcal{O}} c_{K_i} F\left(\frac{\overline{P_i}}{\gamma_i c_s + (1-\gamma_i) c_l}\right) d\mathcal{L}^n = (\gamma_i c_s + (1-\gamma_i) c_l) F\left(\frac{\overline{P_i}}{\gamma_i c_s + (1-\gamma_i) c_l}\right) |\mathcal{O}|\, .$$

From (6.6), $\delta_1 \leq \overline{P_i} \leq \delta_2$ for some fixed positive numbers δ_1 and δ_2 (depending on M). Taking a convergent subsequence (still denoted by i) such that $\overline{P_i} \longrightarrow a, \;\; \gamma_i \longrightarrow 0$, then

$$\max\left\{c_s F\left(\frac{a}{c_s}\right), c_l F\left(\frac{a}{c_l}\right)\right\} \leq \min\left\{c_s F\left(\frac{a}{c_s}\right), c_l F\left(\frac{a}{c_l}\right)\right\} - \varrho,$$

which is absurd. □

To apply the above result, (L, P) will be the crystal and heat distribution right before each minimization and K will be the crystal right after.

6.2.2. Admissible Initial Conditions. *Initial configuration* (K_0, Q_0) *is called* **admissible** *if*

$$\Phi\,(\partial K_0) + \int_{\mathcal{O}} c_{K_0} F(c_{K_0}^{-1} Q_0)\, d\mathcal{L}^n < \min\left\{c_s F(c_s^{-1}\overline{Q_0}),\ c_l F(c_l^{-1}\overline{Q_0})\right\} |\mathcal{O}| - \varrho. \quad (6.8)$$

The fact that there exists such an initial condition can be seen as follows.

Let $0 < r < 1$, $\mathcal{L}^n(K_0) = r\,|\mathcal{O}|$, and $T_0(r)$ be the constant temperature field satisfying

$$(rc_s + (1-r)c_l)T_0(r) = \overline{Q_0} \qquad \text{(preservation of heat content).}$$

We claim the existence of $K_0(r)$, $\overline{Q}_0$, and r such that

$$\begin{aligned} &\Phi(\partial K_0(r)) + (rc_s + (1-r)c_l)\, F(T_0(r))\, |\mathcal{O}| \\ &\quad < \min\left\{c_s F(c_s^{-1}\overline{Q_0}),\ c_l F(c_l^{-1}\overline{Q_0})\right\} |\mathcal{O}| - \varrho. \end{aligned} \quad (6.9)$$

- (Recall that F is uniformly convex, nonnegative, and $F(T_*) = 0$ for some $T_* > 0$.) Pick $\overline{Q_0}$ satisfying

$$\frac{\overline{Q_0}}{c_l} < T_* < \frac{\overline{Q_0}}{c_s}, \qquad \text{i.e.,} \qquad c_s T_* < \overline{Q_0} < c_l T_*.$$

- Choose r giving $T_0(r) = T_*$.
- $K_0(r)$ exists if $|\mathcal{O}|$ is large enough.

6.2.3. Proposition. *Start from each* **initial admissible configuration** *of crystal and heat distribution, at each level of the discrete scheme ($\triangle t = 1/N$), and there is an almost surely positive* **stopping time** τ^N *and a* **random number** γ^N *such that the crystal satisfies*

$$\gamma^N\, |\mathcal{O}| \le \mathcal{L}^n(K^N(t)) \le (1-\gamma^N)\, |\mathcal{O}| \qquad \textit{for } 0 \le t < \tau^N. \quad (6.10)$$

Proof. (The superscript N is suppressed for what follows until the very last.) From the previous proposition, at each minimization step, we want to ensure condition (6.5):

$$\begin{aligned} \mathcal{E}(t_i^-) \ &= \ \Phi(\partial K_i^-) + \int_{\mathcal{O}} c_{K_i^-} F(c_{K_i^-}^{-1} Q_i^-)\, d\mathcal{L}^n \\ &< \ \min\left\{c_s F(c_s^{-1}\overline{Q_i^-}), c_l F(c_l^{-1}\overline{Q_i^-})\right\} |\mathcal{O}| - \varrho \end{aligned} \quad (6.11)$$

The L.H.S. of the above is bounded by (Proposition 5.2.1)

$$\mathcal{E}(0) + \int_0^t \left\langle c_K F'(T) f(T),\ dW_r\right\rangle + \frac{1}{2}\int_0^t \mathrm{Tr}\left[c_K F''(T) f(T) \Lambda f(T)\right] dr. \quad (6.12)$$

We tighten the sufficient condition (6.11) to

$$\begin{aligned} \mathcal{E}(0) \ + \ &\int_0^t \left\langle c_K F'(T) f(T),\ dW_r\right\rangle + \frac{1}{2}\int_0^t \mathrm{Tr}\left[c_K F''(T) f(T) \Lambda f(T)\right] dr \\ &< \min\left\{c_s F(c_s^{-1}\overline{Q_i^-}), c_l F(c_l^{-1}\overline{Q_i^-})\right\} |\mathcal{O}| - \varrho. \end{aligned}$$

Set

$$I_s(t) \;=\; \mathcal{E}(0) + \frac{1}{2}\int_0^t \mathrm{Tr}\left[c_K F''(T)\, f(T)\Lambda f(T)\right] dr$$
$$- c_s F(c_s^{-1}\overline{Q(t)})\, |\mathcal{O}| + \int_0^t \left\langle c_K F'(T)\, f(T),\; dW_r\right\rangle. \tag{6.13}$$

The same definition holds for $I_l(t)$ with c_s replaced by c_l.

All the quantities in the above expression are **continuous** in time. We can then define the stopping time,

$$\tau^N = \min\{t_i : I_s(t_i) \geq -\varrho \;\text{ or }\; I_l(t_i) \geq -\varrho.\} \tag{6.14}$$

Choose initial configuration (K_0, Q_0) such that

$$\mathcal{E}(0) < \min\left\{c_s F(c_s^{-1}\overline{Q_0}),\;\; c_l F(c_l^{-1}\overline{Q_0})\right\} |\mathcal{O}| - \varrho; \tag{6.15}$$

then $\tau^N > 0$ a.s.

The random number γ^N depends on

$$\sup\left\{\int_{\mathcal{O}} T^N(t)^2 + T^N(t)^{-2}\, d\mathcal{L}^n,\;\; t \in [0, 1]\right\}.$$

□

6.3. Tightness of the Discrete Scheme

Using the previous sections, we now show that a **stopped version** of our discretized evolutions are compact in the sense of probability measures.

6.3.1. Review of Notations and Results. Let $X^N = (K^N(t), T^N(t), U^N(t), Q^N(t))_{t\in[0,1]}$ be the evolutions generated by the alternating minimization and heat flow process ($N = 1/\triangle t$). Then they are random variables defined on a probability space $(\Omega, P, \mathcal{F})$ taking values in S.[17] There is also a Wiener Process $\{W_t,\; \mathcal{F}_t;\; 0 \leq t < \infty\}$ defined on Ω. Let μ^N be the law of X^N on S.

6.3.2. Definition of the Stopped Process. As mentioned earlier, in order to make use of the compactness results in Section 6.1 to show the tightness of the μ^N's, the crystals must be nontrivial. This condition is true only up to a stopping time τ^N. We will modify the evolution X^N after τ^N for mathematical convenience. **We will not assert anything about the evolution after τ^N.** We do not know whether such a modification is absolutely necessary. At the moment, it seems quite artificial. In addition, the state space S will be **expanded to include more information**.

[17] Note that the heat evolutions are not quite continuous in time due to the jumps in the minimization steps. We will take care of this in Section 6.3.4.

1. Let K_* be **any fixed crystal** and T_* be **any fixed constant temperature field**. We now introduce the **stopped process**

$$X_*^N(t) = \left(K_*^N(t), T_*^N(t), U_*^N(t), Q_*^N(t)\right)_{t\in[0,1]},$$

where

$$\begin{aligned}
K_*^N(t) &= K^N(t) \quad \text{for } t < \tau^N, \quad K_* \text{ for } t \geq \tau^N, \\
T_*^N(t) &= T^N(t) \quad \text{for } t < \tau^N, \quad T_* \text{ for } t \geq \tau^N, \\
U_*^N(t) &= T_*^N(t)^{-1} \quad \text{for } t \in [0,1], \\
Q_*^N(t) &= Q^N(t) \quad \text{for } t \in [0,1].
\end{aligned}$$

2. Consider the definition for $I_s(t)$ (6.13) (and similarly for $I_l(t)$),

$$\begin{aligned}
I_s^N(t) &= \mathcal{E}(0) + \frac{1}{2}\int_0^t \mathrm{Tr}\left[c_{K^N} F''(T^N) f(T^N) \Lambda f(T^N)\right] dr \\
&\quad - c_s F(c_s^{-1}\overline{Q^N}(t)) \,|\mathcal{O}| + J^N(t),
\end{aligned} \tag{6.16}$$

where $J^N(\cdot)$ is a $\mathcal{F}_t$-martingale with quadratic variation

$$\int_0^t \left\langle \Lambda c_{K^N} F'(T^N) f(T^N),\ c_{K^N} F'(T^N) f(T^N)\right\rangle dr. \tag{6.17}$$

These processes act like indicators to give extra information about the nontriviality of the crystals. From them we define

$$\tau^N = \inf\{t_i\colon\ I_s(t_i) \geq -\varrho, \quad I_l(t_i) \geq -\varrho.\} \tag{6.18}$$

Consider the “revised version” of $I_s^N(t)$ (and similarly for $I_l^N(t)$),

$$\begin{aligned}
I_{*s}^N(t) &= \mathcal{E}(0) + \frac{1}{2}\int_0^t \mathrm{Tr}\left[c_{K_*^N} F''(T_*^N) f(T_*^N) \Lambda f(T_*^N)\right] dr \\
&\quad - c_s F\left(c_s^{-1}\overline{Q_*^N(t)}\right) |\mathcal{O}| + J^N(t).
\end{aligned} \tag{6.19}$$

Denote $J_*^N(\cdot) = J^N(\cdot)$.

What we have done so far is to define the stopped process from the original one:

$$\left\{X^N = (K^N, T^N, U^N, Q^N),\, J^N\right\}_{t\in[0,1]} \longrightarrow \left\{I_s^N, I_l^N\right\}_{t\in[0,1]} \longrightarrow \tau^N \tag{6.20}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\left\{X_*^N = (K_*^N, T_*^N, U_*^N, Q_*^N),\, J_*^N\right\}_{t\in[0,1]} \longrightarrow \left\{I_{*s}^N, I_{*l}^N\right\}_{t\in[0,1]} \longrightarrow \tau^N. \tag{6.20*}$$

By (6.18), τ^N is also a $\mathcal{F}_t^{X_*^N, J_*^N}$-stopping time, where $\mathcal{F}_t^{X_*^N, J_*^N}$ is the filtration generated by X_*^N and J_*^N.

6.3.3. Theorem (Tightness of the Stopped Process). *Let Γ_*^N be the law of*

$$\left\{X_*^N(t),\, J_*^N(t)\right\}_{t\in[0,1]} \qquad \text{on } S \times C([0,1], R). \tag{6.21}$$

Then $\left\{\Gamma_*^N\right\}_{N\geq 1}$ is tight.

Proof. Let μ_*^N be the law of X_*^N on S. Clearly it satisfies the same estimates as μ^N (Theorem 5.2.2). Hence, for all $\epsilon > 0$, there is an $M > 0$ such that

$$\mu_*^N\left\{\sup_{t\in[0,1]} \Phi(\partial K_t) + \|T_t\|_{L^2}^2 + \|U_t\|_{L^2}^2 \leq M\right\} \geq 1-\epsilon/4,$$
$$\mu_*^N\left\{\int_0^1 \|\nabla T_t\|_{L^2}^2 + \|\nabla U_t\|_{L^2}^2\, dt \leq M\right\} \geq 1-\epsilon/4.$$

By Theorem 5.3.1 and Theorem B.2.2 (see the remark following this proof), there is a compact subset B of $C([0,1],\ \mathcal{Q})$ such that $\mu_*^N(\pi_4^{-1}B) \geq 1-\epsilon/4$.[18]

Consider the set

$$A = \left\{\sup_{t\in[0,1]} \Phi(\partial K(t)) + \|T(t)\|_{L^2}^2 + \|U(t)\|_{L^2}^2 \leq M\right\}$$
$$\bigcap\left\{\int_0^1 \|\nabla T(t)\|_{L^2}^2 + \|\nabla U(t)\|_{L^2}^2\, dt \leq M\right\}\bigcap(\pi_4^{-1}B).$$

Clearly, $\mu_*^N(A) \geq 1-3\epsilon/4$. According to Proposition 6.2.1, there is a $\gamma > 0$ (depending on M) such that

$$\gamma\,|\mathcal{O}| \leq \mathcal{L}^n(K(t)) \leq (1-\gamma)\,|\mathcal{O}|\,, \qquad \mu_*^N \text{ a.s. for } 0 \leq t < \tau^N.$$

By Theorem 6.1.5, A is compact in the metric of S. (Note that even though the relationship $Q^N = c_{K^N} T^N$ does not hold for $t \geq \tau^N$, the compactness result is still applicable since the crystals and temperature fields are fixed after that.)

Since $J^N(t) \in C([0,1], R)$, from (6.17) and Proposition 5.3.5, we can invoke B.2.2 again to conclude the existence of a compact subset C of $C([0,1], R)$ such that $\Gamma_*^N(S \times C) \geq 1-\epsilon/4$.

Finally, we have $\Gamma_*^N((A \times C([0,1], R)) \cap (S \times C)) \geq 1-\epsilon$. The theorem follows. □

6.3.4. Remark—Indirect Usage of Theorem B.2.2. The original version of B.2.2 does not apply directly in the above proof, as the sample paths of the heat distributions $Q^N(t)$, strictly speaking, are not continuous in time but have jumps. To overcome this, we use the following twist.

Let $C_1 = C([0,1], \mathcal{Q})$ and C_2 be the collection of heat evolutions which are piecewise continuous with a finite number of jumps at $\left\{jN^{-1}\right\}_{j\geq 0, N\geq 1}$. C_2 is given the **metric of**

[18] π_4 is the projection from S onto its fourth factor $C([0,1], \mathcal{Q})$.

uniform convergence. Let C_3 be the completion of C_2. Then C_3 is a **complete separable metric space**. Let ν_*^N be the law of Q_*^N on C_3.

Now, for all N, $Q^N(\cdot)$ can be decomposed as $Q^N(t)' + R^N(t)'$ with $Q^N(\cdot)' \in C_1$. (**Note that the map** $Q(t) \longrightarrow Q'(t)$ **is a continuous and hence Borel map on** C_3**.**) By Theorem 5.3.1, we apply B.2.2 to $Q^N(\cdot)'$ to conclude the existence of a measure ν_* on C_1 and a subsequence (still denoted by N) such that for all **bounded uniformly continuous** functions f on C_3, we have

$$\int_\Omega f(Q^{N'})\,dP = \int_{C_1} f(Q')\,d\nu_*^N \longrightarrow \int_{C_1} f(Q)\,d\nu_*.$$

But, by Lemma 5.3.3, $R^N(t)' = Q^N(t) - Q^N(t)' \longrightarrow 0$ uniformly in t in probability and so does $f(Q^N) - f(Q^{N'})$ (f is uniformly continuous). By the dominated convergence theorem,

$$\int_\Omega f(Q^N) - f(Q^{N'})\,dP \longrightarrow 0.$$

Hence, we conclude that

$$\int_{C_3} f(Q)\,d\nu_*^N \longrightarrow \int_{C_1} f(Q)\,d\nu_* = \int_{C_3} f(Q)\,d\nu_*.$$

By the Prokhorov criterion (Section B.2), given any $\epsilon > 0$, there is a compact set B in C_3 such that, for all N,

$$\nu_*^N(B) \geq 1 - \epsilon.$$

Thus, the step in the above proof using B.2.2 is justified.[19]

6.4. Formulation on a Common Probability Space

From Theorem 6.3.3 and the Prohkorov Criterion, there is a subsequence $\{\Gamma_*^{N_j}\}_j$ and a probability measure Γ_* on $S \times C([0,1], R)$ such that $\Gamma_*^{N_j} \rightharpoonup \Gamma_*$.

We now formulate this weak convergence in terms of **almost sure convergence on a common probability space** by an extended version of the **Skorokhod Theorem** (which is usually stated for the case of complete separable metric space). The reason for doing this is that later on we will make comparisons between the limiting random variables and the approximated ones, treating them as defined on the same probability space.

6.4.1. Proposition (Skorokhod Theorem—Extended Version). *Let* $\left\{\Gamma^N\right\}_{N\geq 1}$ *be a* **tight** *sequence of probability measures on a* **separable metric space** Y *converging weakly to a probability measure* Γ *on* Y. *Then there is a probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ *and random variables* $\tilde{X}^N$ *and* $\tilde{X}$ *taking values in* Y *such that the law of* $\tilde{X}^N$ *is* Γ^N *and* $\tilde{X}^N \longrightarrow \tilde{X}$ $\tilde{P}$ *a.s. in the metric of* Y.

---

[19] Actually, in the original definition of S, we can even replace $C([0,1], \mathcal{Q})$ by C_3. Everything remains unchanged.

Proof. The proof starts by embedding Y into its completion $\bar{Y}$. Then we can invoke the usual Skorokhod Theorem [IW] to construct random variables $\tilde{X}^N$ and $\tilde{X}$, taking values in $\bar{Y}$ satisfying the stated properties.

By using the tightness of $\left\{\Gamma^N\right\}_{N\geq 1}$, there are compact sets C_i in Y (and hence compact and closed in $\bar{Y}$) such that, for all $N \geq 1$, $\tilde{P}(\tilde{X}^N \in C_i) = \Gamma^N(C_i) \geq 1 - 2^{-i}$. Then, $\tilde{P}(\tilde{X}^N \in \cup C_i) = 1$. Since $\tilde{X}^N \rightharpoonup \tilde{X}$ $\tilde{P}$ a.s., we also have $\tilde{P}(\tilde{X} \in C_i) \geq \limsup_i \tilde{P}(\tilde{X}^N \in C_i) \geq 1 - 2^{-i}$. This leads to $\tilde{P}(\tilde{X} \in \cup C_i) = 1$. Hence, we can treat all the $\tilde{X}^N$'s and $\tilde{X}$ as taking values in Y ($\cup C_i \subset Y$). □

Applying the above to the case of

$$Y = S \times C([0,1], R) = \{X = (K(\cdot), T(\cdot), U(\cdot), Q(\cdot)), J(\cdot)\}_{t\in[0,1]},$$

we conclude that there is a probability space $\tilde{P}$ and random variables $\left(\tilde{X}^N, \tilde{J}^N\right)$ taking values in $S \times C([0,1], R)$ with the same law as (X_*^N, J_*^N) and converging $\tilde{P}$ a.s. to a random variable $(\tilde{X}, \tilde{J})$ in the metric of $S \times C([0,1], R)$. $\tilde{\tau}^N$ is defined the same way as in (6.18).

We are going to study the properties of $(\tilde{X}^N, \tilde{J}^N)$ and $(\tilde{X}, \tilde{J})$. In the following, $\tilde{E}$ is with respect to $\tilde{P}$.

6.4.2. Energy Estimates and Hölder Continuity of Heat Evolution. For all positive integers m, we have

$$\tilde{E}\left\{\sup_{t\in[0,1]} \Phi(\partial\tilde{K}^N(t))^m\right\}, \qquad \tilde{E}\left\{\sup_{t\in[0,1]} \left\|\tilde{T}^N(t)\right\|_{L^2}^m + \left\|\tilde{U}^N(t)\right\|_{L^2}^m\right\}, \qquad \text{and}$$
$$\tilde{E}\left[\left(\int_0^1 \left\|\nabla\tilde{T}^N(t)\right\|_{L^2}^2 + \left\|\nabla\tilde{U}^N(t)\right\|_{L^2}^2 dt\right)^m\right] \leq C_m < \infty. \tag{6.22}$$

Furthermore, $\tilde{Q}^N$ can be decomposed as $\tilde{Q}^{N'} + \tilde{R}^{N'}$, such that

$$\tilde{E}\left\|\tilde{Q}^N(t)' - \tilde{Q}^N(s)'\right\|_{\sim}^{2m} \leq C_m\,|t-s|^m \qquad \text{and} \qquad \tilde{E}\sup_{t\in[0,1]}\left\|\tilde{R}^N(t)'\right\|_{\sim}^{2m} \leq C_m \triangle t^{2m\alpha}. \tag{6.23}$$

In addition,

$$\tilde{E}\left|\tilde{J}^N(t) - \tilde{J}^N(s)\right|^{2m} \leq C_m\,|t-s|^m. \tag{6.24}$$

All the C_m's are independent of N.

6.4.3. Minimizing Property. For each N, we have $\Gamma^N\left(\bigcup_{M,i} B_{M,i}\right) = 1$, where $B_{M,i}$ is the collection of elements $(\tau, X) = (\tau, (K, T, U, Q))$ in $([0,1], S)$ satisfying:

- $\sup_{t\in[0,1]}\left\{\Phi(\partial K(t)) + \|T(t)\|_{L^2}^2 + \|U(t)\|_{L^2}^2\right\} \leq M$.
- $\tau - \triangle t \geq t_i$.

- For all $L \in \mathcal{K}$ and $R \in \mathcal{Q}$ such that $\overline{R} = \overline{Q}_{i^-}$,

$$\Phi\left(\partial K_{i^+}\right) \;+\; \int_{\mathcal{O}} c_{K_{i^+}} F\left(T_{i^+}\right)\, d\mathcal{L}^n + \frac{1}{\triangle t^{\alpha}} \left\| Q_{i^+} - Q_{i^-} \right\|_* \\ \leq \Phi(\partial L) + \int_{\mathcal{O}} c_L\, F(c_L^{-1} R)\, d\mathcal{L}^n + \frac{1}{\triangle t^{\alpha}} \left\| R - Q_{i^-} \right\|_* .$$

It is easy to show that $B_{M,i}$ is closed. Hence, $\bigcup_{M,i} B_{M,i}$ is a Borel set. Thus, we also have

$$\tilde{P}\left(\left(\tilde{\tau}^N,\, \tilde{X}^N\right) \in \bigcup_{M,i} B_{M,i}\right) = 1. \tag{6.25}$$

6.4.4. Martingale Property. Consider the following filtration:

$$\tilde{\mathcal{F}}_t^N = \bigcap_{\delta \downarrow 0} \sigma\left\{\left(\tilde{X}^N(r),\, \tilde{J}^N(r)\right) :\; 0 \leq r \leq t + \delta\right\}, \qquad 0 \leq t < \infty. \tag{6.26}$$

1. Let $\varphi \in C^{\infty}(\mathcal{O})$. We claim that $\tilde{M}_*^N(t,\varphi)'$, defined as

$$\tilde{M}_*^N(t,\varphi)' \;=\; \left\langle \tilde{Q}^N(t \wedge \tilde{\tau}^N)',\; \varphi \right\rangle - \left\langle \tilde{Q}^N(0^+)',\; \varphi \right\rangle \\ - \int_0^{t \wedge \tilde{\tau}^N} \left\langle \Sigma_{\tilde{K}^N} \nabla \tilde{T}^N,\; \nabla\varphi \right\rangle ds, \tag{6.27}$$

   is a $\tilde{\mathcal{F}}_t^N$-martingale. The cross variation process between $\tilde{M}_*^N(t,\varphi)'$ and $\tilde{M}_*^N(t,\psi)'$ is given by

$$\left\langle \tilde{M}_*^N(\cdot,\varphi)',\; \tilde{M}_*^N(\cdot,\psi)' \right\rangle_t = \int_0^{t\wedge\tau^N} \left\langle \Lambda c_{\tilde{K}^N} f(\tilde{T}^N)\varphi,\; c_{\tilde{K}^N} f(\tilde{T}^N)\psi \right\rangle ds. \tag{6.28}$$

2. $\tilde{J}^N(t \wedge \tilde{\tau}^N)$ is a $\tilde{\mathcal{F}}_t^N$-martingale with quadratic variation

$$\left\langle \tilde{J}^N(\cdot \wedge \tilde{\tau}^N) \right\rangle_t = \int_0^{t\wedge\tilde{\tau}^N} \left\langle \Lambda c_{\tilde{K}^N} F'(\tilde{T}^N)\, f(\tilde{T}^N),\; c_{\tilde{K}^N} F'(\tilde{T}^N)\, f(\tilde{T}^N) \right\rangle dr. \tag{6.29}$$

   In addition, the cross variation process between $\tilde{J}^N(\cdot \wedge \tilde{\tau}^N)$ and $\tilde{M}_*^N(t,\varphi)'$ is given by

$$\left\langle \tilde{J}^N(\cdot \wedge \tilde{\tau}^N),\; \tilde{M}_*^N(\cdot,\varphi)' \right\rangle_t = \int_0^{t\wedge\tilde{\tau}^N} \left\langle \Lambda c_{\tilde{K}^N} F'(\tilde{T}^N)\, f(\tilde{T}^N),\; c_{\tilde{K}^N} f(\tilde{T}^N)\varphi \right\rangle dr. \tag{6.30}$$

   (1) can be seen easily by the fact that the process

$$M_*^N(t,\varphi)' = \left\langle Q^N(t\wedge\tau^N)',\; \varphi\right\rangle - \left\langle Q^N(0^+)',\; \varphi\right\rangle - \int_0^{t\wedge\tau^N} \left\langle \Sigma_{K^N} \nabla T^N,\; \nabla\varphi \right\rangle ds \tag{6.31}$$

is a $\mathcal{F}_t$-martingale. (Recall that $\mathcal{F}_t$ is the filtration with respect to which W_t, the Wiener Process, is adapted.)

Let Θ_s be any bounded Borel function defined on $S \times C([0,1], R)$ which is

$$\bigcap_{\delta\downarrow 0} \sigma\left\{(x(r), y(r)) \colon\ 0 \le r \le s+\delta,\ \ x \in S,\ \ y \in C([0,1], R)\right\}\text{-measurable.}$$

Then,

$$\begin{aligned}
&\tilde{E}\left\{\left(\left\langle \tilde{Q}^N(t\wedge\tilde{\tau}^N)',\ \varphi\right\rangle - \left\langle \tilde{Q}^N(0^+)',\ \varphi\right\rangle - \int_0^{t\wedge\tilde{\tau}^N} \left\langle \Sigma_{\tilde{K}^N}\nabla\tilde{T}^N,\ \nabla\varphi\right\rangle dr\right)\Theta_s(\tilde{X}^N,\ \tilde{J}^N)\right\}\\
&= E\left\{\left(\left\langle Q_*^N(t\wedge\tau^N)',\ \varphi\right\rangle - \left\langle Q_*^N(0^+)',\ \varphi\right\rangle - \int_0^{t\wedge\tau^N} \left\langle \Sigma_{K_*^N}\nabla T_*^N,\ \nabla\varphi\right\rangle dr\right)\Theta_s(X_*^N,\ J_*^N)\right\}\\
&= E\left\{\left(\left\langle Q_*^N(s\wedge\tau^N)',\ \varphi\right\rangle - \left\langle Q_*^N(0^+)',\ \varphi\right\rangle - \int_0^{s\wedge\tau^N} \left\langle \Sigma_{K_*^N}\nabla T_*^N,\ \nabla\varphi\right\rangle dr\right)\Theta_s(X_*^N,\ J_*^N)\right\}\\
&= \tilde{E}\left\{\left(\left\langle \tilde{Q}^N(s\wedge\tilde{\tau}^N)',\ \varphi\right\rangle - \left\langle \tilde{Q}^N(0^+)',\ \varphi\right\rangle - \int_0^{s\wedge\tilde{\tau}^N} \left\langle \Sigma_{\tilde{K}^N}\nabla\tilde{T}^N,\ \nabla\varphi\right\rangle dr\right)\Theta_s(\tilde{X}^N,\ \tilde{J}^N)\right\}.
\end{aligned}$$

Similar computations lead to the other assertions.

The final result in this chapter follows (from Section 6.4.2 and Kolmogorov Theorem B.2.1).

6.4.5. Theorem (Energy Estimates and Heat Holder Continuity). *The limit evolution $\tilde{X} = (\tilde{K}, \tilde{T}, \tilde{U}, \tilde{Q})$ satisfies the following estimates:*

$$\tilde{E}\left\{\sup_{t\in[0,1]} \Phi(\partial\tilde{K}(t))^m\right\}, \qquad \tilde{E}\left\{\sup_{t\in[0,1]} \left\|\tilde{T}(t)\right\|_{L^2}^m + \left\|\tilde{U}(t)\right\|_{L^2}^m\right\}, \qquad \textit{and}$$

$$\tilde{E}\left[\left(\int_0^1 \left\|\nabla\tilde{T}(t)\right\|_{L^2}^2 + \left\|\nabla\tilde{U}(t)\right\|_{L^2}^2 dt\right)^m\right] \le C_m < \infty, \tag{6.32}$$

$$\tilde{E}\left\|\tilde{Q}(t) - \tilde{Q}(s)\right\|_{\sim}^{2m} \le C_m\,|t-s|^m, \tag{6.33}$$

$$\tilde{E}\left|\tilde{J}(t) - \tilde{J}(s)\right|^{2m} \le C_m\,|t-s|^m. \tag{6.34}$$

(The above statements hold because all the functionals inside the expectations are (lower-semi)-continuous with respect to the metric of S.)

Thus, $\tilde{Q}(\cdot)$ is **continuous** *in time in the modified Monge-Kantorovich norm. $\tilde{J}(\cdot)$ is also* **continuous** *in time.*

6.4.6. Remark. From now on, we will drop the $\sim$ symbol. It is understood that the random variables $(\tilde{X}^N, \tilde{J}^N)$'s are defined on a common probability space $\tilde{\Omega}$. They satisfy all the previously stated properties and converge to $(\tilde{X}, \tilde{J})$ $\tilde{P}$ almost surely in the metric of $S \times C([0,1], R)$.

7. Limiting Heat Equation

The goal of this chapter is to establish (2.15) in the sense set forth in Section 2.2 (4).

We will follow the usual procedure, which is to convert the problem into a martingale formulation and then construct an extension of the underlying probability space so as to accommodate a Wiener Process. We have already set up the technical devices (especially the almost sure convergence in some space-time topology) to carry out this procedure and, with the special case that our operators are **nondegenerate**, the whole proof is thus quite transparent.

7.1. Martingale Formulation

First, we solve (2.15) in the setting of martingale formulation. We define some notations.

From Remark 6.4.6, we know that P a.s. in the metric of $S \times C([0,1], R)$

$$\left\{X^N = (K^N, T^N, U^N, Q^N),\ \ J^N\right\} \longrightarrow \{X = (K, T, U, Q),\ \ J\}.$$

Consider the following filtrations on P:

$$\mathcal{F}_t^N \ = \bigcap_{\delta\downarrow 0} \sigma\left\{(X^N(r), J^N(r))\colon\ 0 \le r \le t+\delta\right\}, \qquad 0 \le t \le 1, \tag{7.1}$$

$$\mathcal{F}_t \ = \bigcap_{\delta\downarrow 0} \sigma\left\{(X(r), J(r))\colon\ 0 \le r \le t+\delta\right\}, \qquad 0 \le t \le 1. \tag{7.2}$$

We also introduce the filtration on $S \times C([0,1], R)$,

$$\mathcal{B}_t = \bigcap_{\delta\downarrow 0} \sigma\left\{(x(r), y(r))\colon\ 0 \le r \le t+\delta,\ \ x \in S,\ \ y \in C([0,1], R)\right\}, \qquad 0 \le t \le 1. \tag{7.3}$$

Recall the definitions of $I_s^N(t)$, $I_l^N(t)$, τ^N in Section 6.3.2. Similarly, we set

$$I_s(t) \ = \ \mathcal{E}(0) + \frac{1}{2}\int_0^t \mathrm{Tr}\left[c_K F''(T) f(T) \Lambda f(T)\right] dr - c_s F(c_s^{-1}\overline{Q(t)}\,|\mathcal{O}| + J(t)),$$

$$\text{(replace } c_s \text{ by } c_l \text{ for } I_l(t)) \tag{7.4}$$

$$\tau \ = \ \inf\{t\colon\ I_s(t) \ge -\varrho,\ \ I_l(t) \ge -\varrho\}. \tag{7.5}$$

It is a simple matter to check that $I_s^N(\cdot)$ and $I_l^N(\cdot)$ converge to $I_s(\cdot)$ and $I_l(\cdot)$ uniformly in $t \in [0,1]$ P a.s. In addition, τ^N and τ are $\mathcal{F}_t^N$ and $\mathcal{F}_t$ stopping times, respectively.

The result in this section is as follows.

7.1.1. Theorem.

1. $Q = c_K T$, $d\mathcal{L}^1 \times dP$ *a.s. on* $\{(t,\omega)\colon\ t < \tau(\omega)\}$.
2. *For all* $\varphi \in C^\infty(\mathcal{O})$*, the following is a continuous* $\mathcal{F}_t$*-martingale:*

$$M_*(t,\varphi) = \langle Q(t\wedge\tau),\ \varphi\rangle - \langle Q_0,\ \varphi\rangle + \int_0^{t\wedge\tau} \langle \Sigma_K \nabla T,\ \nabla\varphi\rangle\, dr. \tag{7.6}$$

3. *The cross-variation process between $M_*(t,\varphi)$ and $M_*(t,\psi)$ equals*

$$\langle M_*(\cdot,\varphi),\ M_*(\cdot,\psi)\rangle_t = \int_0^{t\wedge\tau} \langle \Lambda c_K f(T)\varphi,\ c_K f(T)\psi\rangle\, dr. \tag{7.7}$$

4. *$J(t\wedge\tau)$ is a continuous $\mathcal{F}_t$-martingale with quadratic variation*

$$\langle J(\cdot\wedge\tau)\rangle_t = \int_0^{t\wedge\tau} \left\langle \Lambda c_K F'(T) f(T),\ c_K F'(T) f(T)\right\rangle dr. \tag{7.8}$$

The cross-variation process between $J(t\wedge\tau)$ and $M_*(t,\varphi)$ is

$$\langle J(\cdot\wedge\tau),\ M_*(\cdot,\varphi)\rangle_t = \int_0^{t\wedge\tau} \left\langle \Lambda c_K F'(T) f(T),\ c_K f(T)\varphi\right\rangle dr. \tag{7.9}$$

Before starting the proof, we present some elementary but useful results that will help in many of the computations later on.

7.1.2. Lemma. *Let $1 < p$. If $\{f_n\}_{n\geq 1}$ are real valued random variables such that $E\,|f_n|^p \leq C < \infty$ for all n and $f_n \longrightarrow f$ P a.s., then $Ef_n \longrightarrow Ef$.*

Proof. It suffices to show that $\{f_n\}_{n\geq 1}$ are uniformly integrable, i.e., given any $\epsilon > 0$, there exists $M > 0$ such that $E\,|f_n|\,1_{\{|f_n|\geq M\}} \leq \epsilon$ for all n. However,

$$\begin{aligned} E\,|f_n|\,1_{\{|f_n|\geq M\}} &\leq \left(E\,|f_n|^p\right)^{1/p} \left(E 1_{\{|f_n|\geq M\}}\right)^{1/q} \qquad (1/p + 1/q = 1.) \\ &\leq CP\left(|f_n| \geq M\right)^{1/q} \leq C\frac{\left(E\,|f_n|^p\right)^{1/q}}{M^{p/q}} \leq \frac{C}{M^{p/q}}. \end{aligned}$$

Hence, M can be chosen independently of n to make $E\,|f_n|\,1_{\{|f_n|\geq M\}}$ arbitrarily small. □

7.1.3. Lemma. *For all positive integers m,*

$$E\left\{\sup_{\lambda\in[0,1]} \left\| Q^N(\lambda)\right\|_{\sim}^m\right\} \leq C_m < \infty. \tag{7.10}$$

Proof. This can be established in the same way as in Lemma 5.3.4. (We just need to apply Burkholder's Inequality to (5.11) and (5.14). Note that all the quadratic variation processes are uniformly bounded by some deterministic number.) □

There is a subtlety about stopping time. We would like to have $\tau^N \longrightarrow \tau$ P a.s., but in general, this is not true. To overcome this, we make use of the following idea (which the author learned about from the preprint [Fun] Lemma 3.1).

7.1.4. Definition. *Let $\eta > 0$ and*

$$\tau_\eta^N = \left\{t_i\colon\ I_s^N(t_i) \geq -\varrho - \eta,\ \ I_l^N(t_i) \geq -\varrho - \eta\right\}, \tag{7.11}$$

$$\tau_\eta = \{t_i\colon\ I_s(t) \geq -\varrho - \eta,\ \ I_l(t) \geq -\varrho - \eta\}. \tag{7.12}$$

η is called a **point of continuity** *if*

$$P\left\{\lim_{\eta'\to\eta}\tau_{\eta'}=\tau_\eta\right\}=1. \tag{7.13}$$

7.1.5. Proposition.

1. *For all but countably many $\eta>0$, it is a point of continuity.*
2. *If η is a point of continuity, then $\tau_\eta^N\longrightarrow\tau_\eta$ P a.s.*

Proof. For (1). This is because τ_η is **decreasing** in η, and hence so is $E\tau_\eta$. Any continuity point η of $E\tau_\eta$ is a point of continuity for our definition.

For (2). Let η be a point of continuity. Without loss of generality, we just need to consider one function I_s. Given any $\epsilon>0$, there is an $\eta'>\eta$ such that $\tau_\eta<\tau_{\eta'}\le\tau_\eta+\epsilon$. But $I_s^N(t)\longrightarrow I_s(t)$ for all $t\in[0,1]$. Hence, for large enough N,

$$I_s^N(\tau_{\eta'})\longrightarrow I_s(\tau_{\eta'})>-\varrho-\eta.$$

Therefore, $\tau_{\eta'}^N<\tau_{\eta'}\le\tau_\eta+\epsilon$ for large N. □

With the above preparations, we will prove Theorem 7.1.1 with τ (τ^N) replaced by τ_η (τ_η^N) for η a point of continuity, and then take a sequence of such $\eta\longrightarrow 0$. Furthermore, the energy estimates in Section 6.4.2 will be kept in mind.

For the simplicity of notations, we use

$$\begin{gathered} c_N=c_{K^N},\qquad c=c_K,\qquad \Sigma_N=\Sigma_{K^N},\\ \Sigma=\Sigma_K,\qquad \tau^N=\tau_\eta^N,\qquad \text{and}\qquad \tau=\tau_\eta. \end{gathered} \tag{7.14}$$

Now we have $\tau^N\longrightarrow\tau$ P a.s.

7.1.6. Proof of 7.1.1 (1)**—Relationship between K, T, and Q.** Let ζ be an arbitrary random bounded function on $\mathcal{O}\times[0,1]$. Consider

$$\begin{aligned} &E\int_{\mathcal{O}\times[0,1]}(Q-cT)1_{\{t<\tau\}}\zeta\,d\mathcal{L}^n\,dt\\ &\quad= E\int_{\mathcal{O}\times[0,1]}(Q-cT)\left(1_{\{t<\tau\}}-1_{\{t<\tau^N\}}\right)\zeta\,d\mathcal{L}^n\,dt\\ &\qquad+E\int_{\mathcal{O}\times[0,1]}\left\{(Q-Q^N)-(c-c_N)T-c_N(T-T^N)+(Q^N-c_NT^N)\right\}1_{\{t<\tau^N\}}\zeta\,d\mathcal{L}^n\,dt. \end{aligned}$$

By the dominated convergence theorem, energy estimates, and Lemma 7.1.2, all the terms tend to zero as $N\longrightarrow\infty$. (Note that the last term is zero for all N.)

7.1.7. Proof of 7.1.1 (2)—Martingale Property of $M_*(t,\varphi)$. What is needed is that, for all $0 \le s \le t$ and any bounded continuous function Θ_s defined on S which is $\mathcal{B}_s$-measurable (see (7.3)), we should have[20]

$$EM_*(t,\varphi)\Theta_s(X) = EM_*(s,\varphi)\Theta_s(X). \tag{7.15}$$

Let

$$\begin{aligned} M_*^N(t,\varphi) &= \left\langle Q^N(t\wedge\tau^N),\ \varphi\right\rangle - \left\langle Q_0^N,\ \varphi\right\rangle + \int_0^{t\wedge\tau^N} \left\langle \Sigma_N \nabla T^N,\ \nabla\varphi\right\rangle dr \\ &= \left\langle Q^N(t\wedge\tau^N),\ \varphi\right\rangle - \left\langle Q_0^N,\ \varphi\right\rangle + \int_0^{t} \left\langle \Sigma_N \nabla T^N,\ \nabla\varphi\right\rangle 1_{\{r<\tau^N\}}\, dr, \quad (7.16) \\ M_*(t,\varphi) &= \langle Q(t\wedge\tau),\ \varphi\rangle - \langle Q_0,\ \varphi\rangle + \int_0^{t\wedge\tau} \langle \Sigma_K \nabla T,\ \nabla\varphi\rangle\, dr \\ &= \langle Q(t\wedge\tau),\ \varphi\rangle - \langle Q_0,\ \varphi\rangle + \int_0^{t} \langle \Sigma_K \nabla T,\ \nabla\varphi\rangle\, 1_{\{r<\tau\}}\, dr. \quad (7.17) \end{aligned}$$

The proof of (7.15) is established after the following two lemmas.

7.1.8. Lemma.

$$EM_*(t,\varphi)\Theta_s(X) = \lim_N EM_*^N(t,\varphi)\Theta_s(X^N). \tag{7.18}$$

Proof. First, since $\left\langle Q^N(t),\ \varphi\right\rangle \longrightarrow \langle Q(t),\ \varphi\rangle$ uniformly in $t\in[0,1]$ and $\tau^N \longrightarrow \tau$, we have $\left\langle Q^N(t\wedge\tau^N),\ \varphi\right\rangle \longrightarrow \langle Q(t\wedge\tau),\ \varphi\rangle$ P a.s. But, due to Lemma 7.1.3,

$$E\left|\left\langle Q^N(t\wedge\tau^N),\ \varphi\right\rangle\right|^m \le C_\varphi E\left\|Q^N(t\wedge\tau^N)\right\|_\sim^m \le C_\varphi.$$

Hence, $E\left\langle Q^N(t\wedge\tau^N),\ \varphi\right\rangle\Theta_s(X^N) \longrightarrow E\langle Q(t\wedge\tau),\ \varphi\rangle\,\Theta_s(X)$ by Lemma 7.1.2.

Next, consider

$$\begin{aligned} &E\left\{\Theta_s(X)\int_0^t \langle\Sigma\nabla T,\ \nabla\varphi\rangle\, 1_{\{r<\tau\}}\,dr - \Theta_s(X^N)\int_0^t \left\langle\Sigma_N\nabla T^N,\ \nabla\varphi\right\rangle 1_{\{r<\tau^N\}}\,dr\right\} \\ &= E\left\{\left(\Theta_s(X)-\Theta_s(X^N)\right)\int_0^t \langle\Sigma\nabla T,\ \nabla\varphi\rangle\, 1_{\{r<\tau\}}\,dr\right\} \\ &\quad + E\left\{\Theta_s(X^N)\left(\int_0^t \langle\Sigma\nabla T,\ \nabla\varphi\rangle\, 1_{\{r<\tau\}} - \left\langle\Sigma_N\nabla T^N,\ \nabla\varphi\right\rangle 1_{\{r<\tau^N\}}\,dr\right)\right\}. \end{aligned}$$

The first term of the above will tend to zero by the dominated convergence theorem.

[20] Since τ is a $\mathcal{F}_t$-stopping time, $M_*(t,\varphi)$ is clearly adapted to $\mathcal{F}_t$. The requirement about $M_*(t,\varphi)$ having finite first moment is easy to check.

For the second term, it is bounded by

$$
\begin{aligned}
&CE\left|\int_0^t \left\langle \Sigma_N \nabla T^N,\ \nabla\varphi\right\rangle 1_{\{r<\tau^N\}} - \langle \Sigma\nabla T,\ \nabla\varphi\rangle\, 1_{\{r<\tau\}}\, dr\right| \\
&\le\ CE\left|\int_0^t \left(\left\langle \Sigma_N \nabla T^N,\ \nabla\varphi\right\rangle - \langle \Sigma\nabla T,\ \nabla\varphi\rangle\right) 1_{\{r<\tau^N\}}\, dr\right| \qquad (7.19)\\
&\quad + CE\left|\int_0^t \langle \Sigma\nabla T,\ \nabla\varphi\rangle \left(1_{\{r<\tau\}} - 1_{\{r<\tau^N\}}\right) dr\right|.
\end{aligned}
$$

It suffices to consider only (7.19). We decompose $\left\langle \Sigma_N \nabla T^N,\ \nabla\varphi\right\rangle - \langle \Sigma\nabla T,\ \nabla\varphi\rangle$ as

$$
\begin{aligned}
&\left\langle \Sigma_N \nabla T^N,\ \nabla\varphi\right\rangle - \langle \Sigma\nabla T,\ \nabla\varphi\rangle \\
&= \left\langle (\Sigma_N - \Sigma)\nabla T^N,\ \nabla\varphi\right\rangle + \left\langle (\Sigma - \xi)\nabla T^N,\ \nabla\varphi\right\rangle - \left\langle (T^N - T),\ \operatorname{div}(\xi\nabla\varphi)\right\rangle \\
&\quad + \langle (\xi - \Sigma)\nabla T,\ \nabla\varphi\rangle, \qquad (7.20)
\end{aligned}
$$

where ξ is a smoothed version of Σ.

Every term in (7.20) can be shown to converge to zero. For example,

$$
\begin{aligned}
E\int_0^t \left\langle (\Sigma_N - \Sigma)\nabla T^N,\ \nabla\varphi\right\rangle dr\ &\le\ CE\int_0^t \|\Sigma_N - \Sigma\|_{L^2}\left\|\nabla T^N\right\|_{L^2} dr \\
&\le\ C\left(E\int_0^t \|\Sigma_N - \Sigma\|_{L^2}^2\, dr\right)^{1/2} \\
&\quad \times\left(E\int_0^t \left\|\nabla T^N\right\|_{L^2}^2 dr\right)^{1/2}. \qquad (7.21)
\end{aligned}
$$

The first factor tends to zero by the dominated convergence theorem, while the second factor is uniformly bounded by the energy estimates.

All the other terms can be handled similarly (upon choosing better and better ξ to approximate Σ). □

7.1.9. Lemma. *We can decompose M_*^N as $M_*^N(t,\varphi) = M_*^N(t,\varphi)' + R_*^N(t,\varphi)$, where $M_*^N(t,\varphi)'$ is a continuous $\mathcal{F}_t^N$-martingale and $R_*^N(t,\varphi)$ is an error term such that $ER_*^N(t,\varphi)^k \le C_\varphi \triangle t^{k\alpha}$. ($k \ge 1$.)*

Proof. Actually, this is similar to the decomposition in Theorem 5.3.1. For simplicity, let $t = t_q^+$. Then we have $M_*^N(t,\varphi) = M_*^N(t,\varphi)' + R_*^N(t,\varphi)$, where

$$
\begin{aligned}
M_*^N(t,\varphi)'\ &=\ \sum_{i=1}^q \left\langle Q^N\left(t_i^- \wedge \tau^N\right),\ \varphi\right\rangle - \left\langle Q^N\left(t_{i-1}^+ \wedge \tau^N\right),\ \varphi\right\rangle \\
&\quad + \int_{t_{i-1}}^{t_i} \left\langle \Sigma_N \nabla T^N,\ \nabla\varphi\right\rangle 1_{\{r<\tau^N\}}\, dr \\
&= \left\langle Q^N(t_q^- \wedge \tau^N)',\ \varphi\right\rangle - \left\langle Q^N(0^+)',\ \varphi\right\rangle \\
&\quad + \int_0^{t_q\wedge\tau^N} \left\langle \Sigma_N \nabla T^N,\ \nabla\varphi\right\rangle dr, \qquad (7.22)
\end{aligned}
$$

$$R_*^N(t,\varphi) \;=\; \sum_{i=1}^{q} \left\langle Q^N\left(t_i^+ \wedge \tau^N\right),\, \varphi\right\rangle - \left\langle Q^N\left(t_i^- \wedge \tau^N\right),\, \varphi\right\rangle. \tag{7.23}$$

$M_*^N(t,\varphi)'$ consists of terms involving the heat flow process. By Section 6.4.4, it is a martingale with quadratic variation

$$\int_0^{t\wedge\tau^N} \left\langle \Lambda^\epsilon c_N\, f_\delta(T^N)\varphi,\; c_N\, f_\delta(T^N)\varphi\right\rangle dr. \tag{7.24}$$

The error term $R_*^N(t,\varphi)$ is handled by Lemma 5.3.3, which immediately leads to

$$E\left|R_*^N(t,\varphi)\right|^k \le C_\varphi \triangle t^{k\alpha}. \tag{7.25}$$

□

7.1.10. The Final Step—Martingale Property of $M_*(t,\varphi)$. From the previous two lemmas, we deduce that

$$\begin{aligned} EM_*(t,\varphi)\Theta_s(X) \;&=\; \lim_N EM_*^N(t,\varphi)\Theta_s(X^N) \\ &=\; \lim_N E\left[\left(M_*^N(t,\varphi)' + R_*^N(t,\varphi)\right)\Theta_s(X^N)\right] \\ &=\; \lim_N E\left[\left(M_*^N(s,\varphi)' + R_*^N(t,\varphi)\right)\Theta_s(X^N)\right] \\ &=\; EM_*(s,\varphi)\Theta_s(X), \end{aligned}$$

i.e., (7.15) is satisfied, and hence $M_*(t,\varphi)$ is an $\mathcal{F}_t$-martingale.

7.1.11. Proof of 7.1.1 (3)—Quadratic Variation of $M_*(t,\varphi)$. The asserted form for the quadratic variation of $M_*(t,\varphi)$, is equivalent to the following:

For all $0 \le s \le t$, $\Theta_s\colon\ S \longrightarrow R$*, bounded, continuous, and* $\mathcal{B}_s$*-measurable,*

$$\begin{aligned} &E\left\{M_*(t,\varphi)M_*(t,\psi) - \int_0^{t\wedge\tau} \langle \Lambda c f(T)\varphi,\; cf(T)\psi\rangle\; dr\right\}\Theta_s(X) \\ &\qquad = E\left\{M_*(s,\varphi)M_*(s,\psi) - \int_0^{s\wedge\tau} \langle \Lambda c f(T)\varphi,\; cf(T)\psi\rangle\; dr\right\}\Theta_s(X). \end{aligned} \tag{7.26}$$

It suffices to verify the above for $\psi = \varphi$.

Note that, by Section 6.4.4,

$$\left\langle M_*^N(\cdot,\varphi)'\right\rangle_t = \int_0^t \left\langle \Lambda^\epsilon c_N\, f_\delta(T^N)\varphi,\; c_N\, f_\delta(T^N)\varphi\right\rangle 1_{\{r<\tau\}}\, dr. \tag{7.27}$$

Consider

$$\begin{aligned} &E\left\{M_*^N(t,\varphi)^2 - \left\langle M_*^N(\cdot,\varphi)'\right\rangle_t\right\}\Theta_s(X^N) \\ &\;=\; E\left\{\left|M_*^N(t,\varphi)' + R_*^N(t,\varphi)\right|^2 - \left\langle M_*^N(\cdot,\varphi)'\right\rangle_t\right\}\Theta_s(X^N) \end{aligned}$$

$$
\begin{aligned}
&= E\left\{M_*^N(t,\varphi)'^2 - \left\langle M_*^N(\cdot,\varphi)'\right\rangle_t\right\}\Theta_s(X^N)\\
&\quad + E\left\{2M_*^N(t,\varphi)'R_*^N(t,\varphi) + R_*^N(t,\varphi)^2\right\}\Theta_s(X^N)\\
&= E\left\{M_*^N(s,\varphi)'^2 - \left\langle M_*^N(\cdot,\varphi)'\right\rangle_s\right\}\Theta_s(X^N)\\
&\quad + E\left\{2M_*^N(t,\varphi)'R_*^N(t,\varphi) + R_*^N(t,\varphi)^2\right\}\Theta_s(X^N).
\end{aligned}
$$

The second part of the above are the error terms which will tend to zero by Lemma 7.1.9. Now,

$$
\begin{aligned}
&E\left(M_*^N(s,\varphi)'^2 - \left\langle M_*^N(\cdot,\varphi)'\right\rangle_s\right)\Theta_s(X^N)\\
&= E\left(\left|M_*^N(s,\varphi) - R_*^N(s,\varphi)\right|^2 - \left\langle M_*^N(\cdot,\varphi)'\right\rangle_s\right)\Theta_s(X^N)\\
&= E\left(M_*^N(s,\varphi)^2 - \left\langle M_*^N(\cdot,\varphi)'\right\rangle_s\right)\Theta_s(X^N)\\
&\quad + \text{error terms involving } R_*^N(t,\varphi).
\end{aligned}
$$

Thus, what needs to be shown is that, for $0 \le s \le t$,

$$
EM_*(t,\varphi)^2\Theta_s(X) = \lim_N EM_*^N(t,\varphi)^2\Theta_s(X^N), \tag{7.28}
$$

$$
\begin{aligned}
&E\Theta_s(X)\int_0^t \langle \Lambda cf(T)\varphi,\ cf(T)\varphi\rangle\, 1_{\{r<\tau\}}\,dr\\
&= \lim_N E\Theta_s(X^N)\int_0^t \langle \Lambda^\epsilon cf_\delta(T)\varphi,\ cf_\delta(T)\varphi\rangle\, 1_{\{r<\tau^N\}}\,dr.
\end{aligned} \tag{7.29}
$$

Proof of (7.28). This is very similar to the proof for Lemma 7.1.8.

Proof of (7.29). As everything is bounded, by the dominated convergence theorem it is enough to show that P a.s.

$$
\int_0^t \left\langle \Lambda^\epsilon c_N f_\delta(T^N)\varphi,\ c_N f_\delta(T^N)\varphi\right\rangle 1_{\{r<\tau^N\}}\,dr \longrightarrow \int_0^t \langle \Lambda cf(T)\varphi,\ cf(T)\varphi\rangle\, 1_{\{r<\tau\}}\,dr.
$$

Since Λ^ϵ is given by a bounded kernel (4.4), the L.H.S of the above can be written as (we omit the harmless τ^N and τ)

$$
\int_0^t \iint_{(x,y)\in(\mathcal{O}\times\mathcal{O})} \left\{\Lambda^\epsilon(x,y)c_N(x)c_N(y)\varphi(x)\varphi(y)\, f_\delta(T^N(x))\, f_\delta(T^N(y))\right\} d\mathcal{L}^n x\, d\mathcal{L}^n y\, dr.
$$

From this, it is clear that the asserted convergence holds because of the following convergence in $L^2(\mathcal{O}\times\mathcal{O})$ and with all the functionals being bounded (recall that f_δ is uniformly Lipschitz in δ):

$$
\Lambda^\epsilon(\cdot,\cdot) \longrightarrow \Lambda(\cdot,\cdot); \qquad c_N(\cdot) \longrightarrow c(\cdot); \qquad f_\delta(T^N(\cdot)) \longrightarrow f(T(\cdot)).
$$

7.1.12. The Remaining Steps of Theorem 7.1.1. Statement (4) of the theorem can be verified in exactly the same way as above.

Finally, we take a sequence of $\eta \longrightarrow 0$ consisting of points of continuity. Then $\tau_\eta \longrightarrow \tau$. By Lemma 7.1.3, the whole theorem remains true.

7.2. Weak Formulation

In this section, we reformulate the previous statement of the heat equation in terms of stochastic integration with respect to an infinite dimensional Wiener Process. Precisely,

7.2.1. Theorem. *There is a probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ *(which is an* **extension** *of* $(\Omega, \mathcal{F}, P)$*) and an* $L^2(\mathcal{O})$*-valued Wiener Process* $\left\{\tilde{W}_t,\ \tilde{\mathcal{F}}_t;\ 0 \le t < \infty\right\}$ *with covariance operator* Λ *such that*

1. $\tilde{P}$ *a.s. for all* $\varphi \in C^\infty(\mathcal{O})$

$$(Q(t \wedge \tau),\ \varphi) = (Q_0,\ \varphi) - \int_0^{t\wedge\tau} \langle \Sigma_K \nabla T,\ \nabla\varphi\rangle\, dr + \int_0^{t\wedge\tau} \Big(c_K\, f(T)\, d\tilde{W}_r,\ \varphi\Big). \tag{7.30}$$

The above can also be written as

$$dQ_t = \operatorname{div}(\Sigma_K \nabla T)\, dt + c_K\, f(T)\, d\tilde{W}_t \qquad \text{for } 0 \le t < \tau. \tag{7.31}$$

2. $$J(t \wedge \tau) = \int_0^{t\wedge\tau} \Big(c_K F'(T)\, f(T),\ d\tilde{W}_r\Big). \tag{7.32}$$

The general technique for converting the martingale formulation to a statement of this sort is standard ([DZ] 8.2 and [MM] p. 77). For completeness, we outline the procedure here, which is simplified due to the existence of the inverse of the multiplicative operator $c_K\, f(T)$—$(c_K\, f(T))^{-1}$.

7.2.2. Step I—Construction of a Continuous $L^2(\mathcal{O})$-Valued Martingale. Let $\{\varphi_i\}_{i\ge1}$ be a O.N.B. of $L^2(\mathcal{O})$ with $\varphi_i \in C^\infty(\mathcal{O})$. Define $H(t) = \sum_i M_*(t, \varphi_i)\varphi_i$. Using the fact that $\sum_i E\, |M_*(t, \varphi_i)|^2 = \sum_i E \int_0^t \langle \Lambda c_K\, f(T)\varphi_i,\ c_K\, f(T)\varphi_i\rangle\, dr < \infty$, we conclude that $H(t)$ is a continuous square integrable $L^2(\mathcal{O})$-valued martingale with covariance operator

$$\langle\!\langle H \rangle\!\rangle_t = \int_0^{t\wedge\tau} c_K\, f(T)\Lambda c_K\, f(T)\, dr. \tag{7.33}$$

In addition, for all $\varphi \in C^\infty(\mathcal{O})$,

$$(Q(t \wedge \tau),\ \varphi) = (Q(0),\ \varphi) - \int_0^{t\wedge\tau} \langle \Sigma_K \nabla T,\ \nabla\varphi\rangle\, dr + (H(t),\ \varphi). \tag{7.34}$$

7.2.3. Step II—Construction of a Wiener Process. Let $(\Omega', \mathcal{F}', P')$ be a new probability space ("independent" of $(\Omega, \mathcal{F}, P)$) equipped with a Wiener Process $\left\{W'_t,\ \mathcal{F}'_t;\ 0 \le t < \infty\right\}$ with covariance operator Λ. Construct the following extension of Ω:

$$\tilde{\Omega} = \Omega \times \Omega' = \left\{(\omega, \omega')\right\}, \qquad \tilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}',$$
$$\tilde{P} = P \otimes P', \qquad \tilde{\mathcal{F}}_i = \mathcal{F}_t \otimes \mathcal{F}'_t, \qquad t \in [0, 1]. \tag{7.35}$$

For $(\omega, \omega') \in \Omega \times \Omega'$, define

$$\tilde{W}'(t, \omega, \omega') = W'(t, \omega'), \tag{7.36}$$

and make the following obvious modifications:

$$\begin{aligned} K(t, \omega, \omega') &= K(t, \omega), \qquad T(t, \omega, \omega') = T(t, \omega), \\ Q(t, \omega, \omega') &= Q(t, \omega), \qquad H(t, \omega, \omega') = H(t, \omega). \end{aligned} \tag{7.37}$$

Note that the multiplicative operator $c_K f(T)$ has an inverse given by $(c_K f(T))^{-1}$ (unbounded). We perform the following operations:

$$\begin{aligned} H(t) &= \int_0^{t\wedge\tau} dH_r = \int_0^{t\wedge\tau} (c_K f(T))(c_K f(T))^{-1} \, dH_r \\ &= \int_0^{t\wedge\tau} c_K f(T) \left\{ 1_{\{r<\tau\}}(c_K f(T))^{-1} \, dH_r + 1_{\{r\geq\tau\}} \, d\tilde{W}'_r \right\}. \end{aligned} \tag{7.38}$$

Set

$$\tilde{W}_t = \int_0^t 1_{\{r<\tau\}}(c_K f(T))^{-1} \, dH_r + \int_0^t 1_{\{r\geq\tau\}} \, d\tilde{W}'_r. \tag{7.39}$$

Compute

$$\begin{aligned} \left\langle\!\left\langle \tilde{W}(\cdot) \right\rangle\!\right\rangle_t &= \int_0^t 1_{\{r<\tau\}} \left[(c_K f(T))^{-1} d \left\langle\!\left\langle H \right\rangle\!\right\rangle_r (c_K f(T))^{-1} \right] + \int_0^t 1_{\{r\geq\tau\}} d \left\langle\!\left\langle \tilde{W}' \right\rangle\!\right\rangle_r \\ &= \int_0^t 1_{\{r<\tau\}} \left[(c_K f(T))^{-1} (c_K f(T)) \Lambda (c_K f(T)) (c_K f(T))^{-1} \right] dr \\ &\quad + \int_0^t 1_{\{r\geq\tau\}} \Lambda \, dr \\ &= \int_0^t 1_{\{r<\tau\}} \Lambda + 1_{\{r\geq\tau\}} \Lambda \, dr = t\Lambda. \end{aligned} \tag{7.40}$$

Hence, $\tilde{W}_t$ is a Wiener Process on $\tilde{\Omega}$ with covariance operator Λ.

Now (7.38) is the same as

$$H(t) = \int_0^{t\wedge\tau} c_K f(T) \, d\tilde{W}_r. \tag{7.41}$$

From (7.34), (7.30) holds.

7.2.4. Step III—Representation of J. Statement (2) about J can be seen by the following computations:

$$\langle J(\cdot) \rangle_{t\wedge\tau} = \int_0^{t\wedge\tau} \left\langle \Lambda c_K F'(T) f(T), \; c_K F'(T) f(T) \right\rangle dr,$$

$$\langle J(\cdot), \; H(\cdot, \varphi) \rangle_{t\wedge\tau} = \langle J(\cdot), \; M_*(\cdot, \varphi) \rangle_{t\wedge\tau} = \int_0^{t\wedge\tau} \left\langle \Lambda c_K F'(T) f(T), \; c_K f(T)\varphi \right\rangle dr,$$

$$\left\langle \int_0^{\cdot} \left(c_K F'(T) f(T), \; d\tilde{W}_r \right) \right\rangle_{t\wedge\tau} = \int_0^{t\wedge\tau} \left\langle \Lambda c_K F'(T) f(T), \; c_K F'(T) f(T) \right\rangle dr.$$

Note that $\tilde{W}_{t\wedge\tau} = \int_0^{t\wedge\tau} \big((c_K f(T))^{-1},\, dH_r\big)$. (See (7.39).) Now compute[21]

$$
\begin{aligned}
&\left\langle J(\cdot),\, \int_0^{\cdot} \Big(c_K F'(T) f(T),\, \tilde{W}_r\Big)\right\rangle_{t\wedge\tau} \\
&= \left\langle J(\cdot),\, \int_0^{\cdot} \big(c_K F'(T) f(T),\, (c_K f(T))^{-1} dH_r\big)\right\rangle_{t\wedge\tau} = \left\langle J(\cdot),\, \int_0^{\cdot} \big(F'(T),\, dH_r\big)\right\rangle_{t\wedge\tau} \\
&= \left\langle J(\cdot),\, \int_0^{\cdot} \sum_i \big(F'(T),\, \varphi_i\big)\,(dH_r,\, \varphi_i)\right\rangle_{t\wedge\tau} \\
&= \sum_i \int_0^{t\wedge\tau} \big(F'(T),\, \varphi_i\big)\big\langle \Lambda c_K F'(T) f(T),\, c_K f(T)\varphi_i\big\rangle\, dr \\
&= \int_0^{t\wedge\tau} \big\langle \Lambda c_K F'(T) f(T),\, c_K F'(T) f(T)\big\rangle\, dr.
\end{aligned}
$$

From the above, we deduce that $\left\langle J(\cdot) - \int_0^{\cdot} \Big(c_K F'(T) f(T),\, d\tilde{W}_r\Big)\right\rangle_{t\wedge\tau}$ equals

$$
\langle J(\cdot)\rangle_{t\wedge\tau} + \left\langle \int_0^{\cdot} \Big(c_K F'(T) f(T),\, d\tilde{W}_r\Big)\right\rangle_{t\wedge\tau} - 2\left\langle J(\cdot),\, \int_0^{\cdot} \Big(c_K F'(T) f(T),\, d\tilde{W}_r\Big)\right\rangle_{t\wedge\tau},
$$

which is zero. Hence,

$$
J(t\wedge\tau) = \int_0^{t\wedge\tau} \Big(c_K F'(T) f(T),\, d\tilde{W}_r\Big). \tag{7.42}
$$

8. The Gibbs-Thomson Condition

Our goal is as follows.

8.0.1. Theorem (Gibbs-Thomson Condition). *P a.s. on $\{(t,\omega):\ t < \tau(\omega)\}$, for all C^1 time-varying random vector fields g,*[22]

$$
\langle \partial K(t),\, g\rangle = \int_{K(t)} \operatorname{div}(H(T(t))g)\; d\mathcal{L}^n, \tag{8.1}
$$

where $\langle \partial K,\, g\rangle = \dfrac{d}{ds}\Phi\left(G_{s\sharp}\partial K\right)\Big|_{s=0}$*, with* $G_s(t,x) = x + sg(t,x)$.

This is the heart of the whole paper. Its proof involves an intricate combination of the estimates from the minimization steps and the smoothed heat flow.

As a by-product of the above theorem, we can also show that, in low dimensions, the ∂K's enjoy some regularity properties. Precisely,

[21] The unboundedness of some of the functionals can be easily dealt with by some cut-off and truncation arguments.

[22] The notations are from Section 2.1.7.

8.0.2. Theorem. *For $d\mathcal{L}^1 \times dP$ a.s. on $\{(t, \omega) \colon t < \tau(\omega)\}$, the following is true.*

$n = 2$**:** *$\partial K(t)$ is a one-dimensional differentiable submanifold of $\mathcal{O}$ without boundary and, for any C^1 vector field g on $\mathcal{O}$,*

$$\frac{d}{ds} \Phi \left(G_{s\sharp} \partial K(t) \right) \Big|_{s=0} = \int_{x \in \partial K(t)} H(T(x,t)) \left\langle n_{K(t)},\ g(x) \right\rangle d\mathcal{H}^1 x. \tag{8.2}$$

$n = 3$**:** *$\partial K(t)$ is the homeomorphic image in $\mathcal{O}$ of a compact two dimensional manifold without boundary.*

For Theorem 8.0.1, we will actually prove the following:

$$E \int_0^\tau \langle \partial K(t),\ g(t) \rangle\ dt = E \int_0^\tau \int_{x \in K(t)} \operatorname{div}\left(H(T(t,x)) g(t,x) \right)\ d\mathcal{L}^n x\ dt. \tag{8.3}$$

Essentially, it says that, for all g, $\langle \partial K(t),\ g \rangle = \int_{K(t)} \operatorname{div}(H(T(t))g)\ d\mathcal{L}^n$, P $a.s.$ $\{(t, \omega) \colon t < \tau(\omega)\}$. By the fact that the space of C^1 vector fields is separable, we can then find an almost sure event (independent of g) in $\{(t, \omega) \colon t < \tau(\omega)\}$ such that (8.1) is true.

8.1. Strategy for Proving the Gibbs-Thomson Condition

The underlying picture is as follows. The Gibbs-Thomson condition is restored after every minimization (Theorem 3.2.3). In between them, the heat is diffused. This will destroy the Gibbs-Thomson condition. In order to prove Theorem 8.0.1, two points need to be taken care of:

- The total error due to heat flow tends to zero as $\triangle t \longrightarrow 0$. For this part, we will make full use of the regularity properties of the temperature fields under heat flow.
- The approximated crystals need to converge in a topology stronger than L^1. The reason is that (8.1) involves the convergence of a quantity defined on the boundary of a crystal that is a lower dimensional set. This condition is too singular for the L^1 norm. In order to achieve the stated result, we will improve the crystal convergence to the **varifold** sense. Under this notion, the tangent planes of the boundary of the convergent crystals also match up with those of the limiting crystal. Precisely, we will show that $\Phi(\partial K) = \lim_N \Phi(\partial K^N)$.[23] The proof exploits the fact that the approximating crystals are minimizers of some energy functionals.

For the following, superscript N denotes the approximations corresponding to $\triangle t = 1/N$, and subscript i means that $t_i = i \triangle t$ and $i^+ = t_i^+$. For all t, let i be such that $t_i \leq t < t_{i+1}$.

[23] The L^1 convergence only gives the lower semicontinuity.

We start from the identities for the discrete time approximations. Consider the following $\epsilon/3$-type argument:

$$
\begin{aligned}
&\int_0^\tau \left| \langle \partial K(t),\, g\rangle - \int_{K(t)} \operatorname{div}(H(T(t)g))\, d\mathcal{L}^n \right| dt \\
&\le \int_0^\tau \left| \langle \partial K(t),\, g\rangle - \left\langle \partial K^N(t),\, g\right\rangle d\mathcal{L}^n \right| dt \\
&\quad + \int_0^\tau \left| \left\langle \partial K^N(t),\, g\right\rangle - \int_{K(t)} \operatorname{div}(H(T(t)g))\, d\mathcal{L}^n \right| dt \\
&\le \int_0^\tau \left| \langle \partial K(t),\, g\rangle - \left\langle \partial K^N(t),\, g\right\rangle d\mathcal{L}^n \right| dt \qquad (8.4)\\
&\quad + \int_0^1 \left| \left\langle \partial K^N(t),\, g\right\rangle - \int_{K(t)} \operatorname{div}(H(T(t)g))\, d\mathcal{L}^n \right| \left| 1_{\{t<\tau\}} - 1_{\{t<\tau^N\}} \right| dt \qquad (8.5)\\
&\quad + \int_0^{\tau^N} \left| \left\langle \partial K^N(t),\, g\right\rangle - \int_{K(t)} \operatorname{div}(H(T(t)g))\, d\mathcal{L}^n \right| dt. \qquad (8.6)
\end{aligned}
$$

By Theorem 3.2.3, right after each minimization step, the Gibbs-Thomson condition holds. Hence, we have $\left\langle \partial K^N_{i^+},\, g\right\rangle = \int_{K^N_{i^+}} \operatorname{div}\left(H(T^N_{i^+})g\right) d\mathcal{L}^n$. Since the crystals do not change shapes in between the minimization steps, (8.6) can then be rewritten and bounded by

$$
\begin{aligned}
&\int_0^1 \left| \int_{K^N(t)} \operatorname{div}\left(H(T^N_{i^+})g\right) d\mathcal{L}^n - \int_{K(t)} \operatorname{div}(H(T(t))g)\, d\mathcal{L}^n \right| dt \\
&\le \int_0^1 \left| \int_{K^N(t)} \operatorname{div}\left(H(T^N(t))g\right) d\mathcal{L}^n - \int_{K(t)} \operatorname{div}(H(T(t))g)\, d\mathcal{L}^n \right| dt \\
&\quad + \int_0^1 \left| \int_{K^N(t)} \operatorname{div}\left(H(T^N_{i^+})g\right) d\mathcal{L}^n - \int_{K^N(t)} \operatorname{div}\left(H(T^N(t))g\right) d\mathcal{L}^n \right| dt \\
&= L_2 + L_3. \qquad (8.7)
\end{aligned}
$$

We are going to show that:

(Section 8.2) $EL_2 \longrightarrow 0$—convergence of the temperature fields.
(Section 8.3) $EL_3 \longrightarrow 0$—vanishing of the error for the Gibbs-Thomson condition during heat flow.
(Section 8.4) $\left\langle \partial K^N_t,\, g\right\rangle \longrightarrow \langle \partial K_t,\, g\rangle\, dt \times dP$ a.s. on $\{(t,\omega)\colon t < \tau\}$—varifold convergence of the crystal positions. This will take care of (8.4).
(Section 8.5) $E\displaystyle\int_0^1 \left|\left\langle \partial K^N_t,\, g\right\rangle\right|^2 dt < C$ for all N—this enables us to use Lemma 7.1.2 to take care of (8.4) and (8.5).

The assertion of the theorem will then follow.

The most difficult parts are Section 8.3 and Section 8.4. The basic ideas follow [AW] Chapter 8, together with the probability estimates. In the following, we implicitly assume the functional form of F and H in Section 3.1.

8.2. Convergence of EL_2 *to* 0

The following restatement of [AW] Theorem 8.1 will lead to the asserted convergence of EL_2.

8.2.1. Theorem. *Suppose $K^N(t)$ and $T^N(t)$ are crystal positions and temperature fields converging to $K(t)$ and $T(t)$ in the following sense ($U = 1/T$):*

1. $E\int_0^1 \left\|K^N(t) - K(t)\right\|_{L^1}\, dt \longrightarrow 0$;
2. $E\int_0^1 \left\|T^N(t) - T(t)\right\|_{L^2}^2\, dt$ *and* $E\int_0^1 \left\|U^N(t) - U(t)\right\|_{L^2}^2\, dt \longrightarrow 0$;
3. $E\int_0^1 \left\|\nabla T^N(t)\right\|_{L^2}^2 + \left\|\nabla U^N(t)\right\|_{L^2}^2\, dt = C < \infty$ *for all N (and hence the same estimate holds for T and U).*

Then, for all random time-varying bounded C^1 vector fields g, we have

$$E\int_0^1 \left|\int_{K^N(t)} \frac{\partial}{\partial x_i}\left(H(T^N(t))g(t)\right)\, d\mathcal{L}^n - \int_{K(t)} \frac{\partial}{\partial x_i}\left(H(T(t))g(t)\right)\, d\mathcal{L}^n\right| dt \longrightarrow 0. \tag{8.8}$$

Proof. Considering the growth rate for $H(T)$ and $H'(T)$ (3.7), we have

$$\begin{aligned}\frac{\partial}{\partial x_i}(H(T)g) &= H'(T)\frac{\partial T}{\partial x_i}g + H(T)\frac{\partial g}{\partial x_i}\\ &= C_2(-6U^3 - 2T + L(T))\frac{\partial T}{\partial x_i}g + C_2(3U^2 - T^2 + J(T))\frac{\partial g}{\partial x_i}\\ &= C_2\left(6U\frac{\partial U}{\partial x_i} - 2T\frac{\partial T}{\partial x_i} + L(T)\frac{\partial T}{\partial x_i}\right)g\\ &\quad + C_1(3U^2 - T^2 + J(T))\frac{\partial g}{\partial x_i}.\end{aligned} \tag{8.9}$$

Hence,

$$\begin{aligned}&K^N\frac{\partial}{\partial x_i}\left(H(T^N)g\right) - K\frac{\partial}{\partial x_i}(H(T)g)\\ &= K^N\left[C_2\left(6U^N\frac{\partial U^N}{\partial x_i} - 2T^N\frac{\partial T^N}{\partial x_i} + L(T^N)\frac{\partial T^N}{\partial x_i}\right)g\right]\\ &\quad - K\left[C_2\left(6U\frac{\partial U}{\partial x_i} - 2T\frac{\partial T}{\partial x_i} + L(T)\frac{\partial T}{\partial x_i}\right)g\right]\\ &\quad + K^N\left[C_1(3U^{N2} - T^{N2} + J(T^N))\frac{\partial g}{\partial x_i}\right]\\ &\quad - K\left[C_1(3U^2 - T^2 + J(T))\frac{\partial g}{\partial x_i}\right].\end{aligned} \tag{8.10}$$

We will consider some "typical" terms and show their convergence. For example,

$$\begin{aligned}
&E\int_0^1\left|\int_{\mathcal{O}} K^N T^N g\frac{\partial T^N}{\partial x_i} - KTg\frac{\partial T}{\partial x_i}\,d\mathcal{L}^n\right| dt\\
&\quad= E\int_0^1\left|\int_{\mathcal{O}}\left(K^N T^N - KT\right)g\frac{\partial T^N}{\partial x_i} + KTg\left(\frac{\partial T^N}{\partial x_i} - \frac{\partial T}{\partial x_i}\right)d\mathcal{L}^n\right| dt\\
&\quad\leq CE\int_0^1\left\|K^N T^N - KT\right\|_{L^2}\left\|\frac{\partial T^N}{\partial x_i}\right\|_{L^2} dt\\
&\qquad+E\int_0^1\left|\int_{\mathcal{O}}(KTg-\psi)\left(\frac{\partial T^N}{\partial x_i} - \frac{\partial T}{\partial x_i}\right)d\mathcal{L}^n\right| dt\\
&\qquad+E\int_0^1\left|\int_{\mathcal{O}}\psi\left(\frac{\partial T^N}{\partial x_i} - \frac{\partial T}{\partial x_i}\right)d\mathcal{L}^n\right| dt,
\end{aligned}$$

where ψ is a smooth approximation of KTg. We will consider each term of the above.

- $\left(CE\int_0^1\left\|K^N T^N - KT\right\|_{L^2}\left\|\frac{\partial T^N}{\partial x_i}\right\|_{L^2} dt\right)^2$ is bounded by

$$\begin{aligned}
&\left(E\int_0^1\left\|K^N T^N - KT\right\|_{L^2}^2 dt\right)\left(E\int_0^1\left\|\frac{\partial T^N}{\partial x_i}\right\|_{L^2}^2 dt\right)\\
&\quad\leq CE\int_0^1\left\|K^N(T^N-T)+(K^N-K)T\right\|_{L^2}^2 dt\\
&\quad\leq CE\int_0^1\left\|T^N-T\right\|_{L^2}^2 dt + CE\int_0^1\left\|(K^N-K)T\right\|_{L^2}^2 dt.
\end{aligned}$$

  The first term tends to zero by assumption, while the second term tends to zero by the dominated convergence theorem.

- $E\int_{\mathcal{O}\times[0,1]}\left|(KTg-\psi)\left(\frac{\partial T^N}{\partial x_i} - \frac{\partial T}{\partial x_i}\right)\right| d\mathcal{L}^n\,dt$ is bounded by

$$\left(E\int_0^1\|KTg-\psi\|_{L^2}^2\,dt\right)^{1/2}\left(E\int_0^1\left\|\frac{\partial T^N}{\partial x_i} - \frac{\partial T}{\partial x_i}\right\|_{L^2}^2 dt\right)^{1/2}.$$

  This can be made as small as possible by choosing better and better ψ and by the assumption on the uniform bound on the Dirichlet integral of the T^N's.

- $E\int_0^1\left|\int_{\mathcal{O}}\psi\left(\frac{\partial T^N}{\partial x_i} - \frac{\partial T}{\partial x_i}\right)d\mathcal{L}^n\right| dt$ can be bounded by

$$E\int_0^1\int_{\mathcal{O}}|\nabla\psi|\left|T^N-T\right|\,d\mathcal{L}^n\,dt,$$

  which will tend to zero by arguments similar to those above.

Each term converges to zero either by assumption or by the dominated convergence theorem.

All the other terms can be handled similarly. □

8.3. Convergence of EL_3 *to* 0

We actually prove the following slightly stronger version (Theorem 8.3.3):

$$\lim_N E\int_0^1 \left|\int_{K^N(t)} \operatorname{div}\left(H(T^N_{i^+})g\right)\, d\mathcal{L}^n - \int_{K^N(t)} \operatorname{div}\left(H(T^N(t))g\right)\, d\mathcal{L}^n\right|^2 dt \longrightarrow 0. \tag{8.11}$$

It is at this point that we will use the regularity estimates of the temperature fields right after minimizations (Theorem 3.2.2) and the estimates concerning the smoothed version of the heat equation (4.3).

First we invoke a tool. It estimates the integration of the trace of a function on a hypersurface.

8.3.1. Proposition (AW, Thm. 8.2). *Let K be a set of finite perimeter; f is an $H^1(\mathcal{O})$ vector field. Then, for all $M > 0$,*

$$\begin{aligned}&\left|\int_{p\in\partial K} \langle f(p),\, n_K\rangle\, d\mathcal{H}^{n-1} p\right|^2 \\ &\quad\le C\left\{M\,\|f\|_{L^2}\,\|\nabla f\|_{L^2} + M^{-1/2}\left(\mathcal{H}^{n-1}(\partial K)+1\right)\|\nabla f\|^2_{L^2}\right\},\end{aligned}$$

where C is a universal constant depending on the size and dimension of $\mathcal{O}$. n_K is the outward normal to ∂K, and $\mathcal{H}^{n-1}$ refers to the $(n-1)$-Hausdorff measure. The L^2 norms on the right-hand side refer to $L^2(\mathcal{O})$.

To make use of it, we let $f_i(t) = \left[H(T^N(t_i^+)) - H(T^N(t))\right]g,\ t_i \le t < t_{i+1}$. Now consider the following form of L_3:

$$\begin{aligned} L_3' &= \int_0^1 \left|\int_{K^N(t)} \operatorname{div}\left(H(T^N(t_i^+))g\right) - \operatorname{div}\left(H(T^N(t))g\right)\, d\mathcal{L}^n\right|^2 dt \\ &= \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \left|\int_{K^N(t)} \operatorname{div}\left(H(T^N(t_i^+))g\right) - \operatorname{div}\left(H(T^N(t))g\right)\, d\mathcal{L}^n\right|^2 dt \\ &= \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \left|\int_{K^N(t)} \operatorname{div}(f_i(t))\, d\mathcal{L}^n\right|^2 dt \\ &= \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \left|\int_{\partial K^N(t)} \langle f_i(t),\, n_{K^N}\rangle\, d\mathcal{H}^{n-1}\right|^2 dt \\ \Longrightarrow EL_3' &= \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} E\left|\int_{\partial K^N(t)} \langle f_i(t),\, n_{K^N}\rangle\, d\mathcal{H}^{n-1}\right|^2 dt; \end{aligned} \tag{8.12}$$

i.e., we have written EL_3' as a sum of all the deviations from the Gibbs-Thomson condition caused by the heat flow process in between the minimizations. We will make use of the above proposition to bound $E|\int_{p\in\partial K^N} \langle f_i(t,p),\, n_{K^N}\rangle\, d\mathcal{H}^{n-1}p|^2$. (In what follows, the N's will be suppressed.)

- $E\left|\int_{p\in\partial K}\langle f_i(t,p),\, n_K\rangle\, d\mathcal{H}^{n-1}p\right|^2$—For all $M>0$,

$$
\begin{aligned}
&E\left|\int_{p\in\partial K}\langle f_i(t,p),\, n_K\rangle\, d\mathcal{H}^{n-1}p\right|^2 \\
&\le C\left\{ME\left(\|f_i\|_{L^2}\|\nabla f_i\|_{L^2}\right)+\frac{1}{M^{1/2}}E\left(\mathcal{H}^{n-1}(\partial K_i)+1\right)\|\nabla f_i\|_{L^2}^2\right\} \\
&\le C\left\{M\left(E\|f_i\|_{L^2}^2\right)^{1/2}\left(E\|\nabla f_i\|_{L^2}^2\right)^{1/2}+\frac{1}{M^{1/2}}\left[E\left(\mathcal{H}^{n-1}(\partial K_i)+1\right)^2\right]^{1/2}\left[E\|\nabla f_i\|_{L^2}^4\right]^{1/2}\right\} \\
&\le C\left\{M\left(E\|f_i\|_{L^2}^2\right)^{1/2}\left(E\|\nabla f_i\|_{L^2}^4\right)^{1/4}+\frac{1}{M^{1/2}}\left[E\left(\mathcal{H}^{n-1}(\partial K_i)+1\right)^2\right]^{1/2}\left[E\|\nabla f_i\|_{L^2}^4\right]^{1/2}\right\}.
\end{aligned}
\tag{8.13}
$$

- $E\|f_i\|_{L^2}^2$—We repeat the notation of (4.15): For $t\in[t_i,t_{i+1})$, $x\in\mathcal{O}$,

$$
m_i=\inf_{(t,x)}\left\{T(t,x),\,T(t_i^+,x)\right\},\qquad M_i=\sup_{(t,x)}\left\{T(t,x),\,T(t_i^+,x)\right\}. \tag{8.14}
$$

From (3.6),

$$
\begin{aligned}
|f_i(t)| &= |(H(T(t))-H(T(t_i)))\,g| \\
&\le C\,\|g\|_\infty\,|T(t)-T(t_i)|\left(\frac{1}{T^2(t_i)T(t)}+\frac{1}{T(t_i)T^2(t)}+T(t)+T(t_i)\right).
\end{aligned}
$$

Hence (C depends on $\|g\|_\infty$),

$$
\begin{aligned}
\|f_i(t)\|_{L^2}^2 &\le C\,\|T(t)-T(t_i)\|_{L^2}^2\left(m_i^{-3}+M_i\right)^2, \qquad t\in[t_i,t_{i+1}), \\
E\|f_i(t)\|_{L^2}^2 &\le CE\left(\|T(t)-T(t_i)\|_{L^2}^2\left(m_i^{-3}+M_i\right)^2\right), \\
&\le C\left(E\|T(t)-T(t_i)\|_{L^2}^8\right)^{1/4}\left[\left(Em_i^{-24}\right)^{1/4}+\left(EM_i^8\right)^{1/4}\right].
\end{aligned}
\tag{8.15}
$$

- $E\|\nabla f_i\|_{L^2}^4$—Now,

$$
\nabla f_i(t)=[\nabla H(T(t))-\nabla H(T(t_i))]\,g+[H(T(t))-H(T(t_i))]\,\nabla g.
$$

This implies that (C depends on $\|g\|_\infty$ and $\operatorname{Lip} g$)

$$
\begin{aligned}
|\nabla f_i(t)| &\le \|g\|_\infty\left\{\left|H'(T(t))\right|\,|\nabla T(t)|+\left|H'(T(t_i))\right|\,|\nabla T(t_i)|\right\} \\
&\quad+(\operatorname{Lip} g)\,|H(T(t))-H(T(t_i))| \\
&\le C\left\{|\nabla T(t)|+|\nabla T(t_i)|+|T(t)-T(t_i)|\right\}\left[m_i^{-3}+M_i\right].
\end{aligned}
$$

Hence,

$$
\|\nabla f_i(t)\|_{L^2}^4 \le C\left\{\|\nabla T(t)\|_{L^2}^4+\|\nabla T(t_i)\|_{L^2}^4+\|T(t)-T(t_i)\|_{L^2}^4\right\}\left[m_i^{-3}+M_i\right]^4
$$

$$\begin{aligned} E\,\|\nabla f_i(t)\|_{L^2}^4 &\leq C\left(E\left\{\|\nabla T(t)\|_{L^2}^8 + \|\nabla T(t_i)\|_{L^2}^8 + \|T(t) - T(t_i)\|_{L^2}^8\right\}\right)^{1/2} \\ &\quad \times \left(E\left[m_i^{-3} + M_i\right]^8\right)^{1/2} \\ &\leq C\left[\left(E\,\|\nabla T(t)\|_{L^2}^8\right)^{1/2} + \left(E\,\|\nabla T(t_i)\|_{L^2}^8\right)^{1/2} + \left(E\,\|T(t) - T(t_i)\|_{L^2}^8\right)^{1/2}\right] \\ &\quad \times \left[\left(E m_i^{-24}\right)^{1/2} + \left(E M_i^8\right)^{1/2}\right]. \end{aligned} \tag{8.16}$$

8.3.2. The Final Step for Convergence of EL_3'. From (8.15) and (8.16), we thus need to estimate

$$E\left(m_i^{-24}\right), \quad E\left(M_i^8\right), \quad E\,\|\nabla T(t)\|_{L^2}^8, \quad E\,\|T(t) - T(t_i)\|_{L^2}^8, \quad t \in [t_i, t_{i+1}).$$

This is exactly the reason we consider the smoothed heat equation (4.3) and prove estimates for the temperature value and gradient.

Now apply Theorems 4.2.2, 4.3.1, and 4.4.1 to (8.15) and (8.16),

$$\begin{aligned} E\,\|f_i(t)\|_{L^2}^2 &\leq C\triangle t^{1/2}\epsilon^{-2n-2}\left(\triangle t^{-\frac{6n\alpha}{3n+2}} + \triangle t^{-2\alpha}\right) \\ &\leq C\triangle t^{1/2-2\alpha}\epsilon^{-2n-2}, \\ E\,\|\nabla f_i(t)\|_{L^2}^4 &\leq C\left[\frac{1}{\epsilon^{4n+4}\triangle t^{4\alpha}} + \frac{\triangle t}{\epsilon^{4n+4}}\right]\left[\triangle t^{-\frac{12n\alpha}{3n+2}} + \triangle t^{-4\alpha}\right] \\ &\leq C\epsilon^{-4n-4}\triangle t^{-8\alpha}. \end{aligned}$$

Substitute the above into (8.13),

$$\begin{aligned} E\left|\int_{p\in\partial K} f_i(t,p)\,d\mathcal{H}^{n-1}p\right|^2 &\leq C\left\{M\triangle t^{1/4-\alpha}\epsilon^{-n-1}\epsilon^{-n-1}\triangle t^{-2\alpha} + M^{-1/2}\epsilon^{-2n-2}\triangle t^{-4\alpha}\right\} \\ &\leq C\left\{M\triangle t^{1/4-3\alpha}\epsilon^{-2n-2} + M^{-1/2}\triangle t^{-4\alpha}\epsilon^{-2n-2}\right\}. \end{aligned}$$

Set $M = \triangle t^{-\beta}$, $\epsilon^{2n+2} = \triangle t^{\gamma}$,[24] and the above becomes

$$C\left\{\triangle t^{-\beta}\triangle t^{1/4-3\alpha}\triangle t^{-\gamma} + \triangle t^{\beta/2}\triangle t^{-4\alpha}\triangle t^{-\gamma}\right\} \leq C\left\{\triangle t^{1/4-\beta-3\alpha-\gamma} + \triangle t^{\beta/2-4\alpha-\gamma}\right\}.$$

Choose $\alpha,\ \beta,\ \gamma$ such that

$$\begin{aligned} 1 &> 4\beta + 12\alpha + 4\gamma, \\ \beta &> 8\alpha + 2\gamma. \end{aligned}$$

As long as $\alpha < 1/44$, such a choice can always be made. For example,

$$\gamma = \frac{1}{64}, \qquad \alpha = \frac{1}{64}, \qquad \beta = \frac{11}{64}.$$

[24] Recall the remark at the beginning of Section 4.3.

Then,

$$E\left|\int_{p\in\partial K} f_i(t,p)\,d\mathcal{H}^{n-1}p\right|^2 \le C\left\{\triangle t^{(16-11-3-1)/64} + \triangle t^{(5.5-4-1)/64}\right\}$$
$$\le C\left\{\triangle t^{1/64} + \triangle t^{0.5/64}\right\} \le C\triangle t^{1/128}.$$

Put the above into (8.12), and we finally get the following.

8.3.3. Theorem.

$$EL_3' \le \int_0^1 C\triangle t^{1/128}\,dt \longrightarrow 0 \qquad \text{as } \triangle t \longrightarrow 0.$$

8.4. Varifold Convergence of the Crystals

The purpose is to show

$$\lim_N \left\langle \partial K_t^N,\ g\right\rangle = \langle \partial K_t,\ g\rangle, \qquad d\mathcal{L}^1 \times dP \text{ a.s. on } \{(t,\omega)\colon t < \tau(\omega)\}. \tag{8.17}$$

This will then lead to the convergence to zero of (8.4).

As mentioned at the beginning of Section 8.1, L^1 norm is not sufficient to conclude the above. Instead, we will prove varifold convergence of the crystals. By Appendix A, the following statement implies varifold convergence.

8.4.1. Theorem (Convergence of Surface Energy).

$$\lim_N \Phi\left(\partial K_t^N\right) = \Phi(\partial K_t), \qquad d\mathcal{L}^1 \times dP \text{ a.s. on } \{(t,\omega)\colon t < \tau(\omega)\}. \tag{8.18}$$

The strategy of proving the above is as follows (heuristically).

Since $K_t^N \longrightarrow K_t$ in L^1, this implies $\Phi(\partial K_t) \le \liminf_N \Phi(\partial K_t)$ by the lower semicontinuity of the surface energy. **But the K_t^N's are the minimizers of some functionals, precisely** (3.2). Thus, with some errors, which can be controlled as $\triangle t \longrightarrow 0$, we have $\Phi(\partial K_t^N) \lesssim \Phi(\partial K_t)$. That means $\limsup_N \Phi(\partial K_t^N) \le \Phi(\partial K_t)$. This gives the desired result. The main step is to control the error in the proof of the upper semicontinuity.

We now go into the details. The scheme is basically the same as [AW] Theorem 8.6. We will need the regularity results of the temperature fields. We set the following notations:

- $K^N = K^N(t) = K^N(t_i^+)$, $\quad t_i^+ \le t < t_{i+1}$; $\quad c_N(t) = c_{K^N(t)}$.
- Let $Q^N(t)$ and $T^N(t)$ be the heat distribution and temperature field of the discrete scheme at time t. ($Q^N(t) = c_N(t)T^N(t)$).
- $P_i^N = Q^N(t_i^+)$, the heat distribution at t_i^+. $S_i^N = T^N(t_i^+)$, the temperature field at t_i^+.
- $K(t)$, $Q(t)$, and $T(t)$ are the limiting crystal position, heat distribution, and temperature field at t. ($Q(t) = c(t)T(t)$).

There are some preliminary lemmas. They are all used to control the error during the heat flow. In the following, we recall the functional form of F—Section 3.1.

8.4.2. Lemma. *For all $t \in [t_i, t_{i+1})$ and $t_{i+1} \leq \tau^N$,*

$$E\left\{\sup_{t\in[t_i,t_{i+1})}\left|\int_{\mathcal{O}} LF(L^{-1}P_i^N)\,d\mathcal{L}^n - LF(L^{-1}Q^N(t))\,d\mathcal{L}^n\right|\right\} \leq C\triangle t^{1/4-\alpha-\delta}, \quad (8.19)$$

where L is the specific heat capacity of any time varying crystal. δ is some very small positive number.

Proof.

$$\begin{aligned}
&\left|\int_{\mathcal{O}} LF(L^{-1}P_i^N)\,d\mathcal{L}^n - LF(L^{-1}Q^N(t))\,d\mathcal{L}^n\right| \\
&\leq C\int_{\mathcal{O}}\left|\left(\frac{P_i^N}{c_N}\right)^2 - \left(\frac{Q^N(t)}{c_N}\right)^2\right| + \left|\left(\frac{c_N}{P_i^N}\right)^2 - \left(\frac{c_N}{Q^N(t)}\right)^2\right| d^n x \\
&\leq C\int_{\mathcal{O}}\left|S_i^N - T^N(t)\right|\left|\frac{1}{{S_i^N}^2 T^N(t)} + \frac{1}{S_i^N T^N(t)^2} + S_i^N + T^N(t)\right| d\mathcal{L}^n \\
&\leq C\left(m_i^{-3} + M_i\right)\left\|S_i^N - T^N(t)\right\|_{L^2} \qquad \text{(using the notation of (8.14))} \\
&\leq C\left(m_i^{-3} + M_i\right)\left\{\sup_{t\in[t_i,t_{i+1})}\left\|S_i^N - T^N(t)\right\|_{L^2}\right\}.
\end{aligned}$$

Now,

$$\begin{aligned}
&E\left(m_i^{-3} + M_i\right)\left\{\sup_{t\in[t_i,t_{i+1})}\left\|S_i^N - T^N(t)\right\|_{L^2}\right\} \\
&\leq C\left[E\left(m_i^{-6} + M_i^2\right)\right]^{1/2}\left(E\left\{\sup_{t\in[t_i,t_{i+1})}\left\|S_i^N - T^N(t)\right\|_{L^2}^8\right\}\right)^{1/8} \\
&\leq C\left(\triangle t^{\frac{-3n\alpha}{3n+2}} + \triangle t^{-\alpha}\right)\left(\frac{\triangle t^{1/4}}{\epsilon^{n+1}}\right) \qquad \text{(Theorems 4.2.2 and 4.4.1)} \\
&\leq C\triangle t^{1/4-\alpha}\epsilon^{-n-1}.
\end{aligned}$$

The result follows by taking $\epsilon = \triangle t^{\gamma}$ with γ small enough.[25] ☐

8.4.3. Lemma.

$$E\int_0^1\left|\int_{\mathcal{O}} c_N F(T^N) - cF(T)\,d\mathcal{L}^n\right| dt \longrightarrow 0.$$

[25] Recall the remark at the beginning of Section 4.3.

Proof. We write

$$\begin{aligned}
&\left|\int_{\mathcal{O}} c_N F(T^N) - cF(T)\, d\mathcal{L}^n\right| \\
&\leq \int_{\mathcal{O}} \left|c_N\left(F(T^N) - F(T)\right) + (c_N - c)F(T)\right| d\mathcal{L}^n \\
&\leq C\int_{\mathcal{O}} \left|T^{N2} - T^2\right| + \left|U^{N2} - U^2\right| + |c_N - c|\, F(T)\, d^n x \\
&\leq C\{\left\|T^N - T\right\|_{L^2} \left\|T^N + T\right\|_{L^2} + \left\|U^N - U\right\|_{L^2} \left\|U^N + U\right\|_{L^2} \\
&\quad + \int_{\mathcal{O}} |c_N - c|\, F(T)\, d\mathcal{L}^n\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&E\int_0^1 \left|\int_{\mathcal{O}} c_N F(T^N) - cF(T)\, d\mathcal{L}^n\right| dt \\
\leq\ & C\left(E\int_0^1 \left\|T^N - T\right\|_{L^2}^2\, dt\right)^{1/2} \left(E\int_0^1 \left\|T^N + T\right\|_{L^2}^2\, dt\right)^{1/2} \\
& + C\left(E\int_0^1 \left\|U^N - U\right\|_{L^2}^2\, dt\right)^{1/2} \left(E\int_0^1 \left\|U^N + U\right\|_{L^2}^2\, dt\right)^{1/2} \\
& + E\int_0^1 \int_{\mathcal{O}} |c_N - c|\, F(T)\, d\mathcal{L}^n\, dt \\
\longrightarrow\ & 0
\end{aligned}$$

(by the convergence of $T^N \longrightarrow T, U^N \longrightarrow U$ and $c_N \longrightarrow c$). □

8.4.4. Lemma. $E\int_0^1 \left|\int_{\mathcal{O}} LF(L^{-1}Q^N) - LF(L^{-1}Q)\, d\mathcal{L}^n\right| dt \longrightarrow 0$, *where* L *is the specific heat capacity of an arbitrary time-varying crystal.*

Proof. The proof is similar to the previous lemma. We write

$$\begin{aligned}
&\left|LF(L^{-1}Q^N) - LF(L^{-1}Q)\right| \\
&\leq C\left\{\left|\frac{1}{Q^{N2}} - \frac{1}{Q^2}\right| + \left|Q^{N2} - Q^2\right|\right\} \\
&= C\left\{\left|c_N^{-2}U^{N2} - c^{-2}U^2\right| + \left|c_N^{-2}T^{N2} - c^{-2}T^2\right|\right\} \\
&\leq C\left\{c_N^{-2}\left|U^{N2} - U^2\right| + \left|c_N^{-2} - c^{-2}\right| U^2 + c_N^2\left|T^{N2} - T^2\right| + \left|c_N^2 - c^2\right| T^2\right\}.
\end{aligned}$$

The asserted convergence follows easily. □

8.4.5. Remark. From the above lemmas (and the fact that $E\int_0^1 \|K^N - K\|_{L^1}\, dt \longrightarrow 0$), there is a subsequence (still denoted by N) such that for $d\mathcal{L}^1 \times dP$ a.e. on $\{(t,\omega)\colon\ t < \tau(\omega)\}$, as $N \longrightarrow 0$, we have

1. $K_t^N \longrightarrow K_t$ in L^1 and hence $\Phi(\partial K_t) \le \liminf_N \Phi(\partial K_t^N)$.
2. $\left|\int_{\mathcal{O}} LF(L^{-1}P_i^N)\, d\mathcal{L}^n - \int_{\mathcal{O}} LF(L^{-1}Q^N(t))\, d\mathcal{L}^n\right| \longrightarrow 0 \qquad$ (where $t \in [t_i, t_{i+1})$).
3. $\int_{\mathcal{O}} c_N F(T_t^N)\, d\mathcal{L}^n \longrightarrow \int_{\mathcal{O}} cF(T_t)\, d\mathcal{L}^n$.
4. $\int_{\mathcal{O}} LF(L^{-1}Q_t^N)\, d\mathcal{L}^n \longrightarrow \int_{\mathcal{O}} LF(L^{-1}Q_t)\, d\mathcal{L}^n$.

8.4.6. Final Steps in the Proof of Theorem 8.4.1. We just need to show that $\Phi(\partial K_t) \ge \limsup_N \Phi(\partial K_t^N)$.

The negation of the above means the existence of ϵ and large N_j (j will be suppressed for simplicity) such that $\Phi(\partial K_t) + \epsilon \le \Phi(\partial K_t^N)$. This implies

$$
\begin{aligned}
&\Phi(\partial K_t) + \int_{\mathcal{O}} cF(c^{-1}P_i^N) + \epsilon \\
&\le \Phi(\partial K_{i^+}^N) + \int_{\mathcal{O}} cF(c^{-1}P_i^N) \qquad \text{(where } t \in [t_i, t_{i+1})\text{)} \\
&= \Phi(\partial K_{i^+}^N) + \int_{\mathcal{O}} c_N F(c_N^{-1}P_i^N) + \int_{\mathcal{O}} cF(c^{-1}P_i^N) - \int_{\mathcal{O}} c_N F(c_N^{-1}P_i^N) \\
&= \Phi(\partial K_{i^+}^N) + \int_{\mathcal{O}} c_N F(c_N^{-1}P_i^N) + \left(\int_{\mathcal{O}} cF(c^{-1}P_i^N) - \int_{\mathcal{O}} cF(c^{-1}Q^N(t))\right) \\
&\quad - \left(\int_{\mathcal{O}} c_N F(c_N^{-1}P_i^N) - \int_{\mathcal{O}} c_N F(c_N^{-1}Q^N(t))\right) \\
&\quad + \left(\int_{\mathcal{O}} cF(c^{-1}Q^N(t)) - \int_{\mathcal{O}} cF(c^{-1}Q(t))\right) \\
&\quad + \left(\int_{\mathcal{O}} cF(c^{-1}Q(t)) - \int_{\mathcal{O}} c_N F(c_N^{-1}Q^N(t))\right).
\end{aligned}
$$

Each of the quantities in the above parentheses tends to zero by the remark before this theorem.

Thus, choosing large enough N implies that

$$\Phi(\partial K_t) + \int_{\mathcal{O}} cF(c^{-1}P_i^N) + \epsilon/2 \le \Phi(\partial K_{i^+}^N) + \int_{\mathcal{O}} c_N F(c_N^{-1}P_i^N).$$

This clearly contradicts the minimality property of the $K_{i^+}^N$ and P_i^N (Section 6.4.3).

Remark. Note that in the above few steps, we can put K, K^N, and other quantities in the same inequality as they are defined on the same probability space. This is due to the use of the Skorokhod Theorem—Section 6.4.

8.5. Uniform Bound for $E \int_0^1 \left|\left\langle \partial K_t^N, g\right\rangle\right|^2 dt$

8.5.1. Proposition. *For all N,*

$$E \int_0^1 \left|\left\langle \partial K_t^N, g\right\rangle\right|^2 dt \leq C < \infty. \tag{8.20}$$

Proof.

$$\begin{aligned}
& E \int_0^1 \left|\left\langle \partial K_t^N, g\right\rangle\right|^2 dt \\
&= E \int_0^1 \left|\int_{\mathcal{O}} \operatorname{div}\left(H(T_{i^+}^N) g\right) d\mathcal{L}^n\right|^2 dt, \qquad (t \in t_i \leq t \leq t_{i+1}) \\
&\quad \text{(The Gibbs-Thomson condition holds right after minimization.)} \\
&\leq CE \int_0^1 \left|\int_{\mathcal{O}} \operatorname{div}\left(H(T_i^N) g\right) d\mathcal{L}^n - \int_{\mathcal{O}} \operatorname{div}\left(H(T^N(t)) g\right) d\mathcal{L}^n\right|^2 dt \\
&\quad + CE \int_0^1 \left|\int_{\mathcal{O}} \operatorname{div}\left(H(T^N(t)) g\right) d\mathcal{L}^n\right|^2 dt \\
&\leq CEL_3' + CE \int_0^1 \left|\int_{\mathcal{O}} \operatorname{div}\left(H(T^N(t)) g\right) d\mathcal{L}^n\right|^2 dt.
\end{aligned}$$

The term EL_3' is handled in Theorem 8.3.3. To estimate the second term, suppressing the N's and making use of (8.9),

$$\begin{aligned}
& CE \int_0^1 \left|\int_{\mathcal{O}} \operatorname{div}\left(H(T(t)) g\right) d\mathcal{L}^n\right|^2 dt \\
&\leq CE \int_0^1 \left|\int_{\mathcal{O}} T\,|\nabla T| + U\,|\nabla U| + |\nabla T| + T^2 + U^2\, d\mathcal{L}^n\right|^2 dt \\
&\leq CE \int_0^1 \left\{\|T\|_{L^2}^2 \|\nabla T\|_{L^2}^2 + \|U\|_{L^2}^2 \|\nabla U\|_{L^2}^2 + \|\nabla T\|_{L^2}^2 + \|T\|_{L^2}^4 + \|U\|_{L^2}^4\right\} dt \\
&\leq \left\{CE\left(\sup_{t\in[0,1]} \|T\|_{L^2}^4 + \|U\|_{L^2}^4\right)\right\}^{1/2} \left\{E\left(\int_0^1 \|\nabla T\|_{L^2}^2 + \|\nabla U\|_{L^2}^2\, dt\right)^2\right\}^{1/2} \\
&\quad + CE \int_0^1 \|\nabla T\|_{L^2}^2\, dt \\
&\leq C \qquad \text{(by the energy estimates).}
\end{aligned}$$

□

8.6. Minimality and Regularity of Limiting Crystals

The method used to prove the varifold convergence in Section 8.4 can also be employed to establish some regularity properties of the limiting crystals—Theorem 8.0.2.

First of all, we prove the following minimizing property of the $K(t)$'s, which has its own interest. (The notations are the same as those in the proof of Theorem 8.4.1.)

8.6.1. Theorem (Limit Crystals as Minimizers). *For* $d\mathcal{L}^1 \times dP$ *a.s. on* $\{(t,\omega)\colon t < \tau(\omega)\}$,

$$\mathcal{E}(K(t), Q(t)) = \inf\{\mathcal{E}(R, Q(t))\colon\ R \in \mathcal{K}\}. \tag{8.21}$$

Proof. The negation of the theorem means the existence of an $R \in \mathcal{K}$ and an $\epsilon > 0$ such that

$$\Phi(\partial R) + \int_{\mathcal{O}} c_R F(c_R^{-1} Q(t)) + \epsilon \le \Phi(\partial K) + \int_{\mathcal{O}} cF(c^{-1}Q(t)).$$

Let $t_i \le t < t_{i+1}$. Now the L.H.S. equals

$$\Phi(\partial R) + \int_{\mathcal{O}} c_R F(c_R^{-1} P_i^N) \ + \ \left(\int_{\mathcal{O}} c_R F(c_R^{-1} Q^N(t)) - \int_{\mathcal{O}} c_R F(c_R^{-1} P_i^N)\right) \\ + \ \left(\int_{\mathcal{O}} c_R F(c_R^{-1} Q(t)) - \int_{\mathcal{O}} c_R F(c_R^{-1} Q^N(t))\right).$$

Using the lower semicontinuity of the surface energy Φ, the R.H.S. can be bounded by (for large enough N)

$$\epsilon/2 + \Phi(\partial K_{i+}^N) \ + \ \int_{\mathcal{O}} c_N F(c_N P_i^N) + \left(\int_{\mathcal{O}} c_N F(c_N^{-1} Q^N(t)) - \int_{\mathcal{O}} c_N F(c_N^{-1} P_i^N)\right) \\ + \ \left(\int_{\mathcal{O}} cF(c^{-1}Q(t)) - \int_{\mathcal{O}} c_N F(c_N^{-1} Q^N(t))\right).$$

By Remark 8.4.5, all the terms in the parentheses will tend to zero. Hence, for large enough N, we get

$$\Phi(\partial R) + \int_{\mathcal{O}} c_R F(c_R^{-1} P_i^N) < \Phi(\partial K_{i+}^N) + \int_{\mathcal{O}} c_N F(c_N^{-1} P_i^N),$$

which contradicts the minimality property of the K_{i+}^N and P_i^N (Section 6.4.3). □

Thus we have shown that the crystals and temperature fields satisfy the hypothesis of [AW] Theorem 8.8, which gives the asserted statements of Theorem 8.0.2.

Appendix A. Varifolds and Sets of Finite Perimeter

We are going to introduce the notion of varifolds and their convergence in the case of co-dimension one.[26]

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[26] A full account of varifold can be found in [All]. A concise introduction is in the Appendix of [AW]. Here we just mention the concepts needed in this paper.

An $(n-1)$ varifold in R^n is a **radon measure** over $R^n \times S^{n-1}$. A sequence of $(n-1)$-varifolds $\{V_i\}_{i\geq 1}$ is said to converge in the **varifold sense** to V, written as $V_i \rightharpoonup V$, if

$$\int_{R^n\times S^{n-1}} \varphi\, dV_j \longrightarrow \int_{R^n\times S^{n-1}} \varphi\, dV \tag{A.1}$$

for all continuous functions φ: $R^n \times S^{n-1} \longrightarrow R$ with compact support.

Given any set of finite perimeter K (Section 2.1.1), we naturally associate with it the following $(n-1)$-varifold:

$$V(\varphi) = \int_{x\in\partial K} \varphi(x, n_K)\, d\mathcal{H}^{n-1}x, \tag{A.2}$$

where ∂K refers to the reduced boundary of K and n_K the (approximate) exterior normal vector to ∂K. We say a sequence of sets of finite perimeter $\{K_i\}_{i\geq 1}$ converges in the varifold sense if their associated varifolds do so.

In [AW] Appendix C, a sufficient condition is established for the varifold convergence of K_i to K. Namely,

1. The union of supports of K_i is bounded.
2. $\sup_i |\partial K_i| < \infty$.
3. $K_i \longrightarrow K$ in L^1.
4. There is an elliptic integrand Φ (Section 2.1.2) such that the Φ surface energies also converge, i.e., $\lim_i \Phi(\partial K_i) \longrightarrow \Phi(\partial K)$.

From this, it is clear that varifold convergence is much stronger than the L^1 convergence. Not only does the set converge, but the normals of the boundaries of the sets do, as well.

It is easy to establish from the definition of Φ first variation (Section 2.1.7) that, if K_i converges to K in the varifold sense, then for all C^1 vector fields g, we have

$$\lim_i \frac{d}{ds}\Phi\left(G_{s\sharp}\partial K_i\right)\bigg|_{s=0} = \frac{d}{ds}\Phi\left(G_{s\sharp}\partial K\right)\bigg|_{s=0},$$

where $G(x) = x + sg(x)$.

Appendix B. Concepts from Probability

We describe very briefly the main concepts from probability theory used in this paper. It is a collection of definitions and notations. We will however elaborate the case of infinite dimensional stochastic calculus. The main references are [KS], [Par], and [KR].

B.1. Basic Definitions

A **probability space** is a collection of elements $\{\omega \in \Omega\}$ equipped with a σ**-algebra** $\mathcal{F}$ of subsets of Ω and a **measure** P on $\mathcal{F}$ such that $P(\Omega) = 1$. Let S be a measurable space with σ-algebra $\mathcal{B}$.[27] An S-valued **random variable (r.v.)** is a measurable map X from Ω to S. In case X is real valued, we use EX to denote the expectation of X with respect to P, i.e.,

$$EX = \int_{\omega \in \Omega} X(\omega)\, dP(\omega). \tag{B.1}$$

A **stochastic process** is a collection of random variables, $\{X_t\}_{t\in I}$, where the index set I is R_+ or $[0, T]$ regarded as a time interval. A stochastic process can also be considered as a map,

$$(t, \omega) \to X_t(\omega) : I \times \Omega \longrightarrow S. \tag{B.2}$$

Upon fixing a $t \in I$, we are observing the random variable X_t at time t. On the other hand, fixing an $\omega \in \Omega$, the collection of elements $\{X_t(\omega)\}_{t\in I}$ is called a **sample path**. X is called **continuous (left or right continuous)** if the sample paths satisfy this property almost surely with respect to P.

A **filtration** $\{\mathcal{F}_t\}_{t\in I}$ is a time parametrized increasing family of sub-σ-algebras of $\mathcal{F}$:

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F} \qquad \text{for all } 0 \le s \le t \text{ and } s, t \in I. \tag{B.3}$$

A stochastic process X is said to be **adapted** to a filtration $\{\mathcal{F}_t\}$ if X_t is $\mathcal{F}_t$-measurable for all t. In this case, we write $X = \{X_t,\ \mathcal{F}_t;\ 0 \le t < \infty\}$. X is called **progressively measurable** to $\{\mathcal{F}_t\}$ if $(s, \omega) \longrightarrow X_s(\omega)$: $([0, t] \times \Omega,\ \mathcal{B}([0, 1]) \otimes \mathcal{F}_t) \longrightarrow (S,\ \mathcal{B})$ is measurable for all $t \in R_+$. **Predictable (completely measurable) sets**[28] are subsets of $[0, \infty) \times \Omega$, which are elements of the smallest σ-algebra relative to which all real $\mathcal{F}_t$-adapted, right-continuous processes with left-hand limit are measurable in (t, ω). A process $X : [0, \infty) \times \Omega \longrightarrow S$ is called **predictable (completely measurable)** if, for any Borel subset $B \in S$, $\{(t, \omega);\ X(t, \omega) \in B\}$ is predictable.

All the processes in this paper are predictable. They are either continuous or can be approximated by piecewise continuous processes.

A positive random variable T is called a **stopping time** (with respect to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$) if for all $0 \le t$, $\{T \le t\} \in \mathcal{F}_t$. This concept is used to indicate the occurrence of some random event.

The **conditional expectation** of a real-valued random variable X with respect to a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ is denoted by $E\,(X|\mathcal{G})$ or $E_{\mathcal{G}}X$.

We assume the well-known definitions of **(continuous) martingale processes**—$\{M_t,\ \mathcal{F}_t;\ 0 \le t < \infty\}$; their associated **quadratic variation processes**—$\langle M\rangle_t$; **(one-dimensional) Brownian motion**—$\{W_t,\ \mathcal{F}_t;\ 0 \le t < \infty\}$; **stochastic integral** with respect to W_t and **Ito's Formula**. However, we single out the following result, which is used frequently in this paper.

Burkholder-Davis-Gundy Inequalities ([KS] 3.3.28)**.** Let M be a continuous martingale with $M_0 = 0$. Then, for all $m > 0$, there are universal positive constants k_m and K_m

[27] When S is a topological space, we always take $\mathcal{B}$ to be the **Borel σ-algebra** of S.

[28] The following definition is from [KR].

(not depending on M) such that

$$k_m E\left(\langle M\rangle_\tau^m\right) \le E\left((M_\tau^*)^{2m}\right) \le K_m E\left(\langle M\rangle_\tau^m\right), \tag{B.4}$$

where τ is any stopping time, $\langle M\rangle_t$ is the quadratic variation of M, and $M_t^* = \sup_{0\le s\le t}|M_s|$.

B.2. Weak Convergence of Probability Measures on Metric Space

Here we take S to be a metric space. A sequence of probability measures $\{P_n\}_{n\ge1} \subset \mathcal{P}(S)$ is said to **converge weakly** to $P \in \mathcal{P}(S)$, denoted by $P = w\lim_n P_n$ or $P \rightharpoonup P_n$ if for all f, bounded and continuous on S,

$$\lim_n \int_S f(x)\, P_n(dx) = \int_S f(x)\, P(dx), \qquad \forall f \in C_b(S). \tag{B.5}$$

In the above definition, we can restrict f to be only uniformly continuous. We have the following important criterion for compactness in $\mathcal{P}(S)$, developed by **Prokhorov**:

A collection of probability measures $\Gamma \subset \mathcal{P}(S)$ is called **tight** *if for all $\epsilon > 0$, there is a compact set $K \subset S$ such that*

$$P(K) \ge 1-\epsilon, \qquad \forall P \in \Gamma. \tag{B.6}$$

If Γ is tight, then it is **relatively compact**. *The reverse is true if S is* **complete**.

Tightness Criteria on $C([0,1], S)$. Let S be a **complete separable metric space**. We describe here an explicit tightness condition on a collection of probability measures on $C([0,1], S)$—the space of continuous functions from $[0,1]$ to S. The condition is summarized by the following two statements.

B.2.1. Theorem (Kolmogorov-Čentsov). *Suppose an S-valued stochastic process defined on $(\Omega, P, \mathcal{F})$ satisfies the condition*

$$E\left|X_t - X_s\right|^\alpha \le C\left|t-s\right|^{1+\beta}, \qquad \textit{for all } 0 \le s, t \le 1, \tag{B.7}$$

where $\alpha, \beta, C > 0$. Then X has a **continuous version** *$\tilde{X}$ which is* **locally Hölder continuous** *with exponent γ for every $0 < \gamma < \beta/\alpha$, i.e.,*

$$P\left\{\omega\colon \sup_{\substack{0<t-s<h(\omega)\\ s,t\in[0,1]}} \frac{\left|\tilde{X}_t(\omega) - \tilde{X}_s(\omega)\right|}{|t-s|^\gamma} \le \delta_\gamma\right\} = 1, \tag{B.8}$$

where $h(\omega)$ is an almost surely positive random variable and δ_γ is some appropriate constant depending on γ.

B.2.2. Theorem. *Let* $\left\{X_t^N\right\}_{t\in[0,1],N\geq 1}$ *be a sequence of continuous process satisfying*

$$E\left|X_0^N\right|^p \leq M < \infty, \tag{B.9}$$

$$E\left|X_t^N - X_s^N\right|^\alpha \leq C\left|t-s\right|^{1+\beta}, \tag{B.10}$$

where $p, \alpha, \beta, M,$ *and* C *are positive numbers independent of* N*.*

Then the collection of measures $\left\{P^N\right\}_{N\geq 1}$ *induced on* $C([0,1],S)$ *by* X^N *is tight.*

B.3. Infinite Dimensional Stochastic Calculus

Here we describe the basic notations and terminology for stochastic calculus when random variables and stochastic processes take values in a **separable Hilbert space** H. Many of the concepts related to real valued stochastic processes can be extended to the present infinite dimensional case. We denote the norm and inner product in H by $\|\cdot\|$ and $(\ ,\ )$. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

A random variable X taking values in H is a measurable map from $(\Omega, \mathcal{F})$ to $(H, \mathcal{B})$ with $\mathcal{B}$ being the Borel σ-algebra of H. If $\int_\Omega \|X\|\, dP < \infty$, we can define $EX = \int_\Omega X\, dP$, and we have $\|EX\| \leq E\|X\|$.

Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Suppose $E\|X\| < \infty$. We define $E(X|\mathcal{G})$ to be the $\mathcal{G}$-measurable H-valued random variable such that for all $h \in H$,

$$(E(X|\mathcal{G}), h) = E((X, h)|\mathcal{G}) \quad P \text{ a.s.} \tag{B.11}$$

Continuous Square Integrable Martingales in H. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration. A stochastic process M_t, adapted to $\mathcal{F}_t$, written as $\{M_t, \ \mathcal{F}_t;\ 0 \leq t < \infty\}$ is called a **martingale** if, for all $0 \leq s \leq t$,

$$E\|X_t\| < \infty \qquad \text{and} \qquad E(X_t|\mathcal{F}_s) = X_s \quad P \text{ a.s.} \tag{B.12}$$

A **continuous martingale** $\{M_t, \ \mathcal{F}_t;\ 0 \leq t < \infty\}$ taking values in H is called **square integrable** if, for all $0 \leq t$, $E\|X_t\|^2 < \infty$. We use $\mathcal{M}_2^c(H)$ to denote such processes with $M_0 = 0$.

Let $M_t \in \mathcal{M}_2^c(H)$. Similar to the real valued case, we can define $\langle M\rangle_t$ to be the **adapted, continuous nondecreasing** process such that $\|M_t\|^2 - \langle M\rangle_t$ is a martingale. However, in the case of infinite dimensional martingales, we have a more general object than just the $\langle M\rangle_t$.

Covariance Operators for $\mathcal{M}_2^c(H)$. Given any $M \in \mathcal{M}_2^c(H)$, there is a **unique adapted, continuous nondecreasing process** (and hence of bounded variations on compact time intervals) $\langle\langle M\rangle\rangle_t$ taking values in the space of **positive trace class operators**,[29] such that for all $x, y \in H$,

$$(\langle\langle M\rangle\rangle_t\, x, y) = \langle (M_\cdot, x), (M_\cdot, y)\rangle_t\,. \tag{B.13}$$

[29] The basic properties of trace class operators will be described in Section B.4.

(The right-hand side is the cross-variation process between (M_t, x) and (M_t, y)—the inner products of M_t with x and y.) $\langle\langle M\rangle\rangle$ is called the **covariance operator** for M.

Let also $N \in \mathcal{M}_2^c(G)$ where G is another Hilbert space. Then there is a unique adapted process of bounded variations on compact time intervals $\langle\langle M, N\rangle\rangle_t$ taking values in the trace class operators from G to H such that, for all $x \in H$ and $y \in G$,

$$(\langle\langle M, N\rangle\rangle_t y, x) = \langle (M_\cdot, x), (N_\cdot, y)\rangle_t . \tag{B.14}$$

$\langle\langle M, N\rangle\rangle$ is called the **cross-covariance operator** for M and N.

Brownian Motion in H. Let Λ be a positive trace class operator. A **Wiener Process** or **Brownian motion** in H with **covariance operator** Λ is a process $W_t \in \mathcal{M}_2^c(H)$ such that

$$\langle\langle W\rangle\rangle_t = t\Lambda. \tag{B.15}$$

The existence of such a process can be demonstrated as follows. Let $\{\lambda_i\}_{i\geq 1}$ be the eigenvalues of Λ with $\{e_i\}_{i\geq 1}$ being the corresponding normalized eigenvectors. Since Λ is a positive trace class operator, we have $\lambda_i \geq 0$ and $\text{Tr}\Lambda = \sum_i \lambda_i < \infty$. Also let $\left\{W_t^i\right\}_{i\geq 1}$ be a sequence of independent real valued Brownian motions. Then the following definition can be shown to satisfy (B.15):

$$W_t = \sum_i \sqrt{\lambda_i} e_i W_t^i. \tag{B.16}$$

Infinite Dimensional Stochastic Integrations. We can define stochastic integrations in the same way as the real valued case. Before giving the formal definition, we first perform two heuristic computations. Take the construction of infinite dimensional Brownian motion from (B.16).

Let $f(\cdot)$ be a predictable function in H. We define

$$\int_0^t (f(s), dW_s) = \sum_i \sqrt{\lambda_i} \int_0^t (f(s), e_i)\, dW_s^i. \tag{B.17}$$

Consider

$$\begin{aligned}\left\langle \int_0^\cdot (f(s), dW_s)\right\rangle_t &= \sum_{i,j} \sqrt{\lambda_i\lambda_j} \left\langle \int_0^\cdot (f(s), e_i)\, dW_s^i, \int_0^\cdot \left(f(s), e_j\right) dW_s^j\right\rangle_t \\ &= \sum_{i,j} \sqrt{\lambda_i\lambda_j} \int_0^t (f(s), e_i)\left\langle f(s), e_j\right\rangle d\left\langle W^i, W^j\right\rangle_s \\ &= \sum_i \lambda_i \int_0^t (f(s), e_i)^2\, ds = \int_0^t (\Lambda f(s), f(s))\, ds.\end{aligned}$$

For another kind of stochastic integration, we replace $f(\cdot)$ by a predictable linear operator $B(\cdot)$ from H to G. We then define

$$M_t = \int_0^t B(s)\, dW_s = \sum_i \sqrt{\lambda_i} \int_0^t B(s) e_i\, dW_s^i. \tag{B.18}$$

For all $x, y \in G$, consider[30]

$$
\begin{aligned}
&\langle (M_t,\ x),\ (M_t,\ y)\rangle_t \\
&= \left\langle \sum_i \sqrt{\lambda_i} \int_0^{\cdot} (B(s)e_i,\ x)\ dW_s^i,\ \sum_j \sqrt{\lambda_j} \int_0^{\cdot} \big(B(s)e_j,\ y\big)\ dW_s^j \right\rangle_t \\
&= \sum_i \lambda_i \int_0^t (B(s)e_i,\ x)\,(B(s)e_i,\ y)\ ds = \sum_i \lambda_i \int_0^t \big(e_i,\ B^*(s)x\big)\big(e_i,\ B^*(s)y\big)\ ds \\
&= \int_0^t \big(\Lambda B^*(s)x,\ B^*(s)y\big)\ ds = \int_0^t \big(B(s)\Lambda B^*(s)x,\ y\big)\ ds \\
&= \left(\left(\int_0^t B(s)\Lambda B^*(s)\, ds\right)x,\ y\right).
\end{aligned}
$$

Hence,

$$
\left\langle\!\!\left\langle \int_0^{\cdot} B(s)\, dW_s \right\rangle\!\!\right\rangle_t = \int_0^t B(s)\Lambda B^*(s)\, ds. \tag{B.19}
$$

In order to make the above computations rigorous, we need to have some integrability conditions. Now we give the formal definition of stochastic integration in the infinite dimensional case as in [Par I.3.2].

Let $\{W_t,\ \mathcal{F}_t;\ 0 \le t < \infty\}$ be an H-valued Wiener Process with covariance operator Λ. G is another Hilbert Space. Suppose $\{B(t),\ \mathcal{F}_t;\ 0 \le t < \infty\}$ is a predictable process taking values in the linear operators from H to G (not necessarily bounded) such that, for all $T > 0$,

$$
E \int_0^T \operatorname{Tr}\big[B(s)\Lambda B(s)^*\big]\ ds < \infty. \tag{B.20}
$$

Then we can define

$$
(B \cdot W)_t = \int_0^t B(s)\, dW_s \tag{B.21}
$$

to be the **unique** element of $\mathcal{M}_2^c(G)$ such that, for all $N \in \mathcal{M}_2^c(K)$ (K is another Hilbert space),

$$
\langle\!\langle (B \cdot W),\, N \rangle\!\rangle_t = \int_0^t B(s)\, d\, \langle\!\langle W,\, N \rangle\!\rangle_s. \tag{B.22}
$$

In particular,

$$
\langle\!\langle B \cdot W \rangle\!\rangle_t = \int_0^t B(s)\Lambda B(s)^*\, ds. \tag{B.23}
$$

---

[30] In the following, B^* denotes the adjoint of B.

Integrands for Stochastic Integrations. We use $\mathcal{N}_\Lambda(H)$ to denote the class of predictable processes f taking values in H such that, for all $T > 0$,

$$E\int_0^T (\Lambda f(s),\ f(s))\, ds < \infty. \tag{B.24}$$

In addition, $\mathcal{N}_\Lambda(L(H, G))$ denotes the class of predictable processes B taking values in the linear operators from H to G (not necessarily bounded) such that, for all $T > 0$,

$$E\int_0^T \mathrm{Tr}\left[B(s)\Lambda B(s)^*\right] ds < \infty. \tag{B.25}$$

Ito's Formula in Infinite Dimension. This is the anolog for the finite dimensional formula.

Let $(V, \|\cdot\|_V)$ be a **separable Banach Space** with dual V^*. The pairing between V and V^* is denoted by $\langle\ ,\ \rangle_V$. V is densely embedded in another Hilbert space $(H, (\ ,\ )_H)$. Then (V, H, V^*) forms a **Gelfand Triple** in the sense that $V \overset{j}{\hookrightarrow} H = H^* \overset{j^*}{\hookrightarrow} V^*$ with each of the embedding being dense and continuous. Furthermore, if $v \in V,\ u \in H$, then $\langle v,\ j^*(u)\rangle_V = (j(v),\ u)_H$.

B.3.1. Theorem. *[KR I.3.1] Let $v(t)$ be a V-valued and $v^*(t)$ be a V^*-valued predictable process such that*

$$P\left(\int_0^T \|v(t)\|_V^2 + \left\|v^*(t)\right\|_{V^*}^2 dt < \infty\right) = 1 \qquad \textit{for all } T > 0. \tag{B.26}$$

Let M_t be an H-valued continuous martingale, and τ be a stopping time. Suppose for every $\xi \in V$, we have for $d\mathcal{L}^1 \times dP$ a.e. on $\{(t, \omega)\colon\ t < \tau(\omega)\}$ that

$$(j(\xi),\ j(v(t)))_H = \int_0^t \left\langle \xi,\ v^*(s)\right\rangle_V ds + (j(\xi),\ M_t)_H\,. \tag{B.27}$$

Then there exists a subset $\Omega' \subset \Omega$ with $P(\Omega') = 1$ and a predictable **continuous** *H-valued process $h(t)$ such that the following statements hold.*

1. *$j(v(t)) = h(t)$ for $d\mathcal{L}^1 \times dP$ a.e. on $\{(t, \omega)\colon\ t \le \tau(\omega)\}$.*
2. *For any $\omega \in \Omega',\ t < \tau(\omega)$ and $\xi \in V$,*

$$(j(\xi),\ h(t))_H = \int_0^t \left\langle \xi,\ v^*(s)\right\rangle_V ds + (j(\xi),\ M_t)_H\,. \tag{B.28}$$

3. *If for some given $t \ge 0$ and any $\xi \in V$, (B.27) is satisfied P a.s. on $\{\omega\colon\ t < \tau(\omega)\}$, then $j(v(t)) = h(t)$ P a.s. on $\{\omega\colon\ t < \tau(\omega)\}$.*
4. *For any $\omega \in \Omega'$ and $t < \tau(\omega)$,*

$$\|h(t)\|_H^2 = \|M_0\|_H^2 + 2\int_0^t \left\langle v(s),\ v^*(s)\right\rangle_V ds + 2\int_0^t (h(s),\ dM_s)_H + \langle M\rangle_t\,. \tag{B.29}$$

This is also called **Ito's Formula for the Norm Square** *of h.*

B.4. Trace Class Operators

From Section B.3, it is clear that trace class operators are naturally associated with Hilbert space valued martingales. Let H be a separable Hilbert space and $L(H)$ denote the space of linear operators on H. The following definitions are from [Kuo].

An operator $A \in L(H)$ is called a **Hilbert-Schmidt Operator** if, for some O.N.B $\{e_i\}_{i\geq 1} \subset H$, we have $\sum_{i=1}^{\infty} \|Ae_i\|^2 < \infty$. We use $L_{(2)}(H)$ to denote the space of all Hilbert-Schmidt operators. Define the **Hilbert-Schmidt Norm** of A by $\|A\|_{(2)} = \left(\sum_{i=1}^{\infty} \|Ae_i\|^2\right)^{1/2}$.

A compact operator A is called a **trace class operator** if $\sum_i \mu_i < \infty$ where the μ_i's are the eigenvalues of $(A^*A)^{1/2}$. We use $L_{(1)}(H)$ to denote the space of all trace class operators. Define the **trace class norm** of A by $\|A\|_{(1)} = \sum_{i=1}^{\infty} \mu_i$. The **trace** of A is defined as $\mathrm{Tr}(A) = \sum_{i=1}^{\infty} \langle Ae_i,\ e_i\rangle$, where $\{e_i\}_{i\geq 1}$ is any O.N.B. of H.

Examples of Hilbert-Schmidt and Trace Class Operators. For use in this paper, we give examples of the above operators in the following setting.

Consider the n-fold product of Hilbert space: $L^2_{(n)}(\mathcal{O}) = L^2(\mathcal{O}) \times \cdots \times L^2(\mathcal{O})$. We denote each element as $\boldsymbol{U}(\cdot) = \left(\boldsymbol{U}^1(\cdot), \ldots, \boldsymbol{U}^n(\cdot)\right)$ with $\boldsymbol{U}^p(\cdot) \in L^2(\mathcal{O})$, $p = 1, 2, \ldots n$. The inner product is defined as $\langle \boldsymbol{U},\ \boldsymbol{V}\rangle = \int_{x\in\mathcal{O}} \sum_p^n \boldsymbol{U}^p(x)\boldsymbol{V}^p(x)\, d\mathcal{L}^n x$.

Let $\boldsymbol{K}$ be an operator on $L^2_{(n)}(\mathcal{O})$ given by an $n \times n$ **matrix valued kernel** $\boldsymbol{K}(x, y) = \left\{\boldsymbol{K}_{pq}(x, y)\right\}_{p,q=1}^{n}$. The operator $\boldsymbol{K}$ is defined as

$$(\boldsymbol{KU})(x) = \int_{y\in\mathcal{O}} \boldsymbol{K}(x, y)\boldsymbol{U}(y)\, d\mathcal{L}^n y, \qquad \boldsymbol{U} \in L^2_{(n)}(\mathcal{O}). \tag{B.30}$$

Define the norm of a matrix $\boldsymbol{A}$ to be

$$\|\boldsymbol{A}\|^2 = \mathrm{Tr}\, \boldsymbol{A}\boldsymbol{A}^T = \sum_{p,q} \boldsymbol{A}^2_{pq}. \tag{B.31}$$

B.4.1. Proposition. *If $\|\boldsymbol{K}(\cdot, \cdot)\| \in L^2(\mathcal{O} \times \mathcal{O})$, then $\boldsymbol{K}$ is Hilbert-Schmidt on $L^2_{(n)}(\mathcal{O})$.*

Proof. Let $\boldsymbol{K}_p(x, y)$ be the p-th row of $\boldsymbol{K}(x, y)$. Then, for all $\boldsymbol{U} \in L^2_{(n)}(\mathcal{O})$,

$$(\boldsymbol{KU})^p(x) = \int_y \left\langle \boldsymbol{K}_p(x, y),\ \boldsymbol{U}(y)\right\rangle d\mathcal{L}^n y.$$

Thus, $\int_x \|\boldsymbol{KU}(x)\|^2\, d\mathcal{L}^n x = \int_x \sum_p^n \left(\int_y \left\langle \boldsymbol{K}_p(x, y),\ \boldsymbol{U}(y)\right\rangle d\mathcal{L}^n y\right)^2 d\mathcal{L}^n x$. Now let $\{\boldsymbol{V}_i\}_{i\geq 1}$ be an O.N.B. for $L^2_{(n)}(\mathcal{O})$. We want to see if the following quantity is finite:

$$\sum_i^{\infty} \|\boldsymbol{KV}_i\|^2 = \sum_i^{\infty} \int_x \sum_p^n \left(\int_y \left\langle \boldsymbol{K}_p(x, y),\ \boldsymbol{V}_i(y)\right\rangle d\mathcal{L}^n y\right)^2 d\mathcal{L}^n x. \tag{B.32}$$

By the hypothesis, $\|\boldsymbol{K}\| \in L^2(\mathcal{O}\times\mathcal{O})$. Using Fubini's Theorem, a.e. $x \in \mathcal{O}$, $\boldsymbol{K}_p(x, \cdot) \in$

$L^2_{(n)}(\mathcal{O})$. Then,

$$\begin{aligned} \boldsymbol{K}_p(x,y) &= \sum_i^\infty \left(\int_y \left\langle \boldsymbol{K}_p(x,y),\ \boldsymbol{V}_i(y) \right\rangle d\mathcal{L}^n y \right) \boldsymbol{V}_i \\ \Longrightarrow \int_y \left\| \boldsymbol{K}_p(x,y) \right\|^2 d\mathcal{L}^n y &= \sum_i^\infty \left(\int_y \left\langle \boldsymbol{K}_p(x,y),\ \boldsymbol{V}_i(y) \right\rangle d\mathcal{L}^n y \right)^2. \end{aligned}$$

Hence,

$$\iint_{(x,y)} \left\| \boldsymbol{K}_p(x,y) \right\|^2 d\mathcal{L}^n y d\mathcal{L}^n x = \sum_i^\infty \int_x \left(\int_y \left\langle \boldsymbol{K}_p(x,y),\ \boldsymbol{V}_i(y) \right\rangle d\mathcal{L}^n y \right)^2 d\mathcal{L}^n x.$$

Finally,

$$\begin{aligned} \infty > \iint_{(x,y)} \|\boldsymbol{K}(x,y)\|^2 d\mathcal{L}^n y\, d\mathcal{L}^n x &= \iint_{(x,y)} \sum_p^n \left\| \boldsymbol{K}_p(x,y) \right\|^2 d\mathcal{L}^n y\, d\mathcal{L}^n x \\ &= \sum_i^\infty \int_x \sum_p^n \left(\int_y \left\langle \boldsymbol{K}_p(x,y),\ \boldsymbol{V}_i(y) \right\rangle d\mathcal{L}^n y \right)^2 d\mathcal{L}^n x. \end{aligned}$$

This is exactly equal to the R.H.S. of (B.32). Hence, $\boldsymbol{K}$ is Hilbert-Schmidt. ☐

B.4.2. Proposition. *Let $\boldsymbol{K}$ be a* **positive** *Hilbert-Schmidt operator, i.e., $\boldsymbol{K}(x,y)^{\boldsymbol{T}} = \boldsymbol{K}(y,x)$, and for any $\boldsymbol{U} \in L^2_{(n)}(\mathcal{O})$, $\int_x \langle (\boldsymbol{KU})(x),\ \boldsymbol{U}(x) \rangle \, d\mathcal{L}^n x \geq 0$. If*

$$\int_x \mathrm{Tr}\left[\boldsymbol{K}(x,x)\right] d\mathcal{L}^n x < \infty, \tag{B.33}$$

then $\boldsymbol{K}$ is a trace class operator on $L^2_{(n)}(\mathcal{O})$, and the above quantity is the trace of $\boldsymbol{K}$.

Proof. Since a Hilbert-Schmidt operator is compact, we can use the Spectral Theorem to decompose $\boldsymbol{K}$ as $\boldsymbol{K}(x,y) = \sum_i^\infty \lambda_i \boldsymbol{V}_i(x) \boldsymbol{V}^T{}_i(y)$, where λ_i's are the eigenvalues of K and $\boldsymbol{V}_i$'s are the corresponding normalized orthogonal eigenvectors written in column form. Note that $\lambda_i \geq 0$. Then,

$$\begin{aligned} \mathrm{Tr}\, \boldsymbol{K}(x,x) &= \sum_i^\infty \lambda_i \mathrm{Tr}\left[\boldsymbol{V}_i(x) \boldsymbol{V}^T{}_i(x)\right] = \sum_i^\infty \lambda_i \left\|\boldsymbol{V}_i(x)\right\|^2 \\ \Longrightarrow \int_x \mathrm{Tr}\, \boldsymbol{K}(x,x)\, d\mathcal{L}^n x &= \sum_i^\infty \lambda_i. \end{aligned}$$ ☐

B.5. Examples of $\langle\langle \int_0^\cdot B(s)\, dW_s \rangle\rangle_t$

By Theorem B.23, we know that for $B \in \mathcal{N}_\Lambda(L(H))$, $\left\langle\!\left\langle \int_0^\cdot B(s)\, dW_s \right\rangle\!\right\rangle_t = \int_0^t B(s) \Lambda B(s)^*\, ds$. Here we compute $B \Lambda B^*$ for some explicit examples of B which will be useful later.

Recall that Λ is given by a kernel $\Lambda(\cdot,\cdot) \in L^\infty(\mathcal{O} \times \mathcal{O})$.

B.5.1. Multiplicative Operator. Given a fixed function $h \in L^2(\mathcal{O})$, let $B : L^2(\mathcal{O}) \longrightarrow L^2(\mathcal{O})$ be defined as

$$(Bf)(x) = h(x)f(x), \qquad f \in L^2(\mathcal{O}). \tag{B.34}$$

Note that B is unbounded. However, $B\Lambda B^*$ is actually a trace class operator on $L^2(\mathcal{O})$ given by a kernel. To compute this kernel, consider

$$\begin{aligned}\left\langle B\Lambda B^* f,\, g\right\rangle &= \left\langle \Lambda B^* f,\, B^* g\right\rangle = \int_x (\Lambda(hf))(x)h(x)g(x)\,d\mathcal{L}^n x\\ &= \int_x\int_y \Lambda(x,y)h(y)f(y)h(x)g(x)\,d\mathcal{L}^n y\,d\mathcal{L}^n x\\ &= \int_x\left(\int_y h(x)\Lambda(x,y)h(y)f(y)\,d\mathcal{L}^n y\right)g(x)\,d\mathcal{L}^n x. \end{aligned}\tag{B.35}$$

Thus, the kernel of $B\Lambda B^*$ is $h(x)\Lambda(x,y)h(y)$. Note that $B\Lambda B^*$ is positive and we have

$$\operatorname{Tr}\left[B\Lambda B^*\right] = \int_x h(x)\Lambda(x,x)h(x)\,d\mathcal{L}^n x \le C\,\|h\|_{L^2}^2\,. \tag{B.36}$$

B.5.2. Smoothing. Let $\phi_\epsilon(x) = \dfrac{1}{\epsilon^n}\phi(\dfrac{x}{\epsilon})$ be a symmetric smoothing function and $[\phi_\epsilon]$ denote the convolution operation with ϕ_ϵ. We compute

$$\begin{aligned}&\left\langle [\phi_\epsilon]\Lambda[\phi_\epsilon]^* f,\, g\right\rangle\\ &= \iint_{(x,y)} \Lambda(x,y)(\phi_\epsilon * f)(y)(\phi_\epsilon * g)(x)\,d\mathcal{L}^n y\,d\mathcal{L}^n x\\ &= \int_x\int_y\int_z\int_w \Lambda(x,y)\phi_\epsilon(y-z)f(z)\phi_\epsilon(x-w)g(w)\,d\mathcal{L}^n w\,d\mathcal{L}^n z\,d\mathcal{L}^n y\,d\mathcal{L}^n x\\ &= \int_z\int_w\left(\int_x\int_y \Lambda(x,y)\phi_\epsilon(y-z)\phi_\epsilon(x-w)\,d\mathcal{L}^n x\,d\mathcal{L}^n y\right)f(z)g(w)\,d\mathcal{L}^n z\,d\mathcal{L}^n w. \end{aligned}$$

Hence, $[\phi_\epsilon]\Lambda[\phi_\epsilon]^*$ is given by the kernel,

$$\Lambda^\epsilon(w,z) = \int_x\int_y \Lambda(x,y)\phi_\epsilon(y-z)\phi_\epsilon(x-w)\,d\mathcal{L}^n x\,d\mathcal{L}^n y \in L^\infty(\mathcal{O}\times\mathcal{O}). \tag{B.37}$$

B.5.3. Differentiation Operator. Let $\partial_p[\phi_\epsilon]$ denote the composition of $[\phi_\epsilon]$ and the differentiation with respect to the p-th coordinate. Then $\partial_p[\phi] = [\partial_p\phi]$. Hence, $(\partial_p[\phi_\epsilon])\Lambda(\partial_p[\phi_\epsilon])^*$ is given by the kernel

$$\begin{aligned}&(\partial_p\Lambda^\epsilon\partial_p)(w,z)\\ &= \int_x\int_y \Lambda(x,y)(\partial_p\phi_\epsilon)(y-z)(\partial_p\phi_\epsilon)(x-w)\,d\mathcal{L}^n y\,d\mathcal{L}^n x\\ &= \int_x\int_y \Lambda(x,y)\frac{1}{\epsilon^{n+1}}(\partial_p\phi)\left(\frac{y-z}{\epsilon}\right)\frac{1}{\epsilon^{n+1}}(\partial_p\phi)\left(\frac{x-w}{\epsilon}\right)d\mathcal{L}^n y\,d\mathcal{L}^n x\\ &\le \frac{C}{\epsilon^{2n+2}}. \end{aligned}\tag{B.38}$$

Appendix C. Solution of Heat Equation (4.3)

C.1. Galerkin's Scheme and Picard's Iteration

Recall the notations from Section 4.1.5. The smoothing parameters δ and ϵ will be suppressed. The following two statements suffice to solve (4.3).

1. *Let* $G(\cdot) \in \mathcal{N}_\Lambda(L(H))$ *such that* $\mathrm{Tr}\,[G(t)\Lambda G^*(t)] \leq C$ *for all* $t \in [0,1]$. *Then there is a unique solution* $\{T(\cdot)\}_{t\geq 0}$ *with initial data* $T_0 \in H$ *for the following equation:*

$$dT(t) = \frac{1}{c_K}\mathrm{div}\,(\Sigma_K \nabla T(t))\ dt + G(t)\,dW_t, \tag{C.1}$$

in the sense that
- $T \in C([0,1], H)$ *and is predictable.*
- $E\left\{\sup_{t\in[0,1]} \|T(t)\|_H^2 + \int_0^1 \|T(s)\|_V^2\ ds\right\} \leq C.$
- *For all* $v \in V$,

$$(T(t),\ v)_H = (T_0,\ v)_H - \int_0^t \langle AT(s),\ v\rangle\ ds + \int_0^t (G(s)\,dW_s,\ v)_H\,. \tag{C.2}$$

Or equivalently,

$$\begin{aligned}(c_K T(t),\ v)_{L^2} \ &= \ (c_K T_0,\ v)_{L^2} - \int_0^t \langle \Sigma_K \nabla T(s),\ \nabla v\rangle_{L^2}\ ds \\ &\quad + \int_0^t (c_K G(s)\,dW_s,\ v)_{L^2}\,.\end{aligned}$$

2. *Let* $\mathcal{X}$ *be the space* $C([0,1],\ H)$ *with norm*

$$\|T\|_{\mathcal{X}} = \sup_{t\in[0,1]} \|T(t)\|_H\,. \tag{C.3}$$

Given any $T^1 \in \mathcal{X}$, *let* $T^2 \in \mathcal{X}$ *be the unique solution of the following equation with initial condition* $T_0 \in H$ *and* B *as defined in (4.10):*

$$dT^2(t) = -AT^2(t)\,dt + B(T^1(t))\,dW_t. \tag{C.4}$$

Because of 1, T^2 *exists. Let* Γ *denote the map from* T^1 *to* T^2. *We claim that* $T^1, T^2, \ldots T^n = \Gamma(T^{n-1}), \ldots$ *will converge to a limiting function* $T \in C([0,1],\ H)$, *which is clearly a solution of (4.2).*
Such a procedure is called **Picard's Iteration**.

We now start the task of solving (C.1) by means of **Galerkin's Scheme**.

C.1.1. Finite Dimensional Approximation. Choose any orthonormal basis (O.N.B.) of H $\{u_i\}_{i\geq 1}$ with $u_i \in V$. Let $T_n(t) = \sum_{i=1}^n c_n^i(t)u_i$ and Π_n be the projection from H

onto the linear span of $u_1, u_2, \ldots u_n$. Consider the following stochastic ODE which can be regarded as a finite dimensional approximation for (C.1):

$$\begin{aligned} dT_n(t) &= -\Pi_n A T_n\, dt + \Pi_n G(t)\, dW_t \\ \Longrightarrow \quad dc_n^i(t) &= -\sum_{j=1}^{n} c_n^i(t) \left\langle Au_j,\, u_i \right\rangle dt + (G(t)\, dW_t,\, u_i)_H\,, \quad i = 1, 2, \ldots n \end{aligned} \tag{C.5}$$

By the fact that G is bounded and independent of the c_i's, the above is a linear stochastic ODE. A unique strong solution exists for $c_n^i(t)$ by the standard techniques of solving such equations ([KS] 5.2.9).

C.1.2. Uniform Energy Estimates for the T_n's. Recall $T_n(t) = \sum_i c_n^i(t) u_i$. Then, $\|T_n(t)\|_H^2 = \sum_i c_n^i(t)^2$. Hence, by Ito's Formula,

$$d\,\|T_n(t)\|_H^2 = \sum_{i=1}^{n} 2c_n^i(t) dc_n^i(t) + \sum_{i=1}^{n} d\left\langle c_n^i(\cdot)\right\rangle_t .$$

Now,

$$\begin{aligned} \sum_{i=1}^{n} 2c_n^i(t) dc_n^i(t) &= -\sum_{i,j}^{n} 2c_n^i(t) c_n^j(t) \left(Au_j,\, u_i\right)_H\, dt + \sum_{i=1}^{n} 2c_n^i(t)\, (G(t)\, dW_t,\, u_i)_H \\ &= -2\,\langle \Sigma_K \nabla T_n(t),\, \nabla T_n(t)\rangle\, dt + 2\,(G(t)\, dW_t,\, T_n(t))_H\,, \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{n} d\left\langle c_n^i(\cdot)\right\rangle_t &= \sum_{i=1}^{n} \left(G(t)\Lambda G(t)^* u_i,\, u_i\right)_H\, dt = \mathrm{Tr}\left[\Pi_n G(t) \Lambda G(t)^* \Pi_n\right] dt \\ &\le C\, dt \qquad \text{(by the boundedness of } G\text{)}. \end{aligned}$$

Hence,

$$\begin{aligned} &\|T_n(t)\|_H^2 + 2\int_0^t \langle \Sigma_K \nabla T_n(s),\, \nabla T_n(s)\rangle\, ds \\ &= \|T_n(0)\|_H^2 + 2\int_0^t (G(s)\, dW_s,\, T_n(s))_H \\ &\int_0^t \mathrm{Tr}\left[\Pi_n G(s) \Lambda G(s)^* \Pi_n\right] ds \end{aligned} \tag{C.6}$$

By **Burkholder's Inequality** (B.4),

$$\begin{aligned} E\left\{\sup_{\lambda\in[0,t]} \|T_n(t)\|_H^2\right\} &\le C + 2E\left\{\sup_{\lambda\in[0,t]} \left|\int_0^t (G(s)\, dW_s,\, T_n(s))_H\right|\right\} \\ &\le C + 2E\left\{\sup_{\lambda\in[0,t]} \left|\int_0^t (G(s)\, dW_s,\, T_n(s))_H\right|^2\right\} \\ &\qquad \text{(by } a \le a^2 + 1) \end{aligned}$$

$$
\begin{aligned}
&= C + DE \int_0^t \left(G(s)\Lambda G(s)^* T_n(s),\ T_n(s)\right)_H \, ds \\
&\le C + D \int_0^t E\, \|T_n(s)\|_H^2 \, ds \\
&\le C + D \int_0^t E \left\{ \sup_{\lambda \in [0,s]} \|T_n(\lambda)\|_H^2 \right\} ds.
\end{aligned}
$$

Applying Gronwall's Inequality for $D_n(t) = E\left\{\sup_{\lambda\in[0,t]} \|T_n(\lambda)\|_H^2\right\}$, we deduce that $D_n(t) \le C$ for $t \in [0, 1]$.

Now (C.6) leads to

$$
E\left\{ \sup_{t\in[0,1]} \|T_n(t)\|_H^2 + \int_0^1 \langle \Sigma_K \nabla T_n(s),\ \nabla T_n(s)\rangle \, ds \right\} \le C. \tag{C.7}
$$

C.1.3. Taking the Limit. Let Y be the space $([0, 1] \times \Omega,\ \mathcal{PM},\ d\mathcal{L}^1 \times dP)$, where $\mathcal{PM}$ is the completion of predictable sets. Then, from (C.7), $\{T_n(t)\}_{t\in[0,1],\, n\ge 1}$ can be considered as elements of $L(Y, H)$ and $L(Y, V)$ with uniform bound on their L^2 norms. Hence, we can extract a subsequence (still denoted by n) such that

$$
\begin{aligned}
T_n &\rightharpoonup \tilde{T} \qquad \text{weakly in } L^2(Y, H), \\
T_n &\rightharpoonup \tilde{S} \qquad \text{weakly in } L^2(Y, V).
\end{aligned}
$$

By the fact that V is densely embedded in H, we have $\tilde{T} = \tilde{S}$ ($d\mathcal{L}^1 \times dP$ a.e.).

Since the T_n's satisfy the following equation:

$$
(T_n(t),\ u_i)_H = (T_0,\ u_i)_H - \int_0^t \langle AT_n(s),\ u_i\rangle \, ds + \int_0^t (\Pi_n G(s)\, dW_s,\ u_i)_H\,, \tag{C.8}
$$

then for any y (bounded function on Y), we have

$$
\begin{aligned}
&E \int_0^1 y(t)\, (T_n(t),\ u_i)_H \, dt \\
&= E \int_0^1 y(t) \left\{ (T_0,\ u_i)_H - \int_0^t \langle AT_n(s),\ u_i\rangle \, ds + \int_0^t (\Pi_n G(s)\, dW_s,\ u_i)_H \right\} dt \\
&= E \int_0^1 y(t) \left\{ (T_0,\ u_i)_H - \int_0^t \langle T_n(s),\ Au_i\rangle \, ds + \int_0^t (\Pi_n G(s)\, dW_s,\ u_i)_H \right\} dt.
\end{aligned} \tag{C.9}
$$

Taking the limit $n \to \infty$, we obtain that

$$
\begin{aligned}
&E \int_0^1 y(t) \left(\tilde{T}(t),\ u_i\right)_H \, dt \\
&\quad = E \int_0^1 y(t) \left\{ (T_0,\ u_i)_H - \int_0^t \left\langle \tilde{S}(s),\ Au_i \right\rangle ds + \int_0^t (G(s)\, dW_s,\ u_i)_H \right\} dt.
\end{aligned}
$$

Hence, for all $v \in V$, $d\mathcal{L}^1 \times dP$ a.e., as elements of V^*, we have

$$\tilde{T}(t) = T_0 - \int_0^t A\tilde{S}(s)\,ds + \int_0^t G(s)\,dW_s = T_0 - \int_0^t A\tilde{T}(s)\,ds + \int_0^t G(s)\,dW_s.$$

By **Ito's Formula in the infinite dimensional case** (B.29), there exists an $\Omega' \subset \Omega$ with $P(\Omega') = 1$ and a $T(t)$, **continuous in** t**, taking values in** H such that $T = \tilde{T}$ ($dP \times d\mathcal{L}^1$ a.e.) and **for all** $t \in [0, 1]$**,** $\omega \in \Omega'$,

$$T(t) = T_0 - \int_0^t AT(s)\,ds + \int_0^t G(s)\,dW_s; \tag{C.10}$$

i.e., (C.1) is solved.

The energy estimate (C.7) is also true for T due to the **lower semicontinuity** of the energy functionals with respect to weak convergence. (It can also be derived from (C.10) by applying Ito's Formula for the norm square of $T(t)$.)

C.1.4. Uniqueness of Solution for (C.1). Let T_1 and T_2 be two solutions. Then,

$$T_1(t) - T_2(t) = -\int_0^t A(T_1(s) - T_2(s))\,ds.$$

Using Ito's Formula again, we have

$$\begin{aligned} \|T_1(t) - T_2(t)\|_H^2 &= -2\int_0^t \langle A(T_1(s) - T_2(s)),\ T_1(s) - T_2(s)\rangle\ ds \\ &= -2\int_0^t \langle \Sigma_K \nabla\,(T_1(s) - T_2(s)),\ \nabla\,(T_1(s) - T_2(s))\rangle\ ds; \end{aligned}$$

i.e., $\|T_1(t) - T_2(t)\|_H^2 = 0$ P a.s. Thus, T_1 and T_2 are indistinguishable.

C.1.5. Picard's Iteration. Now we go into the iteration scheme to solve (4.2).

Let $\{T^n(t)\}_{n\geq 1}$ be such that $dT^n(t) = -AT^n(t)\,dt + B(T^{n-1}(t))\,dW_t$. This implies that

$$d(T^{n+1}(t) - T^n(t)) = -A\left(T^{n+1}(t) - T^n(t)\right)\,dt + \left(B(T^n(t)) - B(T^{n-1}(t))\right)\,dW_t. \tag{C.11}$$

Hence,

$$\begin{aligned} &\left\|T^{n+1}(t) - T^n(t)\right\|_H^2 \\ &= -2\int_0^t \left\langle A\left(T^{n+1}(s) - T^n(s)\right),\ T^{n+1}(s) - T^n(s)\right\rangle\,ds \\ &\quad +2\int_0^t \left(\left(B(T^n(s)) - B(T^{n-1}(s))\right)\,dW_s,\ T^{n+1}(s) - T^n(s)\right)_H \\ &\quad +\int_0^t \mathrm{Tr}\left[\left(B(T^n(s)) - B(T^n(s))\right)\Lambda\left(B(T^n(s)) - B(T^n(s))\right)\right]\,ds. \end{aligned} \tag{C.12}$$

By the positivity of A, we deduce that

$$
\begin{aligned}
&\sup_{\lambda\in[0,t]} \left\|T^{n+1}(\lambda) - T^n(\lambda)\right\|_H^4 \\
&\le\; C \sup_{\lambda\in[0,t]} \left| \int_0^\lambda \left(\left(B(T^n(s)) - B(T^{n-1}(s))\right) dW_s,\; T^{n+1}(s) - T^n(s)\right)_H \right|^2 \\
&\quad + C \left[\int_0^t \mathrm{Tr}\left[\left(B(T^n(s)) - B(T^{n-1}(s))\right) \Lambda \left(B(T^n(s)) - B(T^{n-1}(s))\right)\right] ds \right]^2 \\
\Longrightarrow\quad & E\left\{ \sup_{\lambda\in[0,t]} \left\|T^{n+1}(\lambda) - T^n(\lambda)\right\|_H^4 \right\} \\
&\le\; CE\left\{ \sup_{\lambda\in[0,t]} \left| \int_0^\lambda \left(\left(B(T^n(s)) - B(T^{n+1}(s))\right) dW_s,\; T^{n+1}(s) - T^n(s)\right)_H \right|^2 \right\} \\
&\quad + CE \int_0^t \left(\mathrm{Tr}\left[\left(B(T^n(s)) - B(T^{n-1}(s))\right) \Lambda \left(B(T^n(s)) - B(T^{n-1}(s))\right)\right]\right)^2 ds.
\end{aligned}
\tag{C.13}
$$

Using Burkholder's Inequality (B.4), the first term in the R.H.S. of the above is bounded by

$$
\begin{aligned}
CE \int_0^t &\left(\Lambda \left(B(T^n(s)) - B(T^{n-1}(s))\right) \left(T^{n+1}(s) - T^n(s)\right),\right. \\
&\quad \left.\left(B(T^n(s)) - B(T^{n-1}(s))\right) \left(T^{n+1}(s) - T^n(s)\right)\right) ds,
\end{aligned}
$$

which in turn is bounded by (recall that $\Lambda \in L^\infty(\mathcal{O} \times \mathcal{O})$)

$$
CE \int_0^t \left\|T^n(s) - T^{n-1}(s)\right\|_H^2 \left\|T^{n+1}(s) - T^n(s)\right\|_H^2 \, ds.
$$

Hence, we can rewrite (C.13) as

$$
\begin{aligned}
&E\left\{ \sup_{\lambda\in[0,t]} \left\|T^{n+1}(\lambda) - T^n(\lambda)\right\|_H^4 \right\} \\
&\le\; CE \int_0^t \frac{1}{\theta} \left\|T^n(s) - T^{n-1}(s)\right\|_H^4 + \theta \left\|T^{n+1}(s) - T^n(s)\right\|_H^4 \, ds \\
&\quad + C \int_0^t E \left\|T^n(s) - T^{n-1}(s)\right\|_H^4 \, ds \quad \text{(we have used } ab \le \tfrac{1}{\theta}a^2 + \theta b^2\text{)} \\
&\le\; C' \int_0^t E\left\{ \sup_{\lambda\in[0,s]} \left\|T^n(\lambda) - T^{n-1}(\lambda)\right\|_H^4 \right\} ds \\
&\quad + C\theta \int_0^t E\left\{ \sup_{\lambda\in[0,t]} \left\|T^{n+1}(s) - T^n(s)\right\|_H^4 \right\} ds
\end{aligned}
$$

$$= C' \int_0^t E\left\{\sup_{\lambda\in[0,s]} \left\| T^n(\lambda) - T^{n-1}(\lambda)\right\|_H^4\right\} ds$$
$$+ C\theta t E\left\{\sup_{\lambda\in[0,t]} \left\| T^{n+1}(\lambda) - T^n(\lambda)\right\|_H^4\right\}. \qquad \text{(C.14)}$$

Choose θ small enough such that $C\theta \le 1/2$. Then, for all $0 \le t \le 1$,

$$E\left\{\sup_{\lambda\in[0,t]} \left\| T^{n+1}(\lambda) - T^n(\lambda)\right\|_H^4\right\} \le C' \int_0^t E\left\{\sup_{\lambda\in[0,s]} \left\| T^n(\lambda) - T^{n-1}(\lambda)\right\|_H^4\right\} ds.$$

Let $D^{(n)}(t) = E\left\{\sup_{\lambda\in[0,t]} \left\| T^n(\lambda) - T^{n-1}(\lambda)\right\|_H^4\right\}$. The above says $D^{(n)}(t) \le C\int_0^t D^{(n-1)}(s)\, ds$. Proceeding inductively,

$$\begin{aligned} D^{(n)}(t) &\le C\int_0^t D^{(n-1)}(t_1)\, dt_1 \le C^2 \int_0^t \int_0^{t_1} D^{(n-2)}(t_2)\, dt_2 dt_1 \\ &\vdots \\ &\le C^n \int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-2}} \int_0^{t_{n-1}} dt_n dt_{n-1} dt_{n-2} \cdots dt_1 \\ &\le \frac{C^n t^n}{n!}. \end{aligned} \qquad \text{(C.15)}$$

Hence, $\sum_n P\left\{\sup_{\lambda\in[0,1]} \left\| T^n(\lambda) - T^{n-1}(\lambda)\right\|_H^4 \ge \frac{1}{2^n}\right\} \le \sum_n \frac{2^n C^n}{n!} < \infty$. By the **Borel-Cantelli Lemma**, we conclude that $\{T^n\}_{n\ge1}$ almost surely converges uniformly to some $T \in C([0,1], H)$.

Now, if we go back to (C.12), and use (C.14) and (C.15), we have

$$E\left(\int_0^1 \left\|\nabla T^{n+1}(s) - \nabla T^n(s)\right\|_{L^2}^2 ds\right)^2 \le C'\frac{C^n}{n!} + C'\frac{C^{n+1}}{n!} \le \frac{C^n}{n!}.$$

Hence, $\sum_{n=1}^\infty E\int_0^1 \left\|\nabla T^{n+1}(s) - \nabla T^n(s)\right\|_{L^2}^2 ds \le \sum_{n=1}^\infty \left(\frac{C^n}{n!}\right)^{1/2} < \infty$. Now it is easy to verify that T is a solution for (4.2). The procedure is the same as in Section C.1.3. Uniqueness is very similar to Section C.1.4.

The whole Theorem 4.1.4 is thus proved.

Acknowledgments

I wish to thank my thesis advisor Professor Frederick J. Almgren, Jr. for his guidance and encouragement. The conversations with Professors Erhan Çinlar and Rene Carmona were very helpful. Professor Robert Kohn and the referees of this paper have also given helpful comments in the preparation of the manuscript.

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