Invariant Measures for Stochastic Reaction-Diffusion equations with Weakly Dissipative Nonlinearities

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Abstract

In this paper we establish the existence of an invariant measure for a stochastic reaction-diffusion equation of the type $du = (Au + f(x,u))dt + \sigma(x,u)dW(x,t)$, where A is an elliptic operator, f and σ are nonlinear maps and W is an infinite dimensional Q - Wiener process. Our emphasis is on unbounded domains. In the case of $A = \Delta$, under a very mild dissipation assumption on f, using a general Ito's formula we show the existence of a solution which is bounded in probability. In addition, we investigate a type of equation for which bounded solutions may be obtained as a limit of an iteration scheme. Together with the compactness property of the semigroup, the existence of such solution implies the existence of an invariant measure which is an important step in establishing the ergodic behavior of the underlying physical system. We also address the relation between the existence and uniqueness of invariant measure, and the spectral properties of a general elliptic operator A.

1 Introduction and Main Results

In this paper we study the properties of solutions of the stochastic evolution equation

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = Au(t,x) + f(x,u(t,x)) + \sigma(x,u(t,x))\dot{W}(t,x), t > 0, x \in G \subset \mathbb{R}^d; \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

in particular, the existence and uniqueness of invariant measures. Here A is an elliptic operator, f and σ are measurable functions, the Gaussian noise $\dot{W}(t,x)$ is white in time and

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colored in space (see (6) for the precise definition of $W(t,x)$), and $G \subseteq \mathbb{R}^d$. Our primary goal is to investigate the existence and uniqueness of invariant measures for the equation (1) in the entire space, i.e. $G = \mathbb{R}^d$.

The evolution equations of type (1), both deterministic and stochastic, emerged in the first half of the 20th century as a model of interplay between diffusion and reaction terms. The range of application of (1) includes the population dynamics, chemical physics, biomedical modeling, modeling consumption of resources etc. A comprehensive study of the deterministic version of (1) (when $\sigma(x,u) \equiv 0$) was performed by V. Volpert [24]. Long time behavior in the deterministic case was subsequently analysed e.g., by N. Dirr and N. Yip, see [10] and references therein.

A thorough analysis of stochastic equation (1) has been performed by G. Da Prato and J. Zabczyk (see [8, 9] and references therein). The long time behavior of solutions of (1) is a question of separate interest in these works. The works [8, 9] provide conditions for the existence and uniqueness of invariant measures for (1). Moreover, it is shown that the existence and uniqueness of invariant measures is the crucial item in establishing the ergodic behavior of the underlying physical systems [9, Theorems 3.2.4, 3.2.6]. Using the results of Krylov and Bogoliubov [15] on the tightness of a family of measures, in order to show the existence of an invariant measure, it is sufficient to establish the compactness and Feller properties of the semigroup $S(t)$ generated by A and to find at least one solution which is bounded for $t \in [0,\infty)$ in certain probability sense. The existence of invariant measures using the aforementioned procedure was established in [18, 12, 3], in particular, in the case when $A = \Delta$ in a bounded domain. An alternative approach to establishing the existence of invariant measures, which is based on the coupling method, was suggested by Bogachev and Rockner [2] and C. Mueller [20]. This method can be applied even for space-white noise but only in the case when the space dimension is one.

The existence and uniqueness of the solutions of stochastic reaction-diffusion equations with general elliptic operators A in bounded domains, as well as the existence of an invariant measures for such equations, was studied by S. Cerrai, see [4, 5, 6] and references therein.

In contrast to the case of bounded domains, the solutions of (1) in unbounded domains are studied in weighted Sobolev spaces. Let ρ be a non-negative continuous $L^1(\mathbb{R}^d) \bigcap L^\infty(\mathbb{R}^d)$ function. Following [22], we call ρ to be an *admissible* weight if for every $T > 0$ there exists $C(T) > 0$ such that

$$G(t,\cdot) * \rho \leq C(T)\rho,\ \forall t \in [0,T], \text{ where } G(t,x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}.$$

Some examples of admissible weights include $\rho(x) = exp(-\gamma|x|)$ for $\gamma > 0$, and $\rho(x) = (1 + |x|^n)^{-1}$ for $n > d$. For such weights, define

$$L^2_\rho(\mathbb{R}^d) := \{w : \mathbb{R}^d \to \mathbb{R}, \int_{\mathbb{R}^d} |w(x)|^2 \rho(x)\, dx < \infty\} \tag{2}$$

and

$$\|w\|^2_\rho := \int_{\mathbb{R}^d} |w(x)|^2 \rho(x)\, dx.$$

The long time behavior of the solutions of (1) is significantly different in bounded and unbounded domains. The main reason is that the semigroup of an elliptic operator in a bounded domain G with homogeneous Dirichlet boundary conditions has an exponential contraction property:

$$\|S(t)u\|_{L^2(G)} \le Ce^{-\lambda_1 t}\|u\|_{L^2(G)},\ u \in L^2(G), \tag{3}$$

where $\lambda_1 > 0$ is the first eigenvalue of $-A$ in G (subject to Dirichlet boundary condition $u = 0$ on ∂G). The estimate (3) implies the existence and uniqueness of a stationary solution for a large class of Lipschitz nonlinearities $f(x, u)$ and $\sigma(x, u)$, see e.g. [19] even in unbounded domains. Furthermore, it is not difficult to show using stochastic ODE techniques (see, e.g. [14] and references therein) that under the assumption (3), the corresponding stationary solution is stable. The stability of the stationary solution, in turn, is the crucial step in establishing the uniqueness of invariant measures. However, in unbounded domains the validity of the estimate of type (3) heavily depends on the spectral properties of the operator A. In particular, this estimate does not hold for $A = \Delta$ in $\mathbb{R}^d$ (for example, $\|S(t)u\|_{L^2(\mathbb{R}^d)} \equiv \frac{C}{(t+1)^{d/2}}\|u\|_{L^2(\mathbb{R}^d)}$ for $u(x) = e^{-x^2/4}$). Therefore, generally speaking, the dissipative properties of the diffusion operator in the entire space are often not sufficient for the existence of a stationary solution, and additional dissipative properties of the nonlinearity $f(x, u)$ are needed. The question of the existence of invariant measures in unbounded domains with $A = \Delta$ was studied in [9, 11, 22, 1]. Loosely speaking, the key result of the work [1] states, that there exists invariant measure for (1) provided f satisfies the following dissipation condition

$$uf(u) \le -ku^2 + c \tag{4}$$

for some $k > 0$ and $c \in \mathbb{R}$.

The work [19] established the existence of invariant measure for (1) with $A = \Delta$ in $\mathbb{R}^d$ under different conditions on f. In particular, we show that the invariant measure for (1) exists if f satisfies the global bound:

$$|f(x, u)| \le \varphi(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \forall u \in \mathbb{R}. \tag{5}$$

The case of a nonlocal nonlinearity $f(u)$, which appears, e.g., in the model of nonlocal consumption of resources [25], was also considered in [19].

The goal of the present paper is two-fold: to expand the existing class of nonlinearities $f(x, u)$ and $\sigma(x, u)$, as well as the class of elliptic operators A, for which there is a bounded solution of (1).

In the first part of the paper (Theorems 1 and 2), we consider equation (1) with $A = \Delta$ and $G = \mathbb{R}^d$. We assume that W is given by

$$W(t, x) := \sum_{k=1}^{\infty} \sqrt{a_k} e_k(x)\beta_k(t) \tag{6}$$

where $\beta_k(t)$ are independent Wiener processes, and $\{e_k(x), k \ge 1\}$ is an orthonormal basis in $L^2(\mathbb{R}^d)$ satisfying

$$\sup_{k\ge 1} \|e_k\|_{L^\infty(\mathbb{R}^d)} \le 1. \tag{7}$$

An example of such basis in $L^2(\mathbb{R})$ is

$$e_n^{(k)}(x) := \left\{ \frac{1}{\pi} \sin(nx)\, \chi_{[2\pi k, 2\pi(k+1)]}(x), \cos(nx)\, \chi_{[2\pi k, 2\pi(k+1)]}(x) \right\},\ n \ge 0,\ k \in \mathbb{Z}.$$

The basis in $L^2(\mathbb{R}^d)$ for $d > 1$ can be constructed analogously. We also assume that W has bounded nuclear norm, i.e.

$$\sum_{k=1}^{\infty} a_k := a < \infty. \tag{8}$$

Theorem 1. *Let $u(t, \cdot) \in H^1(\mathbb{R}^d)$ (for a.e. t) be an analytically weak solution of (1) (see Definition 1) with the initial condition $u(0, x) = u_0(x) \in L^2(\mathbb{R}^d)$, where*

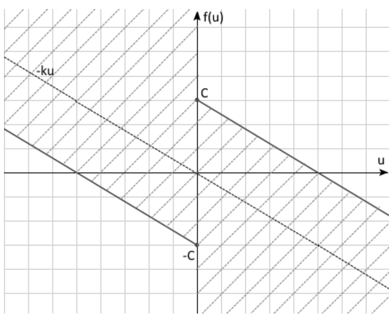
- *$A = \Delta$ in $\mathbb{R}^d$, with $d \geq 3$;*
- *$f(x, u) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is such that $f(x, 0) \in L^2(\mathbb{R}^d)$;*
- *There exists $L > 0$ such that $\forall x \in \mathbb{R}^d$ and $\forall u, v \in \mathbb{R}$ we have*
$$|f(x, u) - f(x, v)|, \ \ |\sigma(x, u) - \sigma(x, v)| \leq L|u - v|;$$
- *There exists $M > 0$ and non-negative $\eta(x) \in L^1(\mathbb{R}^d)$ such that*
$$uf(x, u) \leq \eta(x) \text{ for } |u| > M \text{ and } x \in \mathbb{R}^d; \tag{9}$$
- *There exists $\sigma_0 > 0$ such that $\forall (x, u) \in \mathbb{R}^d \times \mathbb{R}$, $|\sigma(x, u)| \leq \sigma_0$.*

Then

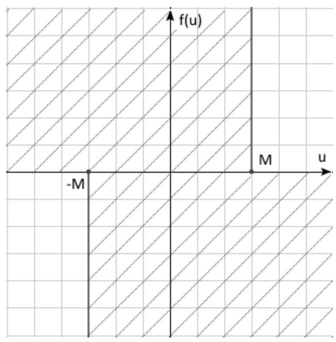
$$\sup_{t \geq 0} \mathbb{E}\|u(t, \cdot)\|_\rho^2 \leq C < \infty \tag{10}$$

for any admissible $\rho \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ (see (38) for the explicit form of C in (10)).

Remark 1. Note that while the condition (9) requires certain dissipation properties for the nonlinearity $f(x, u)$, this dissipation is significantly weaker than the conditions (4) and (5). In particular, (4) can be recovered from (9) if $f(x, u) \equiv f(u)$ and $\eta(x) \equiv 0$.



Values of f that satisfy (4)



Values of f that satisfy (9) with $\eta = 0$

Remark 2. As shown in [22, Proposition 2.1], if two non-negative admissible weights $\rho(x)$ and $\zeta(x)$ in $\mathbb{R}^d$ satisfy

$$\int_{\mathbb{R}^d} \frac{\zeta(x)}{\rho(x)} \, dx < \infty, \tag{11}$$

then

$$S(t) : L^2_\rho(\mathbb{R}^d) \to L^2_\zeta(\mathbb{R}^d) \text{ is a compact map} \tag{12}$$

where $S(t)$ is the semigroup generated by Δ in $\mathbb{R}^d$. Based on this result, the Krylov-Bogoliubov theory yields the existence of invariant measure on $L^2_\zeta(\mathbb{R}^d)$ provided (10) holds ([22, Theorem 3.1] and [1, Theorem 2]).

Our approach to the proof of Theorem 1 is significantly different from the one used in [1]. In our case, an infinite dimensional analog of the classic Ito's formula is applied to a certain, carefully chosen Lyapunov functional of a weak solution of (1). As a result, we obtain a weak formulation of a new stochastic partial differential equation, the analysis of which yields the desired estimates for the norm of the solution of the original equation.

We next consider (1), where the nonlinearities $f(x,u)$ and $\sigma(x,u)$ are Lipschitz in u with the Lipschitz constant having spatial decay. In this case, we construct an iteration scheme, which leads to the existence of a bounded solution. A similar scheme was used in the work [19], which deals with a self - adjoint elliptic operator $Au := \frac{1}{\rho}\mathrm{Div}(\rho\nabla u)$, $u \in L^2_\rho(\mathbb{R}^d)$ with a specific Gaussian weight $\rho(x) = e^{-|x|^2}$. As shown in [19], the semigroup $S(t)$ for such operator, acting in $G = \mathbb{R}^d_+ := \{x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d, x_d > 0\}$ with Dirichlet boundary condition $u = 0$ on ∂G, satisfies (3). The estimate (3) played a crucial role in showing the existence and uniqueness of a bounded solution as a limit of an iteration scheme in [19]. In the present paper, we construct an iteration scheme when (3) does not hold. We have the following result:

Theorem 2. *Let $u(t,x)$ be a mild solution of (1) in $L^2_\rho(\mathbb{R}^d)$ (see Definition 2) with the initial condition $u(0,x) \in L^2(\mathbb{R}^d)$, where*

- *$A = \Delta$ in $\mathbb{R}^d$ with $d \geq 3$;*
- *$|f(x,0)| \leq \varphi(x)$, and $|\sigma(x,0)| \leq \varphi(x)$, where*

$$\frac{\varphi(x)}{\sqrt{\rho(x)}} \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$$

- *f and σ satisfy the following Lipschitz condition:*

$$|f(x,u_1) - f(x,u_2)|, \;\; |\sigma(x,u_1) - \sigma(x,u_2)| \leq L\varphi(x)|u_1 - u_2|$$

Then for sufficiently small L (see (54) for the exact condition on the smallness of L), the solution $u(t,x)$ satisfies

$$\sup_{t\geq 0} \mathbb{E}\|u(t,\cdot)\|^2_\rho < \infty \tag{13}$$

for any admissible $\rho \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

Remark 3. The existence and uniqueness of a mild solution follows from [17].

In the second part of the present work, using the spectral analysis of self-adjoint differential operators, we extend the results of [19] in the linear case (Theorem 5) to a wider class of self-adjoint elliptic operators defined on a Hilbert space H which can be either $L^2(\mathbb{R}^d)$ or $L^2_\rho(\mathbb{R}^d)$. In contrast with [19], here the weight ρ does not have to be Gaussian.

Theorem 3. *(Linear dichotomy) Let H be a separable Hilbert space. Consider*

$$u_t = Audt + dW_b(t) \tag{14}$$

where $u \in H$ and $A : D(A) \subset H \to H$. Assume

- *A is a self adjoint operator generating a C_0-semigroup, with a discrete spectrum $\sigma = \{\lambda_k\}$, and there exists $\varepsilon > 0$ such that $|Re(\lambda_k)| > \varepsilon$ for all k. Let $\{\varphi_k\}$ be the corresponding eigenbasis of A in H;*

- *We have*

$$W_b := \sum_{n=1}^{\infty} b_n \beta_n(t) \tag{15}$$

where $\{\beta_n, n \geq 1\}$ *are independent scalar Wiener processes, and*

$$\sum_{n=1}^{\infty} \|b_n\|_H^2 < \infty.$$

Denote

$$U_A := \overline{Span\ \{\varphi_k\ s.t.\ Re(\lambda_k) < 0\}}$$

to be the stable eigensubspace of A. Then there is a unique invariant measure for (14) in H if and only if $b_n \in U_A$ *for every* $n \geq 1$, *in other words*

$$b_n = \sum_{k: Re(\lambda_k)<0} b_k^{(n)} \varphi_k.$$

In this case, the support of this measure is in U_A.

The paper is organized as follows. In Section 2 we rigorously formulate a version of Ito's formula and use it to prove Theorem 1. Section 3 is devoted to the proof of Theorem 2. Finally, in Section 4 we prove Theorem 3 and provide a example of a class of operators, for which Theorem 3 is applicable.

2 Proof of Theorem 1.

2.1 Preliminary facts.

Define a symmetric kernel $k(x, y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ as

$$k(x, y) := \sum_{k=1}^{\infty} a_k e_k(x) e_k(y). \tag{16}$$

We have

$$\sup_{x \in \mathbb{R}^d} |k(x, x)| \quad = \quad k_0 < \infty, \tag{17}$$

$$\int_{\mathbb{R}^d} k(x, x) dx \quad < \quad \infty, \tag{18}$$

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^d} |k(x, y)|^2 dx dy \quad < \quad \infty. \tag{19}$$

Then the operator R defined by

$$(R\phi)(x) = \int_{\mathbb{R}^d} k(x, y) \phi(y)\, dy, \quad \text{for } \phi \in L^2(\mathbb{R}^d), \tag{20}$$

is Hilbert-Schmidt with the eigenvalues $\{a_k, k \geq 1\}$ and the corresponding eigenvectors $\{e_k, k \geq 1\}$. Furthermore, as shown in [7], for $\sigma_t = \sigma(x, t, \omega)$, which is $\mathcal{F}_t$ measurable for a suitable filtration $\mathcal{F}_t$ and such that

$$\mathbb{E} \int_0^T \int_{\mathbb{R}^d} |\sigma(x, t)|^2 dx dt < \infty, \ T > 0,$$

the stochastic integral

$$\int_0^t \sigma_s dW_s$$

is well defined, and the $\|\cdot\|_R$ norm of $\sigma(x) \in L^2(\mathbb{R}^d)$ is defined as

$$\|\sigma\|_R^2 := \int_{\mathbb{R}^d} k(x,x)|\sigma(x)|^2\, dx.$$

We next proceed with the definitions of solutions and the well-posedness results for (1). Let H be a Hilbert space of functions defined on $\mathbb{R}^d$, $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{F}_t$ be a right-continuous filtration such that $W(t,x)$ is adapted to $\mathcal{F}_t$ and $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $s < t$.

Definition 1. *An $\mathcal{F}_t$-adapted random process $u(t,\cdot) \in H^1(\mathbb{R}^d)$ is called an analytically weak solution of (1), if $\forall \psi \in H^1(\mathbb{R}^d)$ and for a.e. $t \geq 0$*

$$\begin{aligned}\int_{\mathbb{R}^d} u(t,x)\psi(x)dx &= \int_{\mathbb{R}^d} u(0,x)\psi(x)dx - \int_0^t \int_{\mathbb{R}^d} \nabla u(s,x)\nabla\psi(x)dxds \\ &+ \int_0^t \int_{\mathbb{R}^d} f(u(s,x))\psi(x)dx + \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} \sigma(x,u(s,x))e_k(x)\sqrt{a_k}\psi(x)dxd\beta_k(s). \quad (21)\end{aligned}$$

Definition 2. *An $\mathcal{F}_t$-adapted random process $u(t,\cdot) \in L^2(\mathbb{R}^d)$ is called a mild solution of (1) if it satisfies the following integral relation for $t \geq 0$:*

$$u(t,\cdot) = S(t)u_0(\cdot) + \int_0^t S(t-s)f(\cdot,u(s,\cdot))ds + \int_0^t S(t-s)\sigma(u(s,\cdot))dW(s,\cdot) \quad (22)$$

where $\{S(t), t \geq 0\}$ is the semigroup for the linear heat equation, i.e.

$$S(t)u(x) := \int_{\mathbb{R}^d} G(t,x-y)u(y)dy, \quad G(t,x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}. \quad (23)$$

We have the following well-posedness result

Theorem 4. *[7] Consider the stochastic evolution equation:*

$$u_t = (\Delta u + f(x,u))dt + \sigma(x,u)dW(t)$$

If the following conditions hold:

(C1)

$$\|f(\cdot,0)\|_{L^2(\mathbb{R}^d)} < \infty; \ \|\sigma(\cdot,0)\|_R < \infty;$$

(C2) For any $u, v \in H^1(\mathbb{R}^d)$

$$\begin{aligned}\|f(\cdot,u(\cdot)) - f(\cdot,v(\cdot))\|_{L^2(\mathbb{R}^d)} &< L\|u-v\|_{L^2(\mathbb{R}^d)}; \\ \|\sigma(\cdot,u(\cdot)) - \sigma(\cdot,v(\cdot))\|_R &< L\|u-v\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

then (1) has a unique analytically weak solution satisfying (21).

The conditions (C1) and (C2) are straightforward to verify provided f and σ satisfy the Lipschitz condition, thus (1) has an analytically weak solution. In addition, with the above assumptions, following [8], one can check that an analytically weak solution is also a mild solution.

2.2 Ito's formula.

We follow closely the work of Krylov [16], which allows us to apply the Ito's formula to a general functional of integral type. Let $H := L^2(R^d)$, $V := H^1(\mathbb{R}^d)$, and $\Psi : H \to \mathbb{R}$ be a functional satisfying

[i] For any $h\ \xi \in H$, $t \in \mathbb{R}$, the function $\Psi(h + t\xi)$ is twice continuously differentiable as a function of t and the functions

$$\begin{aligned} Psi_{(\xi)}(h) &= \frac{\partial}{\partial t}\Psi(h+t\xi)_{t=0} \\ \Psi_{(\xi)(\xi)}(h) &= \frac{\partial^2}{\partial t^2}\Psi(h+t\xi)_{t=0} \end{aligned}$$

are continuous as functions of $(h, \xi) \in H \times H$;

[ii] For any $R \in (0, \infty)$ there exists $K(R)$ such that for all h and $\xi \in H$ satisfying $\|h\|_H \leq R$ we have

$$|\Psi_{(\xi)}(h)| \leq K(R)\|\xi\|_H \text{ and } |\Psi_{(\xi)(\xi)}(h)| \leq K(R)\|\xi\|_H^2;$$

[iii] If $h \in V$, then $\Psi_{(\cdot)}(h) \in V$, where $\Psi_{(\cdot)}(h)$ is an element in H which satisfies $\Psi_{(\xi)}(h) = (\Psi_{(\cdot)}(h), \xi)_H$. Moreover,

$$\|\Psi_{(\cdot)}(h)\|_V \leq K_1(1 + \|h\|_V)$$

where K_1 is a fixed constant;

[iv] For any $v^* \in V$ the function $(\Psi_{(\cdot)}(v), v^*)_V$ is a continuous function on V (in the metric of V)

We will make use of the following result.

Theorem 5. *[16] Assume $v_t^* \in V$ is $\mathcal{F}_t$ measurable for a.e. $t \geq 0$, $v_0 \in H$, the functional $\Psi : H \to \mathbb{R}$ satisfies the conditions [i]-[iv] above, and $u_t \in H$ satisfies almost surely*

$$(\phi, u_t)_H = (\phi, v_0)_H + \int_0^t (\phi, v_s^*)_V ds + \sum_k \int_0^t \sqrt{a_k}\,(\phi, \sigma e_k)_H\, d\beta_k(s).$$

for all $\phi \in V$. Then

$$\Psi(u_t) = \Psi(u_0) + \int_0^t \Psi_{(\sigma e_k)}(u_s) d\beta_k(s) + \int_0^t \left[(\Psi_{(\cdot)}(u_s) v_s^*)_V + \frac{1}{2}\sum_k \Psi_{(\sigma e_k)(\sigma e_k)}(u_s) \right] ds. \quad (24)$$

The above is applied to the following functional on $L^2(\mathbb{R}^d)$:

$$\Psi[u] := \int_{\mathbb{R}^d} g(u(x))\varphi_b(x)dx,$$

where $\varphi_b(x) \in H^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is an arbitrary fixed bounded function. On one hand, we would like to consider $g(u)$ which grows quadratically, i.e. $g(u) \sim u^2$ for large u, so that $\Psi[u]$ behaves like a square of a weighted L^2 norm of u. However, for technical reasons we need $g(u) \in L^2(\mathbb{R}^d)$ for $u \in L^2(\mathbb{R}^d)$, which is not true if $g(u) = u^2$. To this end, we consider a sequence of approximations $g_n(u)$, which are linear for large u yet converge to a quadratic function as $n \to \infty$:

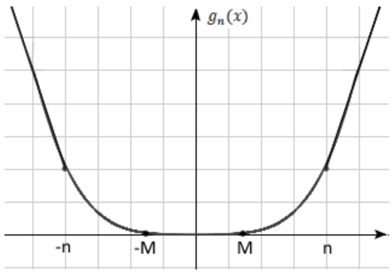
$$\Psi^n[u] := \int_{\mathbb{R}^d} g_n(u(x))\varphi_b(x)dx,$$

where $g_n : \mathbb{R} \to \mathbb{R}$ is

$$g_n(x) = \begin{cases} 0, 0 \le x < M; \\ \approx (x-M)^2,\ M+2 \le x \le n; \\ \approx 2nx, x \ge n+1 \end{cases}$$

$g_n(x) = g_n(-x)$ for $x < 0$. Here M is given in (9) and $n > M+2$. We will also need g_n is $C^2(\mathbb{R})$ with $g_n''(x) \ge 0$ for $x \in \mathbb{R}$. For example, if we use cubic splines to "glue" the linear and quadratic parts on $[M, M+2)$ and $[n, n+1]$, we get

$$g_n(x) = \begin{cases} 0, 0 \le x < M; \\ \frac{1}{6}(x-M)^3, M \le x < M+2; \\ (x-1-M)^2 + \frac{1}{3}, M+2 \le x < n; \\ -\frac{x^3}{3} + (n+1)x^2 + (n^2+2+2M)x + (1+M)^2 + \frac{1+n^3}{3}, n \le x < n+1; \\ (2n-1-2M)(x-n-1) - \frac{2(n+1)^3+1+n^3}{3} + (n^2+2+2M)(n+1) + (1+M)^2, x \ge n+1. \end{cases}$$



The function $g_n(x)$

Notice that by construction $0 \le g_n'' \le 2$. Moreover, since $|g_n(x)| < c_n|x|$ for some $c_n > 0$, if $u(x)$ in $L^2(\mathbb{R}^d)$, then $g_n(u(x))$ is also in $L^2(\mathbb{R}^d)$. Moreover, if $\nabla u \in L^2(\mathbb{R}^d)$, then $|\nabla g_n(u(x))| = |g_n'(u(x))||\nabla u|$ is also in $L^2(\mathbb{R}^d)$ since $|g_n'(x)| \le c_n$. Consequently, if $u \in H^1(\mathbb{R}^d)$, then $g_n(u) \in H^1(\mathbb{R}^d)$ as well.

We now check the conditions (i) - (iv) of Theorem 5 for $\Psi^n[u(x)]$:

[i] We have

$$\begin{aligned} {\Psi^n}_{(\xi)}(h) &= \int_{\mathbb{R}^d} g_n'(h(x))\xi(x)\varphi_b(x)\,dx, \\ {\Psi^n}_{(\xi)(\xi)}(h) &= \int_{\mathbb{R}^d} g_n''(h(x))\xi^2(x)\varphi_b(x)\,dx, \\ {\Psi^n}_{(\cdot)}(h) &= g_n'(h(x))\varphi_b(x). \end{aligned}$$

Then $\Psi^n[u+t\xi]$ is twice continuously differentiable in t and ${\Psi^n}_{(\xi)}$, ${\Psi^n}_{(\xi)(\xi)}$ are continuous.

[ii] $\forall R>0 \exists K>0$ s.t. $\forall h, \xi \in L^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} h(x)^2 dx \le R$ we have

$$\begin{aligned}
\left|\Psi^n{}_{(\xi)}(h)\right| &= \left|\int_{\mathbb{R}^d} g_n^{'}(h)\xi(x)\varphi_b(x)dx\right| \\
&\le \|\xi\|_{L^2(\mathbb{R}^d)}\|g_n^{'}(h)\varphi_b\|_{L^2(\mathbb{R}^d)} \\
&\le \|g_n^{'}\|_{L^\infty(\mathbb{R})}\|\varphi_b\|_{L^2(\mathbb{R}^d)}\|\xi\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Similarly,

$$\left|\Psi^n{}_{(\xi)(\xi)}(h)\right| = \left|\int_{\mathbb{R}^d} g_n^{''}(h)\xi^2(x)\varphi_b(x)dx\right| \le 2\|\varphi_b\|_{L^\infty(\mathbb{R}^d)}\|\xi\|^2_{L^2(\mathbb{R}^d)}.$$

[iii] We have

$$\Psi^n{}_{(\cdot)}(h) = g_n^{'}(h(x))\varphi_b(x)$$

Thus

$$\begin{aligned}
&\|\Psi^n{}_{(\cdot)}(h)\|^2_{H^1(\mathbb{R}^d)} \\
=\;& \int_{\mathbb{R}^d} |g_n^{'}(h(x))\varphi_b(x)|^2 dx + \int_{\mathbb{R}^d} \left|\nabla\left(g_n^{'}(h(x))\varphi_b(x)\right)\right|^2 \\
=\;& \int_{\mathbb{R}^d} |g_n^{'}(h(x))\varphi_b(x)|^2 dx + \int_{\mathbb{R}^d} \left|\left(g_n^{''}(h(x))\nabla h(x)\varphi_b(x) + g_n^{'}(h(x))\nabla\varphi_b(x)\right)\right|^2 \\
\le\;& \int_{\mathbb{R}^d} |g_n^{'}(h(x))\varphi_b(x)|^2 dx + 2\int_{\mathbb{R}^d} |\nabla h(x)|^2 \int_{\mathbb{R}^d}\left|g_n^{''}(h(x))\varphi_b(x)\right|^2 + 2\int_{\mathbb{R}^d}\left|g_n^{'}(h(x))\nabla\varphi_b(x)\right|^2 \\
\le\;& K_1(1+\|h\|_{H^1(\mathbb{R}^d)}).
\end{aligned}$$

[iv] For any $v^* \in H^1(\mathbb{R}^d)$,

$$(v^*, g_n^{'}(h(x))\varphi_b(x))_{H^1(\mathbb{R}^d)}$$

is a continuous functional of h on $H^1(\mathbb{R}^d)$, since $g_n^{'}$ and $g_n^{''}$ are continuous.

For every $\mathcal{F}_t$ - measurable $u \in L^2(0,t,H^1(\mathbb{R}^d))$, for fixed $\omega \in \Omega$ the expression $\Delta u(s,x) + f(u(s,x)) \in L^2(0,t,H^{-1}(\mathbb{R}^d))$ is a linear continuous functional on the Hilbert space $L^2(0,t,H^1(\mathbb{R}^d))$. By Riesz representation theorem, there exists $v_s^* \in L^2(0,t,H^1(\mathbb{R}^d))$ such that for every $\phi(x) \in H^1(\mathbb{R}^d)$,

$$\int_0^t (\phi(x), v_s^*)_{H^1(\mathbb{R}^d)} ds := -\int_0^t \int_{\mathbb{R}^d} \nabla u(s,x)\cdot\nabla\phi(x)dxds + \int_0^t\int_{\mathbb{R}^d}\phi(x)f(u(s,x))dxds. \quad (25)$$

Since $u(t)$ is $\mathcal{F}_t$-measurable, the equality (25) implies that v_t^* is $\mathcal{F}_t$ measurable as well. Now, let $u(t,x)$ satisfy

$$\begin{aligned}
(\phi(x), u(t,x))_{L^2(\mathbb{R}^d)} &= (\phi(x), u(0,x))_{L^2(\mathbb{R}^d)} + \int_0^t (\phi(x), v^*(s,x))_{H^1(\mathbb{R}^d)} ds \\
&\quad + \sum_k \int_0^t \sqrt{a_k}(\phi(x), \sigma(x, u(s,x))e_k(s))_{L^2(\mathbb{R}^d)} d\beta_k(s) \qquad (26)
\end{aligned}$$

for every $\phi(x) \in H^1(\mathbb{R}^d)$, where v^* is implicitly defined by (25). Then by Theorem 5

$$\begin{aligned}
&\int_{\mathbb{R}^d} g_n(u(t,x))\varphi_b(x)\,dx \\
= &\int_{\mathbb{R}^d} g_n(u(0,x))\varphi_b(x)\,dx \\
&+ \sum_k \int_0^t \int_{\mathbb{R}^d} g_n^{'}(u(s,x))\varphi_b(x)\sigma(x,u(s,x))\sqrt{a_k}e_k(x)dxd\beta_k(s) \\
&- \int_0^t \int_{\mathbb{R}^d} \nabla[g_n^{'}(u(s,x))\varphi_b(x)] \cdot \nabla u(s,x)dxds + \int_0^t \int_{\mathbb{R}^d} g_n^{'}(u(s,x))\varphi_b(x)f(u(s,x))dxds \\
&+ \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} g_n^{''}(u(s,x))\sigma^2(x,u(s,x))\sum_k a_k|e_k(x)|^2\varphi_b(x)dxds. \qquad (27)
\end{aligned}$$

To proceed further, we need to extend (27) to hold for any $\varphi \in H^1(\mathbb{R}^d)$. Recall that φ_b is an arbitrary function in $H^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Now, for any fixed $\varphi \in H^1(\mathbb{R}^d)$ we can always find a sequence of $\{\varphi_b^m, m \geq 1\} \in H^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ such that $\varphi_b^m \to \varphi$ strongly in $H^1(\mathbb{R}^d)$ as $m \to \infty$. Notice that due to (7) we have $\|e_k\|_{L^4(\mathbb{R}^d)} \leq 1$. Hence we may pass to the limit in the deterministic terms in (27) as $m \to \infty$. It remains to justify the limit as $m \to \infty$ in the stochastic term in (27). To this end, we shall use the following result for one-dimensional Wiener process $W(t)$, [21] (Theorem 1.3.1): for $T > 0$

$$\sup_{t\in[0,T]} \left| \int_0^t f_m(s)dW(s) - \int_0^t f_0(s)dW(s) \right| \to^P 0 \text{ as } m \to \infty$$

is true if and only if

$$\int_0^T |f_m(s) - f_0(s)|^2\,ds \to^P 0.$$

where $\to^P$ means convergence in probability. In our case, we have

$$\begin{aligned}
&\sum_{k=1}^\infty \int_0^t \left(\int_{\mathbb{R}^d} g_n^{'}(u(s,x))(\varphi_b^m(x) - \varphi(x))\sigma(x,u(s,x))\sqrt{a_k}e_k(x)dx \right)^2 ds \\
\leq \;& C_n \sum_{k=1}^\infty a_k \int_0^t \int_{\mathbb{R}^d} e_k^2(x)dx \int_{\mathbb{R}^d} (\varphi_b^m(x) - \varphi(x))^2 dxds \to 0, \text{ as } m \to \infty,
\end{aligned}$$

implying that we have convergence in probability for the stochastic term. Passing to a subsequence as $m \to \infty$ if necessary, we have the desired result, i.e. (27) holds if $\varphi_b^m \in H^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is replaced with $\varphi \in H^1(\mathbb{R}^d)$. Altogether, we arrived at the following version of Ito's formula for $\int_{\mathbb{R}^d} g_n(u(x))\varphi(x)dx$:

Theorem 6. *Let $u(t,x)$ satisfy (26) for every $\phi(x) \in H^1(\mathbb{R}^d)$. Then for all $\varphi(x) \in H^1(\mathbb{R}^d)$*

and a.e. $\omega \in \Omega$

$$
\begin{aligned}
&\int_{\mathbb{R}^d} g_n(u(t,x))\varphi(x)\,dx \\
&= \int_{\mathbb{R}^d} g_n(u(0,x))\varphi(x)\,dx \qquad (28)\\
&\quad + \sum_k \int_0^t \int_{\mathbb{R}^d} g_n^{'}(u(s,x))\varphi(x)\sigma(x,u(s,x))\sqrt{a_k}e_k(x)dxd\beta_k(s) \\
&\quad - \int_0^t \int_{\mathbb{R}^d} \nabla[g_n^{'}(u(s,x))\varphi(x)]\cdot\nabla u(s,x)dxds + \int_0^t \int_{\mathbb{R}^d} g_n^{'}(u(s,x))\varphi(x)f(u(s,x))dxds \\
&\quad + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} g_n^{''}(u(s,x))\sigma^2(x,u(s,x))\sum_k a_k|e_k(x)|^2\varphi(x)dxds. \qquad (29)
\end{aligned}
$$

2.3 The relation between Ito's formula and weak solutions

Let $u(t,x)$ be a solution of (26). In this subsection we use (29) to derive an equation for $z(t,x) := g_n(u(t,x))$, then use this equation to derive an uniform bounds in t for $g_n(u(t,x))$. The proof of Theorem 1 is completed by passing to the limit as $n \to \infty$.

Lemma 1. *The function* $z(t,x) := g_n(u(t,x))$ *is an analytically weak solution of*

$$
\begin{aligned}
dz(t,x) =& [\Delta z(x,t) + g_n^{'}(u(t,x))f(u(t,x)) - g_n{''}(u(t,x))|\nabla u(t,x)|^2 \\
&+ \frac{1}{2}g_n^{''}(u(t,x))\sum_k a_k\sigma^2(x,u(s,x))|e_k(x)|^2]dt \\
&+ g_n^{'}(u(t,x))\sigma(x,u(t,x))\sum_k \sqrt{a_k}e_k(x)d\beta_k
\end{aligned}
$$

Proof. Observe that

$$
\begin{aligned}
&-\int_0^t \int_{\mathbb{R}^d} \nabla[g_n^{'}(u(s,x))\varphi(x)]\cdot\nabla u(s,x)dxds \\
&= -\int_0^t \int_{\mathbb{R}^d} g_n^{''}(u(s,x))\varphi(x)|\nabla u(s,x)|^2dxds - \int_0^t \int_{\mathbb{R}^d} g_n^{'}(u(s,x))\nabla u(s,x)\cdot\nabla\varphi(x)dxds \\
&= -\int_0^t \int_{\mathbb{R}^d} g_n^{''}(u(s,x))\varphi(x)|\nabla u(s,x)|^2dxds - \int_0^t \int_{\mathbb{R}^d} \nabla[g_n(u(s,x))]\cdot\nabla\varphi(x)dxds.
\end{aligned}
$$

Hence, (27) now reads as

$$
\begin{aligned}
&\int_{\mathbb{R}^d} g_n(u(t,x))\varphi(x)\,dx \\
&= \int_{\mathbb{R}^d} g_n(u(0,x))\varphi(x)\,dx + \sum_k \int_0^t \int_{\mathbb{R}^d} g_n^{'}(u(s,x))\varphi(x)\sqrt{a_k}\sigma(x,u(s,x))e_k(x)dxd\beta_k(s) \\
&\quad - \int_0^t \int_{\mathbb{R}^d} g_n^{''}(u(s,x))\varphi(x)|\nabla u(s,x)|^2dxds - \int_0^t \int_{\mathbb{R}^d} \nabla[g_n(u(s,x))]\cdot\nabla\varphi(x)dxds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} g_n^{'}(u(s,x))f(u(s,x))\varphi(x)dxds \\
&\quad + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} g_n^{''}(u(s,x))\sum_k a_k\sigma^2(x,u(s,x))|e_k(x)|^2\varphi(x)dxds,
\end{aligned}
$$

which concludes the proof of Lemma 1. $\square$

Lemma 2. *If $u(t,x)$ is a solution of (26), then*

$$\mathbb{E}\int_{\mathbb{R}^d} g_n(u(t,x))\rho(x)dx \le \tilde{C} \tag{30}$$

for all $t \ge 0$, where

$$\tilde{C} := \|\rho\|_{L^\infty(\mathbb{R}^d)}\|u_0\|_{L^2(\mathbb{R}^d)} + \left(\sigma_0^2 a + \|\eta\|_{L^1(\mathbb{R}^d)}\left(4+\frac{2}{M}\right)\right)\left(\|\rho\|_{L^\infty(\mathbb{R}^d)} + \frac{1}{(4\pi)^{d/2}}\|\rho\|_{L^1(\mathbb{R}^d)}\frac{2}{d-2}\right). \tag{31}$$

Proof. Since an analytically weak solution is also a mild solution, we have

$$\begin{aligned} g_n(u(t,x)) &= S(t)g_n(u(0,\cdot)) + \int_0^t S(t-s)[g_n'(u(s,\cdot))f(\cdot,u(s,\cdot)) - g_n''(u(s,\cdot))|\nabla u(s,\cdot)|^2 \\ &\quad + \frac{1}{2}g_n''(u(s,\cdot))\sum_k a_k\sigma^2(\cdot,u(s,\cdot))|e_k(\cdot)|^2]ds \\ &\quad + \int_0^t S(t-s)g_n'(u(s,\cdot))\sigma(\cdot,u(s,\cdot))\sum_k \sqrt{a_k}e_k(\cdot)d\beta_k(s) \end{aligned}$$

Using the fact that

$$\mathbb{E}\int_0^t S(t-s)g_n'(u(s,\cdot))\sigma(\cdot,u(s,\cdot))\sum_k \sqrt{a_k}e_k(\cdot)d\beta_k(s) = 0,$$

we have

$$\begin{aligned} \mathbb{E}\; g_n(u(t,x)) &= \mathbb{E}\; S(t)g_n(u(0,\cdot)) + \mathbb{E}\int_0^t S(t-s)[g_n'(u(s,\cdot))f(u(s,\cdot)) - g_n''(u(s,\cdot))|\nabla u(s,\cdot)|^2 \\ &\quad + \frac{1}{2}g_n''(u(s,\cdot))\sum_k a_k\sigma^2(\cdot,u(s,\cdot))|e_k(\cdot)|^2]ds. \end{aligned}$$

Furthermore, since $0 \le g_n(x) \le x^2$, we have

$$\begin{aligned} &\mathbb{E}\int_{\mathbb{R}^d} S(t)g_n(u(0,\cdot))\rho(x)dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t,x-y)g_n(u(0,y))\rho(x)dx\,dy \\ &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t,x-y)u_0^2(y)\rho(x)dydx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t,z)u_0^2(y)\rho(y+z)dydz \le \|\rho\|_{L^\infty(\mathbb{R}^d)}\|u_0\|_{L^2(\mathbb{R}^d)} \end{aligned}$$

and

$$-\mathbb{E}\int_0^t S(t-s)[g_n''(u(s,\cdot))|\nabla u(s,\cdot)|^2]ds \le 0.$$

Therefore

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^d} g_n(u(t,x))\rho(x)dx \\
\leq\ & \|\rho\|_{L^\infty(\mathbb{R}^d)}\|u_0\|_{L^2(\mathbb{R}^d)} \\
&+\frac{1}{2}\mathbb{E}\int_0^t\int_{\mathbb{R}^d} S(t-s)[g_n^{''}(u(s,\cdot))\sum_k a_k\sigma^2(\cdot,u(s,\cdot))|e_k(\cdot)|^2]\rho(x)dxds \\
&+\mathbb{E}\int_0^t\int_{\mathbb{R}^d} S(t-s)[g_n^{'}(u(s,\cdot))f(\cdot,u(s,\cdot))]\rho(x)dxds. \qquad (32)
\end{aligned}$$

We proceed with estimating the second term in (32). Since $g_n^{''}(x)\leq 2$, for $t\geq 1$ we have:

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^d} S(t-s)[g_n^{''}(u(s,\cdot))\sum_k a_k\sigma^2(\cdot,u(s,\cdot))|e_k(\cdot)|^2]\rho(x)dxds \\
\leq\ & 2\sigma_0^2\sum_k a_k\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t-s,x-y)|e_k(y)|^2\rho(x)dydxds \\
=\ & 2\sigma_0^2\sum_k a_k\left(\int_0^{t-1}+\int_{t-1}^t\right)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t-s,x-y)|e_k(y)|^2\rho(x)dydxds \qquad (33)
\end{aligned}$$

with $G(t,x)$ given by (23). Estimating each of the integrals above separately, we have

$$\begin{aligned}
&\int_{t-1}^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t-s,x-y)|e_k(y)|^2\rho(x)dydxds \\
=\ & \int_{t-1}^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t-s,z)|e_k(y)|^2\rho(y+z)dydzds \\
\leq\ & \|\rho\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{t-1}^t G(t-s,z)|e_k(y)|^2dsdydz=\|\rho\|_{L^\infty(\mathbb{R}^d)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{t-1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t-s,x-y)|e_k(y)|^2\rho(x)dydxds \\
=\ & \frac{1}{(4\pi)^{d/2}}\int_1^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{\tau^{d/2}}e^{-\frac{|x-y|^2}{4\tau}}|e_k(y)|^2\rho(x)dydxd\tau \\
\leq\ & \frac{1}{(4\pi)^{d/2}}\|\rho\|_{L^1(\mathbb{R}^d)}\int_1^\infty\frac{d\tau}{\tau^{d/2}}=\frac{1}{(4\pi)^{d/2}}\|\rho\|_{L^1(\mathbb{R}^d)}\frac{2}{d-2},
\end{aligned}$$

hence

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^d} S(t-s)[g_n^{''}(u(s,\cdot))\sum_k a_k\sigma^2(\cdot,u(s,\cdot))|e_k(\cdot)|^2]\rho(x)dxds \\
\leq\ & 2\sigma_0^2 a\left(\|\rho\|_{L^\infty(\mathbb{R}^d)}+\frac{1}{(4\pi)^{d/2}}\|\rho\|_{L^1(\mathbb{R}^d)}\frac{2}{d-2}\right) \qquad (34)
\end{aligned}$$

We finally proceed with the estimate for the third term in (32). Without loss of generality, assume $\eta(x)\geq 0$, and let

$$A(s):=\{y\in\mathbb{R}^d:-\eta(y)\leq u(s,y)f(y,u(s,y))\leq\eta(y)\}.$$

Clearly,

$$
\begin{aligned}
& \mathbb{E}\int_0^t \int_{\mathbb{R}^d} S(t-s)[g_n^{'}(u(s,\cdot))f(\cdot,u(s,\cdot))]\rho(x)dxds \\
\leq \; & \mathbb{E}\int_0^t \int_{\mathbb{R}^d}\int_{A(s)} G(t-s,x-y)g_n^{'}(u(s,y))f(y,u(s,y))\rho(x)dydxds
\end{aligned}
$$

since $g_n^{'}(u)$ has the same sign as u, and therefore everywhere in the complement of $A(s)$ the integrand is negative. Next,

$$
\int_{A(s)} G(t-s,x-y)g_n^{'}(u(s,y))f(y,u(s,y))dy = \sum_{i=1}^{2}\int_{A_i(s)} G(t-s,x-y)g_n^{'}(u(s,y))f(y,u(s,y))dy,
$$

where

$$
A_1(s) := \{y \in \mathbb{R}^d : |u(s,y)| \leq M\},
$$

and

$$
A_2(s) := \{y \in \mathbb{R}^d : |u(s,y)| > M\}.
$$

By the construction of $g_n(u)$,

$$
\int_{A_1(s)} G(t-s,x-y)g_n^{'}(u(s,y))f(y,u(s,y))dy = 0.
$$

Combining the facts that $|u(s,y)||f(y,u(s,y))| \leq \eta(y)$ in A and $|u(s,y)| \geq M$ in A_2, we have $|f(y,u(s,y))| \leq \frac{1}{M}\eta(y)$ in A_2. Moreover, since

$$
|g_n^{'}(u)| \leq 2(|u| + 1 + M)
$$

we get

$$
\int_{A_2(s)} G(t-s,x-y)|g_n^{'}(u(s,y))||f(y,u(s,y))|dy \leq \left(4 + \frac{2}{M}\right)\int_{\mathbb{R}^d} G(t-s,x-y)\eta(y)dy.
$$

Proceeding as in (33), we have

$$
\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t-s,x-y)\eta(y)dy\rho(x)dx\,ds \leq \|\eta\|_{L^1(\mathbb{R}^d)}\left(\|\rho\|_{L^\infty(\mathbb{R}^d)} + \frac{1}{(4\pi)^{d/2}}\|\rho\|_{L^1(\mathbb{R}^d)}\frac{2}{d-2}\right)
$$

thus altogether

$$
\begin{aligned}
& \mathbb{E}\int_0^t \int_{\mathbb{R}^d} S(t-s)[g_n^{'}(u(s,\cdot))f(u(s,\cdot))]\rho(x)dxds \\
\leq \; & \left(4 + \frac{2}{M}\right)\|\eta\|_{L^1(\mathbb{R}^d)}\left(\|\rho\|_{L^\infty(\mathbb{R}^d)} + \frac{1}{(4\pi)^{d/2}}\|\rho\|_{L^1(\mathbb{R}^d)}\frac{2}{d-2}\right). \qquad (35)
\end{aligned}
$$

Combining (32), (34) and (35), the statement of the Lemma follows.

□

To complete the proof of Theorem 1, it remains to pass to the limit as $n \to \infty$ in (30). By Fatou's Lemma, we have

$$\mathbb{E} \int_{\mathbb{R}^d} g^*(u(t,x))\rho(x)dx \le \tilde{C},$$

where

$$g^*(x) = \begin{cases} 0, 0 \le x < M; \\ \frac{1}{6}(x-M)^3, M \le x < M+2; \\ (x-1-M)^2 + \frac{1}{3}, x \ge M+2 \end{cases} \tag{36}$$

and $g^*(x) := g^*(-x), x < 0$. Finally, using the inequality

$$x^2 \le 2(x-1-M)^2 + 2(1+M)^2,$$

we get that for any $t \ge 0$

$$\mathbb{E} \int_{\mathbb{R}^d} u^2(x,t)\rho(x)dx \le 2\mathbb{E} \int_{\mathbb{R}^d} g^*(u(t,x))\rho(x)dx + 2(1+M)^2 \|\rho\|_{L^1(\mathbb{R}^d)} \le C \tag{37}$$

where

$$C = 2\tilde{C} + 2(1+M)^2 \|\rho\|_{L^1(\mathbb{R}^d)} \tag{38}$$

with $\tilde{C}$ given in (31). The proof of Theorem 1 is complete.

3 Proof of Theorem 2

As before, $G(t,x)$ stands for the heat kernel in $\mathbb{R}^d$. The main strategy of the proof it to use an iteration scheme. Set $u^{(0)}(x) = u_0(x) \in L^2(\mathbb{R}^d)$ and let $u^{(n+1)}(t,x)$ be the mild solution of

$$\begin{cases} du = (\Delta u + f(x, u^{(n)}))dt + \sigma(x, u^{(n)})dW(t); \\ u(0,x) = u_0(x). \end{cases} \tag{39}$$

The key ingredients of the proof is to show that the scheme produces a bounded solution at each step, and that the scheme is convergent. This is done in the following two lemmas.

Lemma 3. *Assume*

$$6L^2 \|\rho\|_{L^1(\mathbb{R}^d)} \left((2 + a(1+C_d)) \left\| \frac{\varphi(y)}{\sqrt{\rho(y)}} \right\|^2_{L^\infty(\mathbb{R}^d)} + \frac{8}{(4\pi)^d (d-2)^2} \left\| \frac{\varphi(y)}{\sqrt{\rho(y)}} \right\|^2_{L^2(\mathbb{R}^d)} \right) < 1 \tag{40}$$

where

$$C_d := \int_1^\infty \int_{\mathbb{R}^d} G^2(s,x)dxds < \infty \text{ for } d \ge 3 \tag{41}$$

Then

$$\sup_{t \ge 0} \mathbb{E} \|u^{(n)}(t)\|^2_\rho < C \tag{42}$$

with C independent of n.

Proof. Using induction, we first show that for any $n \geq 1$

$$\sup_{t\geq 0} \mathbb{E}\|u^{(n)}(t)\|_\rho^2 < \infty. \tag{43}$$

Set $u^{(0)}(x,t) \equiv u_0(x)$. Assuming (43) holds for some $n \geq 1$, we proceed with the estimate for $u^{(n+1)}(t)$:

$$u^{(n+1)}(t) = S(t)u_0 + \int_0^t S(t-s)f(\cdot, u^{(n)}(s))ds + \int_0^t S(t-s)\sigma(\cdot, u^{(n)}(s))dW(s)$$

thus

$$\begin{aligned}
& \mathbb{E}\|u^{(n+1)}(t)\|_\rho^2 \\
\leq\ & 3\|S(t)u_0\|_\rho^2 + 3\left\|\int_0^t S(t-s)f(\cdot, u^{(n)}(s))ds\right\|_\rho^2 + 3\left\|\int_0^t S(t-s)\sigma(\cdot, u^{(n)}(s))dW(s)\right\|_\rho^2 \\
:=\ & 3J_0 + 3J_1 + 3J_2
\end{aligned}$$

We now proceed to estimate J_0, J_1 and J_2.

Estimation of J_0.

$$J_0 := \|S(t)u_0\|_\rho^2 \leq \|\rho\|_{L^\infty(\mathbb{R}^d)}\|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{44}$$

Estimation of J_1.

$$\begin{aligned}
J_1 &:= \int_{\mathbb{R}^d} \mathbb{E}\left|\int_0^t S(t-s)f(\cdot, u^{(n)}(s,\cdot))ds\right|^2 \rho(x)dx \\
&\leq \int_{\mathbb{R}^d} \mathbb{E}\left|\int_0^t S(t-s)\varphi(\cdot)(1 + L|u^{(n)}(s,\cdot)|)ds\right|^2 \rho(x)dx \\
&\leq 2\int_{\mathbb{R}^d}\left|\int_0^t S(t-s)\varphi(\cdot)ds\right|^2 \rho(x)dx + 2\int_{\mathbb{R}^d}\mathbb{E}\left|\int_0^t S(t-s)\varphi(\cdot)L|u^{(n)}(s,\cdot)|ds\right|^2 \rho(x)dx \\
&:= 2J_{11} + 2J_{12}.
\end{aligned}$$

Each of the resulting terms is analysed as follows

$$J_{11} := \int_{\mathbb{R}^d}\left|\int_0^t S(t-s)\varphi(\cdot)ds\right|^2 \rho(x)dx \leq 2\left(\int_{\mathbb{R}^d}\left|\int_0^{t-1}\dots\right|^2\rho dx + \int_{\mathbb{R}^d}\left|\int_{t-1}^t\dots\right|^2\rho dx\right) \tag{45}$$

Estimating each component of (45) separately, we have

$$\int_{\mathbb{R}^d}\left|\int_{t-1}^t S(t-s)\varphi(\cdot)ds\right|^2\rho dx \leq \|\rho\|_{L^\infty(\mathbb{R}^d)}\int_{t-1}^t \mathbb{E}\|S(t-s)\varphi(\cdot)\|^2_{L^2(\mathbb{R}^d)}ds \leq \|\rho\|_{L^\infty(\mathbb{R}^d)}\|\varphi\|^2_{L^2(\mathbb{R}^d)}$$

and

$$\begin{aligned}
& \int_{\mathbb{R}^d}\left|\int_0^{t-1} S(t-s)\varphi(\cdot)ds\right|^2\rho dx \\
\leq\ & \int_{\mathbb{R}^d}\int_0^{t-1}\frac{ds}{4\pi(t-s)^{d/2}}\int_0^{t-1}\frac{ds}{(4\pi(t-s)^{d/2})}\left(\int_{\mathbb{R}^d}\frac{\varphi(y)}{\sqrt{\rho(y)}}\sqrt{\rho(y)}dy\right)^2\rho(x)dx \\
\leq\ & \frac{1}{(4\pi)^d}\left(\frac{2}{d-2}\right)^2\|\rho\|^2_{L^1(\mathbb{R}^d)}\left\|\frac{\varphi}{\sqrt{\rho}}\right\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Thus altogether

$$J_{11} \le 2\|\rho\|_{L^\infty(\mathbb{R}^d)}\|\varphi\|^2_{L^2(\mathbb{R}^d)} + \frac{2}{(4\pi)^d}\left(\frac{2}{d-2}\right)^2 \|\rho\|^2_{L^1(\mathbb{R}^d)} \left\|\frac{\varphi}{\sqrt{\rho}}\right\|^2_{L^2(\mathbb{R}^d)} := A_1 \tag{46}$$

Rewriting J_{12} as

$$\begin{aligned} & \int_{\mathbb{R}^d} \mathbb{E}\left(\int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\varphi(y)L|u^{(n)}(s)|dyds|\right)^2 \rho dx \\ \le\ & 2\left(\int_{\mathbb{R}^d} \mathbb{E}\left|\int_{t-1}^t \dots\right|^2 \rho dx + \int_{\mathbb{R}^d} \mathbb{E}\left|\int_0^{t-1} \dots\right|^2 \rho dx\right) \\ :=\ & 2(J_{121} + J_{122}) \end{aligned} \tag{47}$$

we have

$$\begin{aligned} J_{121} &\le \int_{\mathbb{R}^d} \mathbb{E}\int_{t-1}^t \left(\int_{\mathbb{R}^d} G(t-s,x-y)\varphi(y)L|u^{(n)}(s,y)|\frac{\sqrt{\rho(y)}}{\sqrt{\rho(y)}}dy\right)^2 \rho dx \\ &\le \|\rho\|_{L^1(\mathbb{R}^d)}L^2 \int_{t-1}^t \mathbb{E}\int_{\mathbb{R}^d} \frac{\varphi^2(y)}{\rho(y)}|u^{(n)}(s,y)|^2\rho(y)dyds \\ &\le L^2\|\rho\|_{L^1(\mathbb{R}^d)} \left\|\frac{\varphi}{\sqrt{\rho}}\right\|^2_{L^\infty(\mathbb{R}^d)} \sup_{t\ge 0}\mathbb{E}\|u^{(n)}(t)\|^2_\rho, \end{aligned} \tag{48}$$

where we used the induction assumption (43). Next, under the same assumption

$$\begin{aligned} J_{122} &\le \int_{\mathbb{R}^d} \mathbb{E}\left|\int_0^{t-1}\int_{\mathbb{R}^d} \frac{L\varphi(y)|u^{(n)}(s)|}{(4\pi(t-s))^{d/2}}dyds\right|^2 \rho dx \\ &\le L^2 \int_0^{t-1} \frac{ds}{(4\pi(t-s))^{d/2}} \int_{\mathbb{R}^d}\int_0^{t-1} \frac{1}{(4\pi(t-s))^{d/2}}\mathbb{E}\left(\int_{\mathbb{R}^d} \frac{\varphi(y)|u^{(n)}(s,y)|\sqrt{\rho(y)}}{\sqrt{\rho(y)}}\right)^2 ds\, \rho(x)dx \\ &\le L^2 \int_1^\infty \frac{ds}{(4\pi s)^{d/2}} \int_{\mathbb{R}^d}\left(\int_0^{t-1} \frac{1}{(4\pi(t-s))^{d/2}}\int_{\mathbb{R}^d}\frac{\varphi(y)^2}{\rho(y)}dy\ \mathbb{E}\int_{\mathbb{R}^d}|u^{(n)}(s,y)|^2\rho(y)dyds\right)\rho(x)dx \\ &\le L^2\frac{1}{(4\pi)^d}\left(\frac{2}{d-2}\right)^2 \left\|\frac{\varphi}{\sqrt{\rho}}\right\|^2_{L^2(\mathbb{R}^d)} \|\rho\|_{L^1(\mathbb{R}^d)} \sup_{t\ge 0}\mathbb{E}\|u^{(n)}(t)\|^2_\rho. \end{aligned} \tag{49}$$

Estimation of J_2.

$$\begin{aligned} J_2 &= \mathbb{E}\int_{\mathbb{R}^d}\left|\sum_k \sqrt{a_k}\int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)\sigma(y,u^{(n)}(s,y))e_k(y)dyd\beta_k(s)\right|^2 \rho dx \\ &\le \sum_k a_k \int_{\mathbb{R}^d}\int_0^t \mathbb{E}\left|\int_{\mathbb{R}^d} G(t-s,x-y)\sigma(y,u^{(n)}(s,y))e_k(y)dy\right|^2 ds\rho(x)dx \end{aligned} \tag{50}$$

The k-th term in (50) can be estimated as

$$
\begin{aligned}
& a_k \int_{\mathbb{R}^d} \int_0^t \mathbb{E} \left| \int_{\mathbb{R}^d} G(t-s, x-y)\sigma(y, u^{(n)}(s,y)) e_k(y) dy \right|^2 ds \rho(x) dx \\
\leq \ & 2a_k \int_{\mathbb{R}^d} \int_0^t \left| \int_{\mathbb{R}^d} G(t-s, x-y)\sigma(y, 0) e_k(y) dy \right|^2 ds \rho(x) dx \\
& +2a_k \int_{\mathbb{R}^d} \int_0^t \mathbb{E} \left| \int_{\mathbb{R}^d} G(t-s, x-y) L\varphi(y) u^{(n)}(s,y) e_k(y) dy \right|^2 ds \rho(x) dx \\
:= \ & 2a_k (J_{21} + J_{22})
\end{aligned}
$$

Using the bounds $\sigma(x,0) \leq \varphi(x)$, $|e_k(x)| \leq 1$, as well as (46), we have

$$
J_{21} := \int_{\mathbb{R}^d} \int_0^t \left| \int_{\mathbb{R}^d} G(t-s, x-y)\sigma(y, 0) e_k(y) dy \right|^2 ds \rho(x) dx \leq A_1
$$

Now,

$$
\begin{aligned}
J_{22} \ & \leq \ \mathbb{E} \int_0^t \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} G(t-s, x-y) L\varphi(y) |u^{(n)}(s,y)| |e_k(y)| dy \right|^2 \rho(x) dx ds \\
& = \ \mathbb{E} \left[\int_0^{t-1} \dots + \int_{t-1}^t \dots \right] \\
& := \ J_{221} + J_{222}.
\end{aligned}
$$

For we estimate J_{221} as follows.

$$
\begin{aligned}
J_{221} & \leq L^2 \int_0^{t-1} \int_{\mathbb{R}^d} \mathbb{E} \left(\int_{\mathbb{R}^d} G(t-s, x-y) \varphi(y) |u^{(n)}(y)| |e_k(y)| dy \right)^2 \rho(x) dx ds \\
& \leq L^2 \int_0^{t-1} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} G^2(t-s, x-y) e_k^2(y) dy \mathbb{E} \int_{\mathbb{R}^d} \varphi(y)^2 |u^{(n)}(y)|^2 dy \right) \rho(x) dx ds \\
& \leq L^2 C_d \|\rho\|_{L^1(\mathbb{R}^d)} \left\| \frac{\varphi(y)}{\sqrt{\rho(y)}} \right\|^2_{L^\infty(\mathbb{R}^d)} \sup_{t \geq 0} \mathbb{E} \|u^{(n)}(t)\|^2_\rho,
\end{aligned}
$$

where C_d is defined in (41). Similarly,

$$
\begin{aligned}
J_{222} & \leq L^2 \int_{t-1}^t \int_{\mathbb{R}^d} \mathbb{E} \left(\int_{\mathbb{R}^d} G(t-s, x-y) \varphi(y) |u^{(n)}(y)| |e_k(y)| dy \right)^2 \rho(x) ds \\
& \leq L^2 \int_{t-1}^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} G(t-s, x-y) dy \mathbb{E} \int_{\mathbb{R}^d} G(t-s, x-y) \varphi(y)^2 |e_k(y)|^2 |u^{(n)}(y)|^2 dy \right) \rho(x) dx ds \\
& \leq L^2 \|\rho\|_{L^1(\mathbb{R}^d)} \int_{t-1}^t E \int_{\mathbb{R}^d} \frac{\varphi(y)^2}{\rho(y)} |u^{(n)}(y)|^2 \rho(y) dy ds \qquad (51) \\
& \leq L^2 \|\rho\|_{L^1(\mathbb{R}^d)} \left\| \frac{\varphi(y)}{\sqrt{\rho(y)}} \right\|^2_{L^\infty(\mathbb{R}^d)} \sup_{t \geq 0} \mathbb{E} \|u^{(n)}(t)\|^2_\rho. \qquad (52)
\end{aligned}
$$

As a summary of the estimates above, we have

$$\begin{aligned}\|u^{(n+1)}(t)\|_\rho^2 &\le 3(J_0+J_1+J_2) \le 3J_0 + 6(J_{11}+J_{12}+\sum_k a_k J_{21} + \sum_k a_k J_{22})\\ &\le 3J_0 + 6(A_1 + 2J_{121} + 2J_{122} + aA_1 + aJ_{212} + aJ_{222})\\ &:= A_2 + A_3 \sup_{t\ge 0} \mathbb{E}\|u^{(n)}(t)\|_\rho^2, \end{aligned} \tag{53}$$

where $A_2 = 3\|\rho\|_{L^\infty(\mathbb{R}^d)}\|u_0\|^2_{L^2(\mathbb{R}^d)} + 6A_1(1+a)$ and $L>0$ is chosen to be small enough for

$$A_3 := 6L^2\|\rho\|_{L^1(\mathbb{R}^d)}\left((2+a(1+C_d))\left\|\frac{\varphi(y)}{\sqrt{\rho(y)}}\right\|^2_{L^\infty(\mathbb{R}^d)} + \frac{8}{(4\pi)^d(d-2)^2}\left\|\frac{\varphi(y)}{\sqrt{\rho(y)}}\right\|^2_{L^2(\mathbb{R}^d)}\right) < 1. \tag{54}$$

Consequently, we have

$$\sup_{t\ge 0}\mathbb{E}\|u^{(n)}(t)\|_\rho^2 \le \frac{A_2}{1-A_3} < \infty$$

for all $n \ge 1$. The final conclusion follows from the fact that if a nonnegative numerical sequence $\{x_n, n\ge 0\}$ satisfies

$$x_{n+1} \le a + bx_n$$

with $b<1$, then $x_n \le \frac{a}{1-b}$ for all $n \ge 0$. □

Lemma 4. *(Convergence) If $A_3 < 1$, then there exists a limiting function $u^*(t) \in L^2_\rho(\mathbb{R}^d)$ such that*

$$\sup_{t\ge 0}\mathbb{E}\|u^*(t)\|_\rho^2 < \infty$$

and

$$\sup_{t\ge 0}\mathbb{E}\|u^{(n)}(t) - u^*(t)\|_\rho^2 \to 0$$

Proof. For any $n \ge 1$ we have

$$\begin{aligned} &\mathbb{E}\|u^{(n+1)}(t) - u^{(n)}(t)\|_\rho^2\\ \le\ & 2\int_{\mathbb{R}^d}\mathbb{E}\left|\int_0^t S(t-s)[f(\cdot,u^{(n)}(s)) - f(\cdot,u^{(n-1)}(s))]ds\right|^2\rho(x)dx\\ &+2\int_{\mathbb{R}^d}\mathbb{E}\left|\int_0^t S(t-s)[\sigma(\cdot,u^{(n)}(s)) - \sigma(\cdot,u^{(n-1)}(s))]dW(s)\right|^2\rho(x)dx\\ :=\ & 2I_1 + 2I_2 \end{aligned}$$

We now estimate the above two terms.
Estimation of I_1.

$$\begin{aligned} I_1 &\le \int_{\mathbb{R}^d}\mathbb{E}\left(\int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)L\varphi(y)|u^{(n)}(s,y) - u^{(n-1)}(s,y)|dyds\right)^2\rho(x)dx\\ &\le 2\int_{\mathbb{R}^d}\mathbb{E}\left(\int_0^{t-1}\cdots\right)^2 + 2\int_{\mathbb{R}^d}\mathbb{E}\left(\int_{t-1}^t\cdots\right)^2\\ &:= 2I_{11} + 2I_{12} \end{aligned}$$

Proceeding as in (48), we have

$$
\begin{aligned}
I_{11} &\leq \int_{\mathbb{R}^d} \mathbb{E} \int_{t-1}^{t} \left(\int_{\mathbb{R}^d} G(t-s, x-y) L\varphi(y) |u^{(n)}(s,y) - u^{(n-1)}(s,y)| dy \right)^2 ds \rho(x) dx \\
&\leq \|\rho\|_{L^1(\mathbb{R}^d)} \mathbb{E} \int_{t-1}^{t} \left(\int_{\mathbb{R}^d} L^2 \frac{\varphi(y)^2}{\rho(y)} |u^{(n)}(s,y) - u^{(n-1)}(s,y)|^2 \rho(y) dy \right) ds \\
&\leq L^2 \|\rho\|_{L^1(\mathbb{R}^d)} \left\| \frac{\varphi(y)}{\sqrt{\rho(y)}} \right\|^2_{L^\infty(\mathbb{R}^d)} \sup_{t\geq 0} \mathbb{E} \|u^{(n)}(t) - u^{(n-1)}(t)\|^2_\rho.
\end{aligned}
$$

Similarly, using the same reasoning as in (49), we have

$$
\begin{aligned}
I_{12} &\leq \int_{\mathbb{R}^d} \mathbb{E} \left| \int_0^{t-1} \int_{\mathbb{R}^d} \frac{L\varphi(y)|u^{(n)}(s,y) - u^{(n-1)}(s,y)|}{(4\pi(t-s))^{d/2}} dy ds \right|^2 \rho(x) dx \\
&\leq L^2 \frac{1}{(4\pi)^d} \left(\frac{2}{d-2} \right)^2 \|\rho\|_{L^1(\mathbb{R}^d)} \left\| \frac{\varphi(y)}{\sqrt{\rho(y)}} \right\|^2_{L^2(\mathbb{R}^d)} \sup_{t\geq 0} \mathbb{E} \|u^{(n)}(t) - u^{(n-1)}(t)\|^2_\rho
\end{aligned}
$$

Estimation of I_2.

$$
I_2 \leq \sum_k a_k \int_{\mathbb{R}^d} \int_0^t \mathbb{E} \left| \int_{\mathbb{R}^d} G(t-s, x-y) [\sigma(y, u^{(n)}(y)) - \sigma(y, u^{(n-1)}(y))] e_k(y) dy \right|^2 ds \rho(x) dx \tag{55}
$$

The k-th term in (55) may be estimated as

$$
\begin{aligned}
& a_k \int_{\mathbb{R}^d} \int_0^t \mathbb{E} \left| \int_{\mathbb{R}^d} G(t-s, x-y) [\sigma(y, u^{(n)}(s,y)) - \sigma(y, u^{(n-1)}(s,y))] e_k(y) dy \right|^2 ds \rho(x) dx \\
= \; & a_k \int_0^{t-1} \cdots ds + a_k \int_{t-1}^{t} \cdots ds \\
:= \; & a_k I_{21} + a_k I_{22}
\end{aligned}
$$

We estimate I_{21} in a similar manner as it is done in (51):

$$
\begin{aligned}
I_{21} &\leq \int_{t-1}^{t} \int_{\mathbb{R}^d} \rho dx \; \mathbb{E} \int_{\mathbb{R}^d} G(t-s, x-y) L^2 \frac{\varphi(y)^2}{\rho(y)} |u^{(n)}(s,y) - u^{(n-1)}(s,y)|^2 \rho(y) |e_k|^2 dy ds \\
&\leq \|\rho\|_{L^1(\mathbb{R}^d)} L^2 \left\| \frac{\varphi}{\sqrt{\rho}} \right\|^2_{L^\infty(\mathbb{R}^d)} \sup_{t\geq 0} \mathbb{E} \|u^{(n)}(t) - u^{(n-1)}(t)\|^2_\rho
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
I_{22} &\leq \int_0^{t-1} \int_{\mathbb{R}^d} \mathbb{E} \left| G(t-s, x-y) L\varphi(y) |u^{(n)}(s,y) - u^{(n-1)}(s,y)| |e_k(y)| dy \right|^2 \rho(x) dx ds \\
&\leq \|\rho\|_{L^1(\mathbb{R}^d)} L^2 C_d \left\| \frac{\varphi}{\sqrt{\rho}} \right\|^2_{L^\infty(\mathbb{R}^d)} \sup_{t\geq 0} \mathbb{E} \|u^{(n)}(t) - u^{(n-1)}(t)\|^2_\rho.
\end{aligned}
$$

Finally, summarizing the aforementioned estimates, we have

$$
\begin{aligned}
\mathbb{E} \|u^{(n+1)}(t) - u^{(n)}(t)\|^2_\rho &\leq 2I_1 + 2I_2 \leq 4I_{11} + 4I_{12} + 2\sum_k a_k I_{21} + 2\sum_k a_k I_{22} \\
&\leq A_4 \sup_{t\geq 0} \mathbb{E} \|u^{(n)}(t) - u^{(n-1)}(t)\|^2_\rho,
\end{aligned}
$$

where

$$A_4 := 2L^2\|\rho\|_{L^1(\mathbb{R}^d)}\left((2+aC_d+a)\left\|\frac{\varphi}{\sqrt{\rho}}\right\|^2_{L^\infty(\mathbb{R}^d)} + 2\frac{1}{(4\pi)^d}\left(\frac{2}{d-2}\right)^2\left\|\frac{\varphi(y)}{\sqrt{\rho(y)}}\right\|^2_{L^2(\mathbb{R}^d)}\right). \tag{56}$$

Note that $A_4 = \frac{1}{3}A_3 < 1$ where $A_3 < 1$ is defined in (54). It is straightforward to verify that $\{u^{(n)}\}$ in this case is a Cauchy sequence, and thus the statement of the Lemma follows. $\square$

In order to complete the proof of Theorem 2, it remains to show that $u^*(t,x)$ is a mild solution of (1). Using the same reasoning as above,

$$\mathbb{E}\left\|\int_0^t S(t-s)[f(\cdot,u^{(n)}(s,\cdot)) - f(\cdot,u^*(s,\cdot))]ds\right\|^2_\rho \le C\sup_{t\ge 0}\|u^{(n)}(t) - u^*(t)\|^2_\rho \to 0, n\to\infty$$

and

$$\mathbb{E}\left\|\int_0^t S(t-s)[\sigma(\cdot,u^{(n)}(s,\cdot)) - \sigma(\cdot,u^*(s,\cdot))]dW(s)\right\|^2_\rho \le C\sup_{t\ge 0}\|u^{(n)}(t) - u^*(t)\|^2_\rho \to 0, n\to\infty.$$

Hence u^* is a solution of

$$u^*(t) = \int_0^t S(t-s)f(\cdot,u^*(s,\cdot))ds + \int_0^t S(t-s)\sigma(\cdot,u^*(s,\cdot))dW(s)$$

with the initial condition $u^*(0) = u_0(x)$ and it satisfies

$$\sup_{t\ge 0}\mathbb{E}\|u^*(t)\|^2_\rho < \infty.$$

This completes the proof of Theorem 2.

4 Proof of Theorem 3

Throughout the proof of Theorem 3 we will use the notation: for any u and v in H, $u\otimes v$ is a linear operator on H defined as

$$u\otimes v[h] := u\langle v,h\rangle,\ h\in H.$$

where $\langle\cdot,\cdot\rangle$ denotes the dot product in H. Note that $W_b(t)$, defined by (15), has a covariance operator

$$Q := \sum_{i=1}^\infty b_i\otimes b_i.$$

Indeed,

- Q is a bounded linear operator on H;
- Q is a nuclear operator with the trace

$$TrQ := \sum_{j=1}^\infty \langle Qe_j, e_j\rangle = \sum_{i=1}^\infty \|b_i\|^2 < \infty. \tag{57}$$

- Q is the covariance operator for W_b: for any $h \in H$

$$CovW_b[h] := \mathbb{E}(W_b \otimes W_b)[h] = tQ[h].$$

By Theorem 11.7 [8], the invariant measure for (14) exists if and only if

$$\sup_{t \geq 0} Tr(Q_t) < \infty, \text{ where } Q_t = \sum_{i=1}^{\infty} \int_0^t [S(r)b_i \otimes S(r)b_i] dr.$$

Hence the desired condition is equivalent to

$$\sum_{i=1}^{\infty} \int_0^{\infty} \|S(t)b_i\|^2 dt < \infty. \tag{58}$$

The following Lemma shows the the semigroup $S(t)$ has the exponential contraction property on

$$U_A := \overline{\text{Span } \{\varphi_k \text{ s.t. } \text{Re}(\lambda_k) < 0\}},$$

which is essential in establishing Theorem 3.

Lemma 5. *Let $\{\lambda_k, k \geq 1\}$ and $\{\varphi_k, k \geq 1\}$ be the corresponding eigenvalues and eigenbasis of A. Then $\|S(t)u_0\|_H \leq e^{-\mu t}\|u_0\|_H$ for any $u_0 \in U_A$, where $-\mu = \max\{2Re\lambda_k\}$ over the set of $\{k : Re\lambda_k < 0\}$*

Proof. Write

$$u_0 := \sum_{k=1}^{\infty} c_k^0 \varphi_k$$

with $c_k^0 = 0$ if $\text{Re}\lambda_k < 0$. Let $u(t)$ be the solution of

$$\begin{cases} u_t(t) = Au(t); \\ u(0) = u_0 \end{cases}$$

Decomposing $u(t)$ in the eigenbasis $\{\varphi_k\}$ we have

$$u(t) = \sum_{k=1}^{\infty} c_k(t) \varphi_k$$

thus

$$\sum_{k=1}^{\infty} c_k^{'}(t)\varphi_k = \sum_{k=1}^{\infty} c_k(t) A\varphi_k = \sum_{k=1}^{\infty} \lambda_k c_k(t)\varphi_k$$

and consequently

$$c_k(t) = c_k^0 e^{\lambda_k t}.$$

Finally

$$\|u(t)\|_H^2 = \sum_{k=1}^{\infty} |c_k(t)|^2 = \sum_{k:\text{Re}\lambda_k<0} (c_k^0)^2 e^{2\text{Re}\lambda_k t} \leq e^{-\mu t} \sum_{k=1}^{\infty} (c_k^0)^2 = e^{-\mu t}\|u_0\|_H.$$

□

It remains to show that the stable subspace U_A is an invariant subspace for the solutions of (14). In other words, we show that if $u_0 := u(0) \in U_A$ then $u(t) \in U_A$ for all $t \geq 0$. Recall that

$$u(t) = S(t)u_0 + \int_0^t S(t-s)dW_b(s)$$

where $W_b(t) = \sum_k b_k \beta_k(t)$ and $b_k \in U_A$. Then $Q : H \to U_A$, since $\forall h \in H$ $Qh = \sum_i b_i \langle b_i, h\rangle \in U_A$. Furthermore, since Q is self adjoint and positive definite (due to (57)) we have $\{\mu_k > 0\}$ and $\{\psi_k \in U_A\}$ such that

$$Q\psi_k = \mu_k \psi_k.$$

Then $W_b(t) = \sum_i \sqrt{\mu_i}\psi_i \beta_i(t)$, and

$$\int_0^t S(t-s)dW_b(s) = \sum_i \sqrt{\mu_i} \int_0^t S(t-s)\psi_i d\beta_i(s).$$

On the other hand, by definition of U_A, φ_k are the basis functions in U_A, so we can write

$$\psi_i = \sum_k c^0_{k,i}\varphi_k.$$

Analogously to the proof of Lemma 5, we have

$$S(t-s)\psi_i = \sum_k c^0_{k,i} e^{\lambda_k(t-s)}\varphi_k \in U_A.$$

hence

$$\int_0^t S(t-s)dW_b(s) \in U_A.$$

Finally, since $u_0 \in U_A$, we have $S(t)u_0 \in U_A$, implying that U_A is an invariant subspace. Moreover, using the contraction property of $S(t)$ on U_A, we can show the existence and uniqueness of the invariant measure on U_A the same way as it is done in the proof of [19, Theorem 5]. Finally, since $|\text{Re}\lambda_k| > \varepsilon > 0$, the operator A has the exponential dichotomy property ([13, Sec. 7.6]). Thus for all $b \notin U_A$, we have $\|S(t)b\| \to \infty$, $t \to \infty$.

Example 1. *Let $Au := \frac{1}{\rho} div(\rho \nabla u)$, where $u \in L^2_\rho(\mathbb{R}^d)$ and $\rho(x) \in L^1(\mathbb{R}^d) \cap C^2(\mathbb{R}^d)$ satisfies*

$$\frac{1}{2}\frac{\Delta\rho}{\rho} - \frac{|\nabla\rho|^2}{\rho^2} \to +\infty, |x| \to \infty \tag{59}$$

Then A has a discrete spectrum, and U_A is the span of all but finitely many eigenfunctions of A.

Let $Bu := \Delta u - q(x)u$, where $q(x)$ is a real valued function on $\mathbb{R}^d$. We will make use of the following result, which characterizes the spectrum of B.

Theorem 7. *([23, Sec. 16.4]) Let $q(x) \to +\infty$ as $|x| \to \infty$. Then the spectrum of B in $L^2(\mathbb{R}^d)$ is discrete. In addition, if $q(x) \geq q_0$, the eigenvalues $\{\lambda_n, n \geq 1\}$ are contained in the interval $(-\infty, -q_0]$, and $\lambda_n \to -\infty$ as $n \to \infty$.*

Consider now the following spectral problem: find $\lambda \in \mathbb{R}$ and $u \in L^2_\rho(\mathbb{R}^d)$ which satisfy

$$\frac{1}{\rho}\text{div}(\rho\nabla u) + \lambda u = 0, x \in \mathbb{R}^d. \tag{60}$$

After the substitution $u = v\rho^{-1/2}$, (60) reads as

$$\Delta v + \left(\frac{|\nabla\rho|^2}{2\rho^2} - \frac{\Delta\rho}{4\rho}\right) v + \lambda v = 0, x \in \mathbb{R}^d. \tag{61}$$

By Theorem 7, if (59) holds, the problem (61) has discrete spectrum $\{\lambda_k\}$ with corresponding eigenfunctions $\{v_k(x) \in L^2(\mathbb{R}^d)\}$, or, equivalently, $\{\lambda_k\}$ and $\{u_k = v_k\rho^{-1/2}\} \in L^2_\rho(\mathbb{R}^d)$ are the eigenvalues and eigenfunctions of (60).

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