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CONVERGENCE OF SPACE-TIME DISCRETE THRESHOLD DYNAMICS TO ANISOTROPIC MOTION BY MEAN CURVATURE

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Abstract. We analyze the continuum limit of a thresholding algorithm for motion by mean curvature of one dimensional interfaces in various space-time discrete regimes. The algorithm can be viewed as a time-splitting scheme for the Allen-Cahn equation which is a typical model for the motion of materials phase boundaries. Our results extend the existing statements which are applicable mostly in semi-discrete (continuous in space and discrete in time) settings. The motivations of this work are twofolds: to investigate the interaction between multiple small parameters in nonlinear singularly perturbed problems, and to understand the anisotropy in curvature for interfaces in spatially discrete environments. In the current work, the small parameters are the spatial and temporal discretization step sizes: $\triangle x = h$ and $\triangle t = \tau$. We have identified the limiting description of the interfacial velocity in the (i) sub-critical ($h \ll \tau$), (ii) critical ($h = O(\tau)$), and (iii) super-critical ($h \gg \tau$) regimes. The first case gives the classical isotropic motion by mean curvature, while the second produces intricate pinning and de-pinning phenomena, and anisotropy in the velocity function of the interface. The last case produces no motion (complete pinning).

1. **Introduction and Main Results.** The current paper addresses convergence issues related to a thresholding scheme for motion by mean curvature. The key is the analysis of the algorithm in the *space-time discrete* setting in which there are *two small parameters* - the step sizes in the spatial and temporal directions. The ultimate results depend on the relative sizes of these parameters.

The analysis of motion by mean curvature (in which the normal velocity of a moving manifold is given by its mean curvature) is an active area. Not only it is interesting in geometry in its own right, it also finds many applications in materials science and image processing. It is a prototype of a gradient flow with respect to the area functional. Due to the possibility of singularity formation and topological changes of the evolving surface, elaborate approaches need to be used. These include

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(i) varifold formulation, (ii) the viscosity solutions, and (iii) singularly perturbed reaction diffusion equations.

The thresholding scheme is a particularly simple algorithm to capture the key feature of (iii). It is essentially a time splitting scheme. The first step is linear diffusion while the second step is thresholding to mimick the fast reaction due to the nonlinear term. This is heuristically proposed in [4] and rigorously analyzed in [8, 2] in the continuous space and discrete time setting. See also the work [6] for an analysis of an reaction diffusion equation in which both space and time variables are discrete. However, so far all the rigorous results essentially works in the case when the interfacial structure is well-resolved. We call this the "sub-critical" regime. When this is not the case, intricate pinning and depinning of the interface can happen. This is analogous to a gradient flow in a *highly wiggling* or *oscillatory energy landscape*. The motion also demonstrates anisotropy of the normal velocity. The motivation of the current paper is to capture these phenomena quantitatively and relate them to the underlying small parameters in the algorithm.

The most relevant reaction diffusion equation for motion by mean curvature is the following Allen-Cahn equation:

$$\frac{\partial u}{\partial t} = \triangle u - \frac{1}{\epsilon^2} W'(u). \tag{1}$$

In the above W is the double well potential $W(u) = (1-u^2)^2$ and ϵ is a small parameter. The qualitative behavior of the solution is that the underlying ambient space is quickly partitioned into two domains on which u takes on the values 1 and -1 which are the minima of W. The function u also makes a smooth but rapid transition with thickness $O(\epsilon)$ between the two domains. The key is then to understand in the limit of $\epsilon \longrightarrow 0$ the dynamics of this transition layer. It is proved in various settings that the limiting motion is motion by mean curvature [5, 12, 6, 10, 9].

As the thresholding scheme is very simple to implement and describe, we embark on its analysis demonstrating the interplay between two small parameters. The scheme is a time splitting approach to solve (1) (in the regime $\epsilon \ll 1$). Given an initial shape Ω_0, and its boundary $\Gamma_0 = \partial\Omega_0$ which is often called the *interface*, a sequence of functions $\{u_k\}_{k\geq 0}$ is constructed in the following manner: for $k = 0$, we define

$$u_0(x) = \mathbf{1}_{\Omega_0}(x) - \mathbf{1}_{\Omega_0^c} = \begin{cases} 1 & \text{for} \quad x \in \Omega_0, \\ -1 & \text{for} \quad x \in \Omega_0^c, \end{cases} \tag{2}$$

then the following two steps are alternately performed (for $k = 0, 1, 2, \ldots$),

- **diffusion step:**

$$\begin{aligned} \frac{\partial v}{\partial t} &= \triangle v, \text{ for } 0 < t < \tau, \\ v(x, 0) &= u_k(x); \end{aligned} \tag{3}$$

- **thresholding step:**

$$u_{k+1}(x) = \operatorname{sign}(v(x, \tau)). \tag{4}$$

Note that the second step above is to mimick the fast reaction term which drives u to 1 or -1, the minima of W. The solution of the problem is described by the sequence of subsets $\{x : u_k(x) = 1\}$. Precisely, we define the time dependent set and interface as

$$\Omega^\tau(t) = \Big\{x : u_k(x) \geq 0, \text{ for } k\tau \leq t < (k+1)\tau\Big\} \text{ and } \Gamma^\tau(t) = \partial\Omega^\tau(t).$$

Then as $\tau \longrightarrow 0$, $\Omega^\tau(t)$ (or $\Gamma^\tau(t)$) has been shown to converge to motion by mean curvature in the viscosity setting [8, 2].

Now we describe some notations and the algorithm for the space-time discrete version of the above thresholding scheme (3) and (4). Let $\Omega \subset \mathbb{R}^2$ be a bounded, smooth domain, and $\Gamma = \partial\Omega$ be its boundary. Let $h > 0$ be the spatial discretization step size. Define

$$\Omega^h := \Big\{(m,n) \in \mathbb{Z}^2 : \mathrm{dist}\big[(nh, mh), \Omega\big] \leq h\Big\} \tag{5}$$

which are the indices of the lattice points within distance h of Ω. Let again $\tau > 0$ be the size of the time step. Given an initial set Ω_0 and its lattice representation Ω_0^h, the discrete thresholding scheme produces $\{u_k^{m,n}\}_{k\geq 0,(m,n)\in\mathbb{Z}^2}$ as follows. Let

$$u_0^{m,n} = \mathbf{1}_{\Omega_0^h}(m,n) - \mathbf{1}_{(\Omega_0^h)^c}(m,n) = \begin{cases} 1 & \text{for} \quad (m,n) \in \Omega_0^h, \\ -1 & \text{for} \quad (m,n) \in (\Omega_0^h)^c. \end{cases} \tag{6}$$

For simplicity, we will also use u to denote the discrete function $\big\{u^{m,n} : (m,n) \in \mathbb{Z}^2\big\}$. Similar to the continuous space case, the following two steps are alternately performed (for $k = 0, 1, 2, \ldots$):

- **solution of semi-discrete heat equation**: for $(m,n) \in \mathbb{Z}^2$ and $0 < t \leq \tau$,

$$\begin{aligned} \frac{d}{dt} w^{m,n}(t) &= \frac{1}{h^2}[w^{m+1,n}(t) + w^{m-1,n}(t) + w^{m,n+1}(t) + w^{m,n-1}(t) - 4w^{m,n}(t)], \\ w^{m,n}(0) &= u_k^{m,n}. \end{aligned} \tag{7}$$

- **thresholding step**: for $n, m \in \mathbb{Z}$,

$$u_{k+1}^{m,n} := \mathrm{sign}\big[w^{m,n}(\tau)\big] = \mathrm{sign}\Big(\big(S^h(\tau)[u_k]\big)^{m,n}\Big), \tag{8}$$

  where the $S^h(\cdot)$ is the solution operator of (7), i.e. $w(t) = S^h(t)[u_k]$.

Note that the above space time discrete scheme involves two small parameter $\tau > 0$ and $h > 0$. We assume that τ and h are related through

$$h = C\tau^\gamma, \quad \text{where } \gamma > 0 \text{ and } C > 0. \tag{9}$$

The following three cases are considered:

Case 1 (sub-critical): $\gamma > 1$, i.e. $h \ll \tau$;
Case 2 (critical): $\gamma = 1$, i.e. $\tau = \mu h$ for some constant μ;
Case 3 (super-critical): $\gamma < 1$, i.e. $h \gg \tau$.

Roughly speaking, the main result for Case 1 is that the level sets of a discrete heat equation move according to the motion by mean curvature. This gives the same result as [2]. Case 2 gives a version of anisotropic curvature dependent motion which demonstrates pinning of the interface when the curvature of the interface is too small. In Case 3, there is no motion at all, i.e. the interface is completely pinned.

It is worth noting that the critical regime is given by $h = O(\tau)$ and not $h = O(\sqrt{\tau})$ as one would originally guess based on the self-similarity property of heat equation. The reason is that outside our domain of interest $\Omega(t)^\tau$, the discrete heat kernel (defined in Section 2) is well-controlled (exponentially small) if $\frac{\tau}{h}$ is bounded (see Lemma 1 for details).

1.1. **Curvature Dependent Motion and Viscosity Solutions.** As mentioned earlier, singularities and topological changes can occur for motion by mean curvature. Different mathematical approaches are invented to define the solution for all time. Due to the presence of maximum principle, we find the viscosity solution to be the most suitable and convenient approach for our problem. We spend a moment to briefly describe this method which can produce a "unique" global in time solution.

The essential idea is to represent the moving interface Γ_t as the zero level set of a function $u(x,t)$:

$$\Gamma_t = \{x : u(x,t) = 0\}. \tag{10}$$

The function u is thus often called the *level set function*. It solves an appropriate partial differential equations related to the motion law of Γ_t. The main result in this approach is that in the space of uniformly continuous functions, there is a unique solution u and the set Γ_t does not depend on the initial data $u(\cdot,0)$ as long as it correctly captures the interior and exterior domains of Γ_0. On the other hand, this set-up does not a priori ensure that Γ_t corresponds to a manifold in any geometric sense. This can happen if Γ_t has positive n-dimensional Lebesgue measure in which case Γ_t is said to *fatten* or develop *non-empty interior*. It also means that the solution of the geometric evolution can be non-unique as $\partial\{x : u(x,t) > 0\} \neq \partial\{x : u(x,t) < 0\}$. Conditions preventing this from happening are discussed in [3]. On the other hand, a definition of *generalized front* is used so that a "unique solution" can be defined. This approach defines the interface as the following triplet of objects:

$$\Gamma_t = \{x : u(x,t) = 0\}, \quad D_t^+ = \{x : u(x,t) > 0\}, \quad D_t^- = \{x : u(x,t) < 0\}. \tag{11}$$

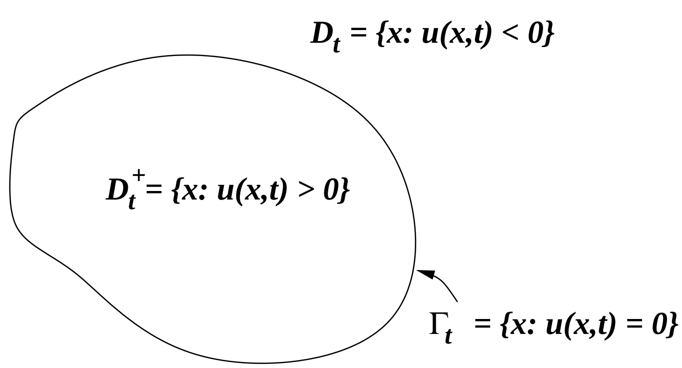

Figure 1.

(See Fig. 1.) Then D_t^+ (D_t^-) is called the *interior (exterior)* of the front Γ_t. It is shown that the map:

$$E_t : (\Gamma_0, D_0^+, D_0^-) \longrightarrow (\Gamma_t, D_t^+, D_t^-) \tag{12}$$

is well defined. We refer to [14] for more detailed description.

Next we describe the equation for general curvature dependent front propagation. We follow the exposition in [11]. Given an interface (hypersurface) Γ in $\mathbb{R}^n$. We consider its motion described by a normal velocity function V of the following form:

$$V = V(D\nu, \nu) \tag{13}$$

where ν is the unit (outward) normal of Γ. In the above, D is the gradient operator on $\mathbb{R}^n$ and $\nu : \mathbb{R}^n \longrightarrow \mathbb{S}^{n-1}$ is defined to be the unit normal vector function of any foliation of hypersurfaces containing Γ. In this way, $D\nu$ is a symmetric matrix and its value does not depend on the choice of the foliation. If $V = \mathrm{div}(D\nu)$, then the motion is called *(isotropic) motion by mean curvature*. In general, the motion law (13) is sometimes called *anisotropic curvature motion*. If we want to represent the moving interface by (10) or (11), then the function u needs to solve the following partial differential equation:

$$u_t + F(D^2u, Du) = 0 \tag{14}$$

where the function F is related to V in the following way:

$$F(X,p) = -|p|\,V\left(-|p|^{-1}(I-\widehat{p}\times\widehat{p})\,X\,(I-\widehat{p}\times\widehat{p}),-\widehat{p}\right) \tag{15}$$

where $\widehat{p} = |p|^{-1}p$. In order to apply the viscosity solution approach, the following monotonicity condition for V is crucial:

$$V \text{ is nondecreasing, i.e. for all } X \leq Y \text{ and } p, \text{ then } V(X,p) \geq V(Y,p). \tag{16}$$

The above property is translated to the function F as

$$\text{for all } X \leq Y \text{ and } p, \text{ then } F(X,p) \geq F(Y,p). \tag{17}$$

With the above set-up, then it can be shown that equation (14) is well-posed in the space of *uniformly continuous functions*. Given an initial manifold $\Gamma_0 = \partial\Omega_0$, a usual choice of the initial data $u(x,0) = u_0(x)$ for (14) is given by the sign distance function to Γ_0:

$$d_0(x) = \mathrm{sdist}(x,\Gamma_0) = \begin{cases} \mathrm{dist}(x,\Gamma_0), & x \in \Omega_0, \\ -\mathrm{dist}(x,\Gamma_0), & x \in \Omega_0^c, \end{cases} \tag{18}$$

The map E_t in (12) is independent of any uniformly continuous initial data as long as it has the same sign as d_0: $u_0 > 0$ on Ω_0 and $u_0 < 0$ on Ω_0^c.

The definition of viscosity solution of (14) is given in Section 1.2. Of the *consistency* and *stability* conditions generally required for most of the proofs for convergence of numerical schemes, for the current algorithm, the latter is quite easy to satisfy by means of maximum principle. The crux of the matter is the former condition which is the key result of our paper. Once we have this, then we can more or less quote some general convergence result.

1.2. **Main result: Sub-Critical Case.** Our most complete result is the sub-critical case for which we can prove convergence to motion by mean curvature in the viscosity sense. In this case, the velocity function is given by $V(D\nu,\nu) = \mathrm{Div}(D\nu)$. Then (14) becomes

$$\frac{\partial u}{\partial t} = |\nabla u|\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \Delta u - \frac{(D^2uDu|Du)}{|Du|^2}. \tag{19}$$

For convenience, we give the definition of viscosity solution specifically for this case.

Definition 1. *A locally bounded upper semicontinuous (usc) function (respectively, lower semicontinuous (lsc)) function u is a viscosity subsolution (respectively, supersolution) of (19), if for all $\phi \in C^2(\mathbb{R}^N\times(0,+\infty))$, and if $(x,t)\in\mathbb{R}^N\times(0,+\infty)$ is a local maximum point of $u-\phi$, then one has*

$$\frac{\partial\phi}{\partial t}(x,t) - \left(\triangle\phi - \frac{(D^2\phi D\phi|D\phi)}{|D\phi|^2}\right)(x,t) \leq 0, \quad \textit{if } D\phi(x,t)\neq 0, \tag{20}$$

and

$$\frac{\partial\phi}{\partial t}(x,t) - \triangle\phi(x,t) + \lambda_{min}(D^2\phi(x,t)) \leq 0, \quad \textit{if } D\phi(x,t) = 0, \tag{21}$$

where $\lambda_{min}(D^2\phi(x,t))$ is the least eigenvalue of $D^2\phi(x,t)$. (Respectively, if for all $\phi \in C^2(\mathbb{R}^N \times (0,+\infty))$, and if $(x,t) \in \mathbb{R}^N \times (0,+\infty)$ is a local minimum point of $u-\phi$, then one has

$$\frac{\partial \phi}{\partial t}(x,t) - \left(\triangle \phi - \frac{\left(D^2\phi D\phi | D\phi\right)}{|D\phi|^2} \right)(x,t) \geq 0, \quad \textit{if } D\phi(x,t) \neq 0, \tag{22}$$

and

$$\frac{\partial \phi}{\partial t}(x,t) - \triangle\phi(x,t) + \lambda_{max}(D^2\phi(x,t)) \geq 0, \quad \textit{if } D\phi(x,t) = 0, \tag{23}$$

where $\lambda_{max}(D^2\phi(x,t))$ is the maximum (or principle) eigenvalue of $D^2\phi(x,t)$.)

On the other hand, a simpler characterization can be given.

Proposition 1. [2, Prop. 2.2] *A locally bounded upper semicontinuous (usc) function u is a viscosity subsolution (respectively supersolution) of (19) if and only if it satisfies* (20) *and*

$$\frac{\partial \varphi}{\partial t}(x,t) \leq 0 \quad \textit{if } D\varphi(x,t) = 0 \quad \textit{and } D^2\varphi(x,t) = 0, \tag{21'}$$

respectively, (22) *and*

$$\frac{\partial \varphi}{\partial t}(x,t) \geq 0 \quad \textit{if } D\varphi(x,t) = 0 \quad \textit{and } D^2\varphi(x,t) = 0. \tag{23'}$$

The consistency proof of the thresholding scheme relies on the following result.

Proposition 2. [2, Prop 4.1] *If $(\phi_h)_h$ is a sequence of smooth functions bounded in $C^{2,1}$ and converging locally in $C^{2,1}$ to a function ϕ and (x_h, t_h) is a sequence of points converging to $(x,t) \in \mathbb{R}^N \times (0,\infty)$ such that $\phi_h(x_h,t_h) = 0$, then if $D\phi(x,t) \neq 0$,*

$$\begin{aligned} \liminf_h \frac{1}{h^{\frac{1}{2}}} \left(\frac{1}{2} - \frac{1}{(4\pi h)^{\frac{N}{2}}} \int_{\{\phi_h(\cdot, t_h - h) \geq 0\}} \exp\left(- \frac{|x_h - y|^2}{4h} \right) dy \right) \\ \geq \frac{1}{2\sqrt{\pi}\,|D\phi(x,t)|} \left(\frac{\partial\phi}{\partial t} - \triangle\phi + \frac{\left(D^2\phi D\phi | D\phi\right)}{|D\phi|^2} \right)(x,t). \end{aligned} \tag{24}$$

Moreoever, if $D\phi(x,t) = 0$ and $D^2\phi(x,t) = 0$ and if the inequality

$$\frac{1}{2} - \frac{1}{(4\pi h)^{\frac{N}{2}}} \int_{\{\phi_h(\cdot, t_h - h) \geq 0\}} \exp\left(- \frac{|x_h - y|^2}{4h} \right) dy \leq 0 \tag{25}$$

holds for a sequence of h converging to 0, then

$$\frac{\partial\phi}{\partial t}(x,t) \leq 0. \tag{26}$$

With the above preparation, we are ready to present our result. We start by introducing

$$\underline{u}(x,t) = \liminf_{k\tau \to t; (mh,nh) \to x} u_k^{m,n} \tag{27}$$

and

$$\bar{u}(x,t) = \limsup_{k\tau \to t; (mh,nh) \to x} u_k^{m,n}. \tag{28}$$

Theorem 1 (Sub-critical case)**.** *Assume $\gamma > 1$. Then the functions $\bar{u}(x,t)$ and $\underline{u}(x,t)$ are viscosity subsolutions and supersolutions of (19), respectively.*

A few remarks about the consequence of the above statement.

1. Let u be the solution of (19) with the initial data u_0 given for example by (18). Then the sets Γ_t, D_t^+, and D_t^- produced by the map E_t (12) is well-defined. As shown in [2, Thm. 1.2], the above statement implies that
$$\underline{u}(x,t) = 1 \text{ in } D_t^+ \text{ and } \bar{u}(x,t) = -1 \text{ in } D_t^-. \tag{29}$$
In other words,
$$\partial\left\{x : \underline{u}(x,t) \geq 1\right\}, \ \ \partial\left\{x : \bar{u}(x,t) \leq -1\right\} \subset \Gamma_t.$$
It is in this sense that we say the zero level set of u_k in the limit moves according to the motion by mean curvature. Note that in general we can only say the limiting interface is *contained in but might not equal to* Γ_t. See the next item of remark.
2. As we are dealing with *discontinuous* initial data and functions u_k, $k \geq 0$, so in general the limit (as $h, \tau \longrightarrow 0$) can be non-unique. In fact, we can infer using [2, Thm. 1.1, Cor. 1.3] that we have a unique limit if
$$\bigcup_{t>0} \Gamma_t \times \{t\} = \partial\left\{(x,t) : u(x,t) > 0\right\} = \partial\left\{(x,t) : u(x,t) < 0\right\}$$
i.e. with no fattening phenomena, and in which case the boundary of the zero level set for u_k converges to $\bigcup_{t>0} \Gamma_t \times \{t\}$ in the sense of Hausdorff distance.

1.3. **Main Results: Critical and Super-Critical Cases.** We next describe the results in the critical case, i.e. $\tau = \mu h$ for fixed $\mu > 0$. To concentrate on the key ideas of our approach, we assume that locally near the origin, the boundary of the initial set Ω_0 is represented by a graph. More specifically, we assume that for some $c_0 > 0$,
$$\Omega_0 \cap (-c_0, c_0) \times (-c_0, c_0) = \Big\{(x,y) : |x| \leq c_0 : -c_0 < y \leq f(x) < c_0\Big\} \tag{30}$$
where $f(x)$ is a C^2, even-function satisfying
$$f(0) = f'(0) = 0, \ \ f''(0) = -\kappa \leq 0. \tag{31}$$
The quantity κ represents the curvature of $\partial\Omega_0$ at $(0,0)$. (See Fig. 2.)

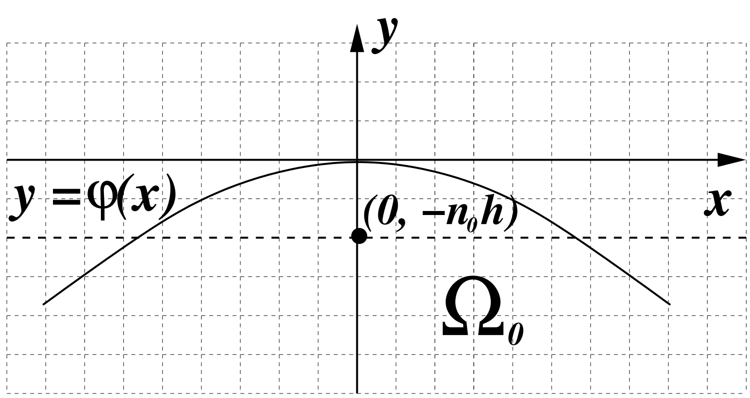

Figure 2.

A note about convention. For simplicity of presentation of the results in this section, we consider an equivalent initialization $\tilde{u}_0^{m,n} := \frac{1}{2} + \frac{1}{2}u_0^{m,n}$ and the threshholding step $\tilde{u}_{k+1}^{m,n} := \frac{1}{2} + \frac{1}{2}u_k^{m,n}$, where $\tilde{u}_0^{m,n}$ and $\tilde{u}_{k+1}^{m,n}$ are given by (6) and (8) respectively. In other words, we replace the "-1 & 1" thresholding scheme with a "0 & 1" one. This new scheme will still be denoted with $u_0^{m,n}$ and $u_{k+1}^{m,n}$.

With the above notation, let w be the function obtained from (7) for $k = 0$ and $n_0 \in \mathbb{Z}$ be such that

$$w^{0,-n_0}(\tau) \le \frac{1}{2} \quad \text{and} \quad w^{0,-n_0-1}(\tau) > \frac{1}{2}. \tag{32}$$

We call n_0 the "discrete normal velocity" of the boundary point $(0,0) \in \partial\Omega_0$. The true physical normal velocity V is defined as:

$$V = \lim_{h,\tau\to 0} \frac{n_0 h}{\tau} = \lim_{h,\tau\to 0} \frac{n_0}{\mu}. \tag{33}$$

We will show that V exists and is still given by a function of the curvature κ. Precisely,

Theorem 2 (Critical case)**.** *Assume $\kappa > 0$. For sufficiently small $\tau > 0$, $w^{0,-n}(\tau) \le \frac{1}{2}$ holds if and only if*

$$\sum_{k=1}^{n-1} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx + \frac{1}{2}\int_0^{\sqrt{\frac{2n}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx \le \frac{1}{2}\int_{\sqrt{\frac{2n}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx + \sum_{k=n+1}^{\infty} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx. \tag{34}$$

In particular, if $w^{0,-n_0}(\tau) = \frac{1}{2}$, then

$$\sum_{k=1}^{n_0-1} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx + \frac{1}{2}\int_0^{\sqrt{\frac{2n_0}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx = \frac{1}{2}\int_{\sqrt{\frac{2n_0}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx + \sum_{k=n_0+1}^{\infty} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx. \tag{35}$$

Note that the discrete velocity $n_0 = n_0(\mu,\kappa)$ is an implicit function of μ and κ, or more precisely $\mu\kappa$. It is also straightforward to see from (35) that if $\kappa_1 < \kappa_2$, we have $n_0(\mu,\kappa_1) \le n_0(\mu,\kappa_2)$. Furthermore, if κ is small enough, then $n_0 = 0$, i.e. the interface is *pinned*. This can be extended to the case $\kappa = 0$ giving $n_0 = 0$. The last two statements are formally stated in the following corollary which essentially covers the super-critical regime.

Corollary 1 (Super-critical case)**.** *Assume $\tau = \mu h$. If $\mu\kappa$ is sufficiently small, then $n_0 = 0$, i.e. the front does not move.*

From numerical calculation, we find that the smallness condition is quantified by $\mu\kappa \le 0.8218$.

We next show that the result in the critical case converges to that of the sub-critical case when $\mu \to \infty$:

Theorem 3. *Let n_0, μ and κ satisfy (32). Then*

$$\lim_{\mu\to\infty} \frac{n_0}{\mu} = \kappa,$$

i.e. we get the mean curvature motion as in the subcritical case.

Finally, we obtain an extension of Theorem 2 to reveal the anisotropic dependence on curvature. In particular, we want to calculate the normal velocity of a boundary point if the normal line at the point forms an angle with the coordinate axis. For concreteness, let the normal line at $(0,0) \in \partial\Omega_0$ forms an angle θ with the x-axis (measure in the counter-clockwise sense). Without loss of generality, we assume $0 < \theta \leq \frac{\pi}{4}$. We consider the case that $\tan\theta = \frac{p}{q}$ for some positive integers $p \leq q$.

We assume that locally in some neighborhood of $(0,0)$, $\partial\Omega_0$ is given by

$$\partial\Omega_0 = (x, g(x)), \quad \text{where } g(x) = \frac{p}{q}x - v_0 x^2 \text{ with } v_0 = \frac{\kappa(q^2+p^2)^{3/2}}{2q^3} \tag{36}$$

so that $\frac{g''}{(1+(g')^2)^{3/2}} = \kappa$ which gives the curvature of $\partial\Omega_0$ at $(0,0)$. In this setting the notion of normal motion has to be defined somewhat differently from the one in Theorem 2 since the line in the normal direction intersects the grid only at the points $(np, nq), n \in \mathbb{Z}$, and thus bypasses the grid points in between.

To quantify the motion in the normal direction, we use S to denote the following set of grid points confined in some strip:

$$S := \left\{(s,j) \in \mathbb{Z}^2 : 0 < s < q, \ -\infty < j < \frac{p}{q}s\right\}. \tag{37}$$

For every $(s,j) \in S$, let $d(s,j) := \frac{|jq - sp|}{\sqrt{p^2+q^2}}$ be the distance from (s,j) to the line $y = \frac{p}{q}x$. This is illustrated in Figure 3.

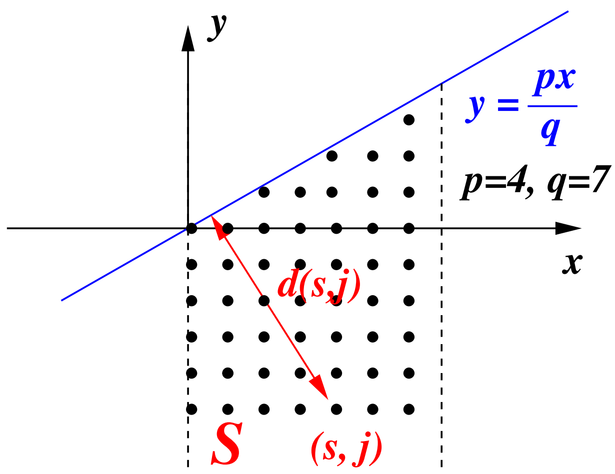

Figure 3

Next, we re-order the elements in the set S with respect to d as follows:

$$S := \{(s_1, j_1), (s_2, j_2), (s_3, j_3), ...\} \tag{38}$$

such that for $d_i := d(s_i, j_i)$, we have

$$0 < d_1 < d_2 < d_3 < ...$$

For example, if $p = 1$, the points in S will be ordered as

$$\Big\{(1,0),(2,0),...,(q-1,0),(0,-1),(1,-1),(2,-1),...,(q-1,-1),(0,-2),(1,-2),...\Big\}.$$

In this setting, we have the following result:

Theorem 4 (Anisotropic curvature motion, critical case)**.** *Given positive intergers p and q, and positive numbers κ and μ, and the set S be defined and ordered as in (37) and (38). Fix an integer $n_0 \geq 0$. Then, for sufficiently small t, $w_{s_{n_0}, j_{n_0}}(\tau) \leq \frac{1}{2}$ holds if and only if*

$$\frac{1}{2}\int_0^{\sqrt{\frac{2d_{n_0}}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx+\sum_{i=1}^{n_0-1}\int_0^{\sqrt{\frac{2d_i}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx \le \frac{1}{2}\int_{\sqrt{\frac{2d_{n_0}}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx+\sum_{i=n_0+1}^{\infty}\int_{\sqrt{\frac{2d_i}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx. \tag{39}$$

In particular, if $w_{s_{n_0},j_{n_0}}(\tau)=\frac{1}{2}$*, then*

$$\frac{1}{2}\int_0^{\sqrt{\frac{2d_{n_0}}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx+\sum_{i=1}^{n_0-1}\int_0^{\sqrt{\frac{2d_i}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx = \frac{1}{2}\int_{\sqrt{\frac{2d_{n_0}}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx+\sum_{i=n_0+1}^{\infty}\int_{\sqrt{\frac{2d_i}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx. \tag{40}$$

In this case, $d_{n_0}h$ *is the normal displacement at time* τ *and* $\frac{d_{n_0}}{\mu}$ *is the normal velocity.*

Here we make some remarks about our results.

1. As mentioned earlier, the key of the convergence proof in the current approximating scheme is the consistency statement. Our results in the critical case essentially gives an analytical (though implicit) expression for the displacement in one iteration step. In principle we can follow the approach in [13] and [11] to prove the convergence similar to the sub-critical case. However, the velocity function is required to be continuous. This is not the case here. To keep the paper within reasonable scope, we do not pursuit the full convergence result.
2. Our result about the anisotropy of the velocity function is only for rational angles. It is certainly interesting to extend the result to the irrational case. From the simulation in Section 5 (Figures 12 and 13), it seems that the angular dependence of the velocity function can be quite intricate. We will investigate further this issue in a future work.

As the proofs of our results rely heavily on the asymptotics of some discrete heat kernel, we first collect their key properties and investigate their connection to the continuum heat kernel.

For convenience, we will use the following common notations:

1. $A=O(B)$ means there exists a finite positive constant C such that $|A|\le C\,|B|$;
2. $A=o(B)$ or $A\ll B$ means $\lim\dfrac{|A|}{|B|}=0$;
3. $A\gg B$ means $\lim\dfrac{|A|}{|B|}=+\infty$.

In the above, the lim-operations will be specified or are self-explanatory when the notations are used.

2. **Properties of discrete heat kernels.**

2.1. **Derivation of discrete heat kernel and its elementary properties.**

We first consider a one-dimensional analog of (7),

$$\begin{cases} u_t^n(t)=\dfrac{1}{h^2}[u^{n+1}(t)+u^{n-1}(t)-2u^n(t)], & n\in\mathbb{Z},\ \ t\ge 0,\\ u^n(0)=u_0(nh), \end{cases} \tag{41}$$

where the initial data u_0 is an L^∞ function. The solution of the above is given by

$$u^m(t) = \sum_{k=-\infty}^{\infty} G_{m-k}\left(\frac{2t}{h^2}\right) u_0^k, \tag{42}$$

where

$$G_n(\alpha) := \frac{1}{2\pi}\int_{-\pi}^{\pi} cos(n\xi)e^{\alpha(cos\xi-1)}d\xi = e^{-\alpha}I_{|n|}(\alpha) \tag{43}$$

and $I_{|n|}(\alpha)$ is the *modified Bessel function*

$$I_n(\alpha) = \sum_{m=0}^{\infty} \frac{\left(\frac{\alpha}{2}\right)^{2m+n}}{m!(m+n)!}, \quad \text{for} \quad n \geq 0.$$

For the rest of this paper, we will use the following notation,

$$\alpha = \frac{2t}{h^2}. \tag{44}$$

To establish (42), define the Fourier transform of a sequence $u^m(t)$ as

$$\hat{v}(\xi, t) := \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi}u^m(t).$$

Using (41), we conclude that

$$\begin{aligned}\frac{\partial \hat{v}}{\partial t} = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi}u_t^m(t) = \frac{1}{h^2\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} (e^{i\xi} + e^{-i\xi} - 2)e^{-im\xi}u^m(t) \\ = \frac{2}{h^2}(cos\xi - 1)\hat{v},\end{aligned}$$

hence

$$\hat{v}(\xi, t) = \hat{v}(\xi, 0)e^{\frac{2t}{h^2}(cos\xi-1)}.$$

Taking the inverse Fourier transform of $\hat{v}(\xi, t)$, we can recover $u^m(t)$:

$$\begin{aligned}u^m(t) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{im\xi}\hat{v}(\xi, t)d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{im\xi}\hat{v}(\xi, 0)e^{\frac{2t}{h^2}(cos\xi-1)}d\xi \\ = \sum_{k=-\infty}^{\infty} G_{m-k}\left(\frac{2t}{h^2}\right) u_0(kh).\end{aligned}$$

In two spatial dimensions, using separation of variables, the solution to the original problem (7) is given by

$$w^{m,n}(t) = \sum_{s,j=-\infty}^{\infty} G_{s-m}\left(\frac{2t}{h^2}\right) G_{j-n}\left(\frac{2t}{h^2}\right) w^{s,j}(0). \tag{45}$$

We further list the following elementary properties of the discrete heat kernel (43).

$$\sum_{k=-\infty}^{\infty} G_k(\alpha) = 1, \tag{46}$$

$$G_k(\alpha) = G_{-k}(\alpha), \quad \text{for} \quad k \geq 1, \tag{47}$$

$$\frac{1}{2}G_0(\alpha) + \sum_{k=1}^{\infty} G_k(\alpha) = \frac{1}{2}. \tag{48}$$

2.2. **Decay properties.** The following result is used to control the Green's function outside a fixed macroscopic domain.

Lemma 1. *Let $G_n(\alpha)$ be the discrete heat kernel (43) with $\alpha = \frac{2\tau}{h^2}$. For any fixed $\mu > 0$, suppose $\tau \le \mu h$. Then for $h \longrightarrow 0$, we have*

$$\sum_{k=\left[\frac{3\mu}{h}\right]}^{\infty} G_k(\alpha) = o\left(\frac{e}{3}\right)^{\frac{\mu}{h}}. \tag{49}$$

We remark before going into the proof that the above result is reminiscent to [8, p. 548, (63)] which shows that the L^1-norm of the heat kernel outside the domain of interest, which in our case is Ω^h, is exponentially small. It also shows that the correct regime for the critical case is indeed $h = O(\tau)$: even though the argument α of G involves like $\frac{\tau}{h^2}$, the summation introduces another factor of h.

Proof. To simplify the notation, we omit the integer part brackets, denoting $\left[\frac{3\mu}{h}\right]$ in (49) simply by $\frac{3\mu}{h}$. Then,

$$\sum_{k=\frac{3\mu}{h}}^{\infty} G_k(\alpha) = e^{-\frac{2\tau}{h^2}} \sum_{k=\frac{3\mu}{h}}^{\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{\tau}{h^2}\right)^{2n+k}}{n!(n+k)!}.$$

Shifting the summation $m = k - \frac{3\mu}{h}$ and using the elementary inequality $(i+j)! \ge i!j!$, we have

$$\begin{aligned}\sum_{k=\frac{3\mu}{h}}^{\infty} G_k(\alpha) &= e^{-\frac{2\tau}{h^2}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{\tau}{h^2}\right)^{2n+m+\frac{3\mu}{h}}}{n!(n+m+\frac{3\mu}{h})!} \\ &\le \frac{\left(\frac{\tau}{h^2}\right)^{\frac{3\mu}{h}}}{\left(\frac{3\mu}{h}\right)!} e^{-\frac{2\tau}{h^2}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{\tau}{h^2}\right)^{2n+m}}{n!(n+m)!} \le \frac{\left(\frac{\tau}{h^2}\right)^{\frac{3\mu}{h}}}{\left(\frac{3\mu}{h}\right)!},\end{aligned} \tag{50}$$

where we have used (46),

$$e^{-\frac{2\tau}{h^2}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{\tau}{h^2}\right)^{2n+m}}{n!(n+m)!} \le \sum_{m=-\infty}^{\infty} G_m\left(\frac{2\tau}{h^2}\right) = 1.$$

Next we have the following Stirling type approximation for a lower bound for the factorial [1],

$$\left(\frac{3\mu}{h}\right)! \ge \sqrt{2\pi\frac{3\mu}{h}} \left(\frac{3\mu}{eh}\right)^{\frac{3\mu}{h}}.$$

Hence by comparing the terms $\frac{\tau}{h^2}$ and $\frac{\mu}{h}$ in (50), upon assuming, $\tau \le \mu h$, we have the desired estimate (49):

$$\sum_{k=\frac{3\mu}{h}}^{\infty} G_k(\alpha) \le \frac{1}{\sqrt{2\pi\frac{3\mu}{h}}} \left(\frac{e\tau}{3\mu h}\right)^{\frac{3\mu}{h}} = o\left(\frac{e}{3}\right)^{\frac{\mu}{h}}.$$

∎

2.3. **Asymptotic expansions.** In what follows, we list the asymptotic expansions for the modified Bessel function. We follow [1, p. 199] and [15]:

(1) The expansion for $I_\nu(z)$ for any fixed index ν is given by:

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \left(1 - \frac{4\nu^3 - 1}{8z} + O\left(\frac{1}{z^2}\right)\right), \quad \text{as} \quad z \to \infty. \tag{51}$$

(2) The following Meissel formula [15] holds uniformly for $z \in \mathbb{R}, z > 0$ and $\nu \geq 1$:

$$I_\nu(\nu z) = \frac{\nu^\nu}{e^\nu \Gamma(\nu+1)} \frac{e^{\nu\eta}}{(1+z^2)^{\frac{1}{4}}} \left(1 + \sum_{k=1}^{\infty} \frac{v_k(\xi)}{\nu^k}\right) \tag{52}$$

where

$$\xi = \frac{1}{\sqrt{1+z^2}}, \quad \eta = \sqrt{1+z^2} + \ln \frac{z}{1+\sqrt{1+z^2}},$$

and $v_k(\xi)$ is a polynomial of the form

$$v_k(\xi) = \sum_{s=0}^{k} a_s^{(k)} \xi^{k+2s}, \tag{53}$$

which are defined through the recursive relation $v_0(\xi) = 1$ and

$$v_{k+1}(\xi) = \frac{1}{2}\xi^2(1-\xi^2)v_k^{'}(\xi) + \frac{1}{8}\int_0^\xi (1-5s^2)v_k(s)ds.$$

Proposition 3. *Assume $\tau = \tau(h)$ is such that $\frac{\tau}{h^2} \to \infty$ as $h \to 0$. Then the following asymptotic expansions hold.*

- *For fixed $n \geq 0$, as $h \longrightarrow 0$,*

$$G_n\left(\frac{2\tau}{h^2}\right) = \frac{1}{\sqrt{4\pi}} \frac{h}{\sqrt{\tau}} \left(1 - \frac{h^2}{\tau} \frac{4n^3 - 1}{16} + O\left(\frac{h^4}{\tau^2}\right)\right). \tag{54}$$

- *Let $x \geq \frac{h}{\sqrt{\tau}}$. Then there exists an absolute constant $C_0 > 0$ and $0 \leq C_{h,x} \leq C_0$ such that*

$$G_{\frac{\sqrt{\tau}x}{h}}\left(\frac{2\tau}{h^2}\right) = \frac{1}{\sqrt{4\pi}} \frac{h}{\sqrt{\tau}} e^{-\frac{x^2}{4}} \left(1 + C_{h,x}\frac{h}{x\sqrt{\tau}}\right). \tag{55}$$

Proof. Expansion (54) follows from (43) and (51):

$$G_n\left(\frac{2\tau}{h^2}\right) = e^{-\frac{2\tau}{h^2}} I_n\left(\frac{2\tau}{h^2}\right). \tag{56}$$

For (55), we will use the Meissel formula (52). Using the notations there, we set

$$\nu = \frac{x\sqrt{\tau}}{h}, \quad z = \frac{2\sqrt{\tau}}{xh}.$$

Then

$$\eta = \sqrt{1+z^2} + \ln \frac{z}{1+\sqrt{1+z^2}} = \frac{\sqrt{x^2h^2 + 4\tau}}{xh} + \ln \frac{2\sqrt{\tau}}{hx + \sqrt{x^2h^2+4\tau}} \tag{57}$$

so that

$$e^{\eta\nu} = e^{\frac{\sqrt{\tau}}{h^2}\sqrt{x^2h^2+4\tau}} \left(\frac{2\sqrt{\tau}}{hx + \sqrt{x^2h^2+4\tau}}\right)^{\frac{x\sqrt{\tau}}{h}}. \tag{58}$$

Next, we write,

$$G_{\left[\frac{\sqrt{\tau}x}{h}\right]}(\frac{2\tau}{h^2}) = e^{-\frac{2\tau}{h^2}} I_{\left[\frac{\sqrt{\tau}x}{h}\right]}(\frac{2\tau}{h^2}) \tag{59}$$

$$= \frac{\nu^\nu}{e^\nu \Gamma(\nu+1)} \frac{\sqrt{x}\sqrt{h}}{(4\tau)^{\frac{1}{4}}} \cdot e^{\frac{\sqrt{\tau}\sqrt{x^2h^2+4\tau}-2\tau}{h^2}} \cdot \left(\frac{2\sqrt{\tau}}{hx+\sqrt{x^2h^2+4\tau}}\right)^{\frac{x\sqrt{\tau}}{h}} \cdot \frac{1}{(1+\frac{h^2x^2}{4\tau})^{\frac{1}{4}}} \cdot$$

$$\left(1+\sum_{k=1}^{\infty}\frac{v_k(\xi)}{\nu^k}\right) = \frac{\nu^\nu}{e^\nu\Gamma(\nu+1)}\frac{\sqrt{x}\sqrt{h}}{(4\tau)^{\frac{1}{4}}} \cdot \mathbf{I}\cdot\mathbf{II}\cdot\mathbf{III}\cdot\mathbf{IV}. \tag{60}$$

Using Stirling approximation [1], we have that for all $\nu \geq 1$, the pre-factor in (60) can be estimated as:

$$\frac{\nu^\nu}{e^\nu\Gamma(\nu+1)} = \frac{1}{\sqrt{2\pi\nu}}\left(1+\frac{C_\nu}{\nu}\right)$$

where $0 \leq C_\nu \leq C_0$ and c_0 is an absolute constant. As a consequence,

$$\frac{\nu^\nu}{e^\nu\Gamma(\nu+1)}\frac{\sqrt{x}\sqrt{h}}{(4\tau)^{\frac{1}{4}}} = \frac{h}{\sqrt{4\pi\tau}}\left(1+\frac{C_\nu h}{x\sqrt{\tau}}\right).$$

We now proceed with the analysis of each term in (60).

Term I $:= e^{\frac{\sqrt{\tau}\sqrt{x^2h^2+4\tau}-2\tau}{h^2}}$. Consider

$$\frac{\sqrt{\tau}\sqrt{x^2h^2+4\tau}-2\tau}{h^2} = \frac{2\tau}{h^2}\left(\sqrt{1+\frac{h^2x^2}{4\tau}}-1\right) = \frac{x^2}{4}-\frac{1}{64}\frac{h^2x^4}{\tau}+O\left(\frac{h^4x^6}{\tau^2}\right).$$

Hence

$$\mathbf{I} = e^{\frac{x^2}{4}}e^{-\frac{h^2x^4}{64\tau}}e^{O\left(\frac{h^4x^6}{\tau^2}\right)} = e^{\frac{x^2}{4}}\left(1-\frac{h^2x^4}{64\tau}+O\left(\frac{h^4x^8}{\tau^2}\right)\right). \tag{61}$$

Term II $:= \left(\frac{2\sqrt{\tau}}{hx+\sqrt{x^2h^2+4\tau}}\right)^{\frac{x\sqrt{\tau}}{h}}$. It can be written as

$$\mathbf{II} = \frac{1}{\left(1+\frac{hx}{2\sqrt{\tau}}\right)^{\frac{2\sqrt{\tau}}{xh}\frac{x^2}{2}}\left(1+\frac{x^3h}{8\sqrt{\tau}}+o\left(\frac{x^3h}{8\sqrt{\tau}}\right)\right)}.$$

Using the expansion

$$(1+y)^{\frac{a}{y}} = e^a - \frac{1}{2}ae^ay + O(e^aa^2y^2),$$

we get

$$\left(1+\frac{hx}{2\sqrt{\tau}}\right)^{\frac{2\sqrt{\tau}}{xh}\frac{x^2}{2}} = e^{\frac{x^2}{2}}\left(1-\frac{x^3h}{8\sqrt{\tau}}+o\left(\frac{x^3h}{\sqrt{\tau}}\right)\right).$$

Altogether,

$$\mathbf{II} = e^{-\frac{x^2}{2}}\left(1+O(\frac{x^6h^2}{\tau})\right). \tag{62}$$

Term III $:= \frac{1}{(1+\frac{h^2x^2}{4\tau})^{\frac{1}{4}}}$. Using Taylor expansion, it becomes:

$$\mathbf{III} = 1-\frac{1}{16}\frac{h^2x^2}{\tau}+O\left(\frac{h^4x^4}{\tau^2}\right). \tag{63}$$

Term IV $:= 1+\sum_{k=1}^{\infty}\frac{v_k(\xi)}{\nu^k}$. In view of (53), for $k\geq 1$, we have

$$\frac{v_k(\xi)}{\nu^k}=a_0^{(k)}\left(\frac{\xi}{\nu}\right)^k+o\left(\frac{\xi}{\nu}\right)^k.$$

Since $\xi=\frac{xh}{\sqrt{x^2h^2+4\tau}}$, we have $\frac{\xi}{\nu}=\frac{h^2}{\sqrt{x^2h^2+4\tau}\sqrt{\tau}}\leq\frac{h^2}{2\tau}$. Hence

$$\mathbf{IV}=1+\frac{1}{8}\frac{\xi}{\nu}+o\left(\frac{\xi}{\nu}\right)=1+O\left(\frac{h^2}{\tau}\right). \tag{64}$$

Combining the four expansions (61)–(64) above, for $x\geq\frac{h}{\sqrt{\tau}}$, (59) reads as

$$\begin{aligned}&G_{\left[\frac{\sqrt{\tau}x}{h}\right]}\left(\frac{2\tau}{h^2}\right)=\frac{h}{\sqrt{4\pi}}\frac{1}{\sqrt{\tau}}e^{-\frac{x^2}{4}}\cdot\\&\left(1+C_\nu\frac{h}{x\sqrt{\tau}}\right)\left(1+O\left(\frac{h^2x^4}{\tau}\right)\right)\left(1+O\left(\frac{h^2x^6}{\tau}\right)\right)\left(1+O\left(\frac{h^2}{\tau}\right)\right)\end{aligned} \tag{65}$$

for $x\geq\frac{h}{\sqrt{\tau}}$, which gives the expansion (55) up to $O\left(\frac{h}{\sqrt{\tau}}\right)$ order terms. $\square$

3. **Proof of Theorem 1 - Sub-Critical Case.** We prove here that the function $\bar{u}$, defined in (28), is a viscosity subsolution of the equation (19). The fact that (27) is a supersolution can be established analogously. The strategy of proof follows [2] closely. Essentially we will reduce the discrete computation to [2, Prop. 4.1] which is the key consistency result needed.

Step I. We establish here the discrete analog of [2, (11)]. First let $\varphi(x,t)$ be a smooth test function satisfying

$$\lim_{|x|+t\to\infty}\varphi(x,t)=+\infty. \tag{66}$$

Let $(\bar{x},\bar{t})$ be a strict global maximum point of $\bar{u}-\varphi$. Suppose $\bar{u}(\bar{x},\bar{t})=-1$. As $\bar{u}$ is upper semicontinuous taking values in $\{-1,1\}$, we have that $\bar{u}=-1$ in a neighborhood of $(\bar{x},\bar{t})$. Consequently, $D\varphi(\bar{x},\bar{t})=0$ and $\frac{\partial\varphi}{\partial\tau}(\bar{x},\bar{t})=0$. By (21'), the conclusion follows. Similar reasoning yields the desired result in the case when $(\bar{x},\bar{t})$ is in the interior of the set $\{\bar{u}=1\}$. Therefore, without loss of generality, we will assume that $(\bar{x},\bar{t})$ is at the boundary of the set $\{\bar{u}=1\}$. We claim that in this case, there exists a sequence $\{(m_hh,n_hh);k_\tau\tau\}$ converging to $(\bar{x}=(\bar{x}_1,\bar{x}_2);\bar{t})$ such that

$$u_{k_\tau}^{m_h,n_h}-\varphi(m_hh,n_hh;k_\tau\tau)=\max_{((m,n),k)\in\mathbb{Z}^2\times\mathbb{N}}(u_k^{m,n}-\varphi(mh,nh;k\tau)) \tag{67}$$

and

$$\lim_{h\to 0}u_{k_\tau}^{m_h,n_h}=1. \tag{68}$$

Indeed, it follows from the coercivity condition (66) that $\max_{\mathbb{Z}^2\times\mathbb{N}}(u_k^{m,n}-\varphi(mh,nh;k\tau))$ is attained at some $m=m_h$, $n=n_h$ an $k=k_\tau$. Furthermore, up to a subsequence, there exist $\bar{x}=(\bar{x}_1,\bar{x}_2)$ and $\bar{t}$ such that $hm_h\to\bar{x}_1$, $hn_h\to\bar{x}_2$ and $\tau k_\tau\to\bar{t}$. This point $(\bar{x},\bar{t})$ must be the global maximum of $\bar{u}-\varphi$. In addition, (68) holds, for otherwise it will contradict (67) and the definition of $\bar{u}$.

To continue, since $u_{k_\tau}^{m_h,n_h}$ takes only values of 1 and -1, it follows from (68) that $u_{k_\tau}^{m_h,n_h} \equiv 1$ for sufficiently small h. For such h, the fact that $(m_h, n_h; k_\tau)$ is a maximum point implies that for all m, n, and k in $\mathbb{Z}$, we have

$$u_k^{m,n} \leq 1 - \varphi(m_h h, n_h h; k_\tau \tau) + \varphi(mh, nh; k).$$

By the same reasoning as in [2, p. 490], the above inequality leads to

$$S^h(\tau)\left[\operatorname{sign}^*\Big(\varphi(\cdot,(k_\tau - 1)\tau) - \varphi(m_h h, n_h h; k_\tau \tau)\Big)\right](m_h h, n_h h) \geq 0$$

where sign^* is the upper semi-continuous envelope of the sign function. The above is equivalent to

$$S^h(\tau)\left[\left(\frac{1+\operatorname{sign}^*}{2}\right)\Big(\varphi(\cdot,(k_\tau - 1)\tau) - \varphi(m_h h, n_h h; k_\tau \tau)\Big)\right](m_h h, n_h h) \geq \frac{1}{2}.$$

In terms of the discrete kernels $G_n(\alpha)$, the above inequality reads as

$$\sum_{(m,n)\in Q_h} G_{m-m_h}(\alpha) G_{n-n_h}(\alpha) \geq \frac{1}{2}, \tag{69}$$

where

$$\alpha = \frac{2\tau}{h^2} \quad \text{and} \quad Q_h = \Big\{(m,n) \in \mathbb{Z}^2 : \varphi(mh, nh; (k_\tau - 1)\tau) - \varphi(m_h h, n_h h; k_\tau \tau) \geq 0\Big\}. \tag{70}$$

Step II. Here we will re-write the discrete inequality (69) into an integral form. For this, we write

$$\sum_{(m,n)\in Q_h} G_{m-m_h}(\alpha) G_{n-n_h}(\alpha) = \int_{\widetilde{Q}_h} G_{[\tilde{x}_1]}(\alpha) G_{[\tilde{x}_2]}(\alpha) d\tilde{x}_1\, d\tilde{x}_2, \tag{71}$$

with

$$\widetilde{Q}_h := \Big\{(\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2 : ([\tilde{x}_1] - m_h, [\tilde{x}_2] - n_h) \in Q_h\Big\}.$$

Next, introducing $x_1 = \frac{h\tilde{x}_1}{\sqrt{\tau}}$ and $x_2 = \frac{h\tilde{x}_2}{\sqrt{\tau}}$, the inequality (69) can be written as

$$\frac{\tau}{h^2}\int_{\widehat{Q}_h} G_{\left[\frac{\sqrt{\tau}x_1}{h}\right]}(\alpha) G_{\left[\frac{\sqrt{\tau}x_2}{h}\right]}(\alpha) dx_1\, dx_2 \geq \frac{1}{2}, \tag{72}$$

where

$$\widehat{Q}_h = \frac{h}{\sqrt{\tau}}\widetilde{Q}_h = \left\{(x_1, x_2) \in \mathbb{R}^2 : \left(\left[\frac{\sqrt{\tau}x_1}{h}\right] - m_h, \left[\frac{\sqrt{\tau}x_2}{h}\right] - n_h\right) \in Q_h\right\}. \tag{73}$$

To continue, we first state an error estimate between the Riemann sum and its integral representation for Gaussian function. For any $G \subset \mathbb{R}^2$, $f : G \to \mathbb{R}$ and $\delta > 0$, denote $RS_\delta[f, G]$ to be the Riemann sum for f in G with step δ, i.e.

$$RS_\delta[f, G] := \delta^2 \sum_{(m,n)\in\mathbb{Z}^2, (m\delta, n\delta)\in G} f(m\delta, n\delta).$$

For $f = e^{-\frac{|x|^2}{4}}$ and $G \subset \mathbb{R}^2$ we have the following error estimate between the Riemann sum and the integral:

$$\left| RS_\delta[e^{-\frac{|x|^2}{4}}, G] - \int_G e^{-\frac{|x|^2}{4}} dx \right| \leq C\delta \tag{74}$$

for some $C > 0$ which does not depend on G and δ. The above follows from the one-dimensional inequality

$$\left| RS_\delta[e^{-\frac{x^2}{4}},[a,b]] - \int_a^b e^{-\frac{x^2}{4}}dx \right| \le 2\delta \tag{75}$$

which holds uniformly for all $-\infty \le a < b \le +\infty$.

Now we proceed to analyze (72). By (55), we have

$$\begin{aligned}\frac{\tau}{h^2}\int_{\widehat{Q}_h} G_{\left[\frac{\sqrt{\tau}x_1}{h}\right]}(\alpha)G_{\left[\frac{\sqrt{\tau}x_2}{h}\right]}(\alpha)dx_1\,dx_2 = RS_{\frac{h}{\sqrt{\tau}}}\frac{1}{4\pi}\left[e^{\frac{-x_1^2-x_2^2}{4}},\widehat{Q}_h\right] + \\ C_1\frac{h}{\sqrt{\tau}}RS_{\frac{h}{\sqrt{\tau}}}\left[\frac{1}{|x_1|}e^{\frac{-x_1^2-x_2^2}{4}},\widehat{Q}_h\right] + C_2\frac{h}{\sqrt{\tau}}RS_{\frac{h}{\sqrt{\tau}}}\left[\frac{1}{|x_2|}e^{\frac{-x_1^2-x_2^2}{4}},\widehat{Q}_h\right] + o\left(\frac{h}{\sqrt{\tau}}\right).\end{aligned} \tag{76}$$

By (54), for $i = 1, 2$, we have,

$$\begin{aligned}RS_{\frac{h}{\sqrt{\tau}}}\left[\frac{1}{|x_i|}e^{\frac{-x_1^2-x_2^2}{4}},\widehat{Q}_h\right] \le \int_{\mathbb{R}^2\setminus[-\frac{h}{\sqrt{\tau}},\frac{h}{\sqrt{\tau}}]^2}\frac{1}{|x_i|}e^{\frac{-x_1^2-x_2^2}{4}}dx_1\,dx_2 + O\left(\frac{h}{\sqrt{\tau}}\right) \\ = O\left(\ln\frac{h}{\sqrt{\tau}}\right).\end{aligned} \tag{77}$$

Applying (74) to (76) and making use of (77), inequality (72) now becomes

$$\frac{1}{4\pi}\int_{\widehat{Q}_h} e^{\frac{-x_1^2-x_2^2}{4}}dx_1\,dx_2 + M_h \ge \frac{1}{2}, \tag{78}$$

with

$$M_h = O\left(\frac{h}{\sqrt{\tau}}\ln\frac{h}{\sqrt{\tau}}\right). \tag{79}$$

Note that $\widehat{Q}_h$ is a discretized set. In order to apply [2, Prop. 4.1], we will make use of a version of (78) in which $\widehat{Q}_h$ is replaced by its continuum analog with well-controlled error. For this purpose, consider,

$$\begin{aligned}Q_h^{\text{cont}} := \Big\{(x_1,x_2)\in\mathbb{R}^2, \varphi(\sqrt{\tau}x_1 - m_h h, \sqrt{\tau}x_2 - n_h h; (k_\tau - 1)\tau) \\ - \varphi(m_h h, n_h h; k_\tau\tau) \ge 0\Big\}.\end{aligned}$$

Since φ is a smooth function satisfying the growth condition (66), we have that the length of the boundary set $\partial\widehat{Q}_h$ is bounded independently of h. Moreoever, for any $(x_1,x_2)\in\partial\widehat{Q}_h$, we have $\operatorname{dist}((x_1,x_2),\partial Q_h^{\text{cont}}) \le \frac{h}{\sqrt{\tau}}$. Hence

$$\mathcal{L}^2\left\{(Q_h^{\text{cont}}\setminus\widehat{Q}_h)\cup(\widehat{Q}_h\setminus Q_h^{\text{cont}})\right\} \le C\frac{h}{\sqrt{\tau}}$$

which in turn implies that

$$\left|\frac{1}{4\pi}\int_{\widehat{Q}_h} e^{\frac{-x_1^2-x_2^2}{4}}dx_1\,dx_2 - \frac{1}{4\pi}\int_{Q_h^{\text{cont}}} e^{\frac{-x_1^2-x_2^2}{4}}dx_1\,dx_2\right| \le C\frac{h}{\sqrt{\tau}} \tag{80}$$

with C independent of h. After rescaling, we have

$$\frac{1}{4\pi}\int_{Q_h^{\text{cont}}} e^{\frac{-x_1^2-x_2^2}{4}}dx_1\,dx_2 = \frac{1}{4\pi\tau}\int_{Q_h^{\text{resc}}} e^{\frac{-(x_1-m_h h)^2-(x_2-n_h h)^2}{4\tau}}dx_1\,dx_2 \tag{81}$$

where

$$Q_h^{\text{resc}} := \Big\{(x_1,x_2) \in \mathbb{R}^2, \varphi(x_1,x_2;(k_\tau-1)\tau) - \varphi(m_h h, n_h h; k_\tau \tau) \geq 0\Big\}.$$

Combining (78) with (80) and (81), we have

$$\begin{aligned} 0 \leq \frac{1}{2} - \sum_{(m,n)\in Q_h} & G_{m-m_h}(\alpha) G_{n-n_h}(\alpha) \\ &= \frac{1}{2} - \frac{1}{4\pi\tau}\int_{Q_h^{\text{resc}}} e^{\frac{-(x_1-m_h h)^2-(x_2-n_h h)^2}{4\tau}}\, dx_1\, dx_2 + M_h. \end{aligned} \tag{82}$$

Step III. We show here how (82) leads to (21') and (23') from Proposition 1. We consider two cases. If $D\phi((\bar{x}_1,\bar{x}_2);\bar{t}) \neq 0$, we apply the result of [2, Prop. 4.1] with $\phi_h((x_1,x_2);t) := \phi((x_1,x_2);t) - \phi((m_h h, n_h h); k_\tau \tau)$:

$$\begin{aligned} 0 &\geq \liminf_h \frac{1}{\sqrt{\tau}} \left(\frac{1}{2} - \sum_{(m,n)\in Q_h} G_{m-m_h}(\alpha) G_{n-n_h}(\alpha) \right) \\ &= \liminf_h \frac{1}{\sqrt{\tau}} \left(\frac{1}{2} - \frac{1}{4\pi\tau} \int_{Q_h^{\text{resc}}} e^{\frac{-(x_1-m_h h)^2-(x_2-n_h h)^2}{4\tau}}\, dx_1\, dx_2 + M_h \right) \\ &\geq \frac{1}{2\sqrt{\pi}|D\phi((\bar{x}_1,\bar{x}_2),t)|} \left(\frac{\partial \phi}{\partial t} - \Delta\phi + \frac{(D^2\phi D\phi|D\phi)}{|D\phi|^2} \right)((\bar{x}_1,\bar{x}_2),\bar{t}) \end{aligned}$$

which yields the desired result.

It remains to show that in the case when $D\phi((\bar{x}_1,\bar{x}_2);\bar{t}) = 0$, $D^2\phi((\bar{x}_1,\bar{x}_2);\bar{t}) = 0$, and

$$\frac{1}{2} - \frac{1}{4\pi\tau}\int_{Q_h^{\text{resc}}} e^{\frac{-(x_1-m_h h)^2-(x_2-n_h h)^2}{4\tau}}\, dx_1\, dx_2 + M_h \leq 0 \tag{83}$$

with M_h satisfying (79) for sufficiently small h, we have $\frac{\partial\phi}{\partial t}((\bar{x_1},\bar{x_2});\bar{t}) \leq 0$. The condition (83), strictly speaking, is different from the one in Proposition 4.1 [2]. However, we can still apply the technique in [2]. The following three cases are possible:

I. Along some subsequence $|D\phi(m_h h, n_h h; k_\tau \tau)| \neq 0$, and $\frac{\sqrt{\tau}}{|D\phi(m_h h, n_h h; k_\tau\tau)|} \to 0$;

II. Along some subsequence $|D\phi(m_h h, n_h h; k_\tau \tau)| \equiv 0$ or $\frac{\sqrt{\tau}}{|D\phi(m_h h, n_h h; k_\tau\tau)|} \to \infty$;

III. Along some subsequence $|D\phi(m_h h, n_h h; k_\tau \tau)| \neq 0$, and $\frac{\sqrt{\tau}}{|D\phi(m_h h, n_h h; k_\tau\tau)|} \to l > 0$.

In Case I, arguing as in [2], we deduce that for any $c > 0$,

$$\begin{aligned} \frac{|D\phi(m_h h, n_h h; k_\tau\tau)|}{\sqrt{\tau}} & \left(\frac{1}{2} - \frac{1}{4\pi\tau}\int_{Q_h^{\text{resc}}} e^{\frac{-(x_1-m_h h)^2-(x_2-n_h h)^2}{4\tau}}\, dx_1\, dx_2 \right) \\ &\geq \frac{1}{2\sqrt{\pi}} \left(\frac{\partial\phi}{\partial t}(x_1,x_2;t) - c \right) + o(1) \end{aligned} \tag{84}$$

However, since

$$\frac{|D\phi(m_h h, n_h h; k_\tau\tau)|}{\sqrt{\tau}} M_h = o\left(\frac{h}{\tau}\ln\frac{h}{\sqrt{\tau}}\right) \to 0,$$

passing to the limit in (84) as $h \to 0$, using (83), we have $\frac{\partial\phi}{\partial t}(x_1,x_2;t) - c \leq 0$ for all $c > 0$, which yields the desired result.

The conclusion in the Cases II and III follows from Proposition 4.1 [2] without any change.

4. Motion in the critical case: $\tau = \mu h$.

This section proves the statements for the critical case, most notably Theorems 2 and 4.

4.1. Proof of Theorem 2.

We refer to page 6385, in particular (30) and (31), for the notation concerning the set Ω_0. Theorem 2 will follow from the following statements.

(i) **(Lower bound)** If $w^{0,-n_0}(\tau) \geq \frac{1}{2}$, then

$$\sum_{k=1}^{n_0-1} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx + \frac{1}{2}\int_0^{\sqrt{\frac{2n_0}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx \geq \frac{1}{2}\int_{\sqrt{\frac{2n_0}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx + \sum_{k=n_0+1}^{\infty}\int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx. \tag{85}$$

(ii) **(Upper bound)** If $w^{0,-n_0}(\tau) \leq \frac{1}{2}$, then

$$\sum_{k=1}^{n_0-1} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx + \frac{1}{2}\int_0^{\sqrt{\frac{2n_0}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx \leq \frac{1}{2}\int_{\sqrt{\frac{2n_0}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx + \sum_{k=n_0+1}^{\infty}\int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx. \tag{86}$$

To prepare the the proof, let $n_0 \geq 0$ be such that $w^{0,-n_0}(\tau) \geq \frac{1}{2}$. By Lemma 1, we have

$$\frac{1}{2} \leq w^{0,-n_0}(\tau) = \sum_{(sh,jh)\in\Omega} G_s(\alpha)G_{j+n_0}(\alpha) = \sum_{s=-\infty}^{\infty}\sum_{j=-\infty}^{\left[\frac{\varphi(sh)}{h}\right]} G_s(\alpha)G_{j+n_0}(\alpha) + o\left(\frac{e}{3}\right)^{\frac{1}{h}}.$$

Shifting the summation indices gives

$$\frac{1}{2} \leq \sum_{s=-\infty}^{\infty}\sum_{j=-\infty}^{\left[\frac{\varphi(sh)}{h}\right]+n_0} G_s(\alpha)G_j(\alpha) + o\left(\frac{e}{3}\right)^{\frac{1}{h}}. \tag{87}$$

Similarly, if $w^{0,-n_0}(\tau) \leq \frac{1}{2}$, then

$$\frac{1}{2} \geq \sum_{s=-\infty}^{\infty}\sum_{j=-\infty}^{\left[\frac{\varphi(sh)}{h}\right]+n_0} G_s(\alpha)G_j(\alpha) + o\left(\frac{e}{3}\right)^{\frac{1}{h}}. \tag{88}$$

We first outline the idea of the proof for (85) and (86). Note that the relations (87) and (88) implicitly define n_0 in terms of h. Our goal is to obtain a description of n_0 which is independent of h. To this end, we will establish the following asymptotic expansion of the sum in the right hand sides of (87) and (88): as $h \to 0$, it holds that

$$\sum_{s=-\infty}^{\infty}\sum_{j=-\infty}^{\left[\frac{\varphi(sh)}{h}\right]+n_0} G_s(\alpha)G_j(\alpha) = \frac{1}{2} + \frac{1}{\sqrt{4\pi}}\Phi(n_0,\mu,\kappa)\sqrt{h} + o(\sqrt{h}), \tag{89}$$

where

$$\Phi(n_0,\mu,\kappa) := \sum_{k=1}^{n_0-1}\int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx + \frac{1}{2}\int_0^{\sqrt{\frac{2n_0}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx - \frac{1}{2}\int_{\sqrt{\frac{2n}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx - \sum_{k=n_0+1}^{\infty}\int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx. \quad (90)$$

Then the desired results will follow.

It is convenient to group the indices in the summation (89) into the following sets:

$$\begin{aligned} I_1 &:= \{(s,j) : 0 < jh < \varphi(sh) + n_0 h\},\\ I_2 &:= \{(s,j) : \varphi(sh) + n_0 h \le jh < 0\},\\ I_3 &:= \{(s,j) : jh < \min\{\varphi(sh) + n_0 h, 0\}\}, \end{aligned} \quad (91)$$

and

$$\begin{aligned} bI_1 &:= \{(s,0) : 0 < \varphi(sh) + n_0 h\},\\ bI_2 &:= \{(s,0) : 0 \ge \varphi(sh) + n_0 h\}. \end{aligned} \quad (92)$$

(See Fig. 4.)

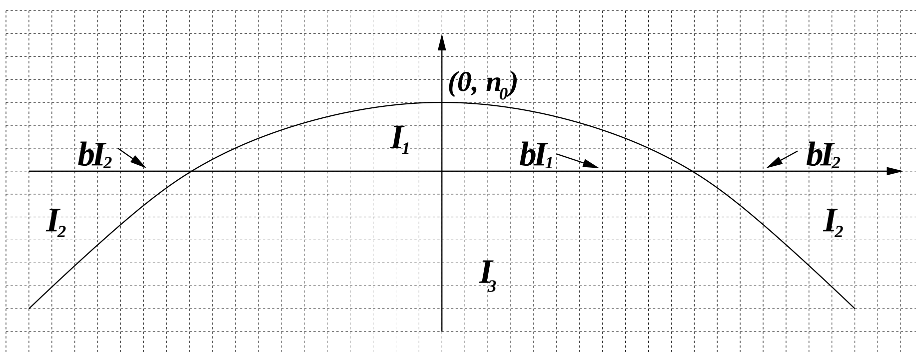

Figure 4.

With the above, we have

$$\sum_{s=-\infty}^{\infty}\sum_{j=-\infty}^{\left[\frac{\varphi(sh)}{h}\right]+n_0} G_s(\alpha)G_j(\alpha) = \sum_{(s,j)\in I_1} G_s(\alpha)G_j(\alpha) + \sum_{(s,j)\in I_3} G_s(\alpha)G_j(\alpha) + \sum_{(s,j)\in bI_1} G_s(\alpha)G_j(\alpha).$$

On the other hand, it follows from the properties of the discrete heat kernel that

$$\sum_{(s,j)\in I_2} G_s(\alpha)G_j(\alpha) + \sum_{(s,j)\in I_3} G_s(\alpha)G_j(\alpha) + \frac{1}{2}\sum_{(s,j)\in bI_1} G_s(\alpha)G_j(\alpha) + \frac{1}{2}\sum_{(s,j)\in bI_2} G_s(\alpha)G_j(\alpha) = \frac{1}{2}. \quad (93)$$

In view of (87), we have

$$\sum_{(s,j)\in I_1} G_s(\alpha)G_j(\alpha) + \sum_{(s,j)\in I_3} G_s(\alpha)G_j(\alpha) + \sum_{(s,j)\in bI_1} G_s(\alpha)G_j(\alpha) \ge \frac{1}{2} + o\left(\frac{e}{3}\right)^{\frac{1}{h}}. \quad (94)$$

Subtracting (94) from (93), we obtain

$$\sum_{(s,j)\in I_1} G_s(\alpha)G_j(\alpha) + \frac{1}{2}\sum_{(s,j)\in bI_1} G_s(\alpha)G_j(\alpha) \geq \sum_{(s,j)\in I_2} G_s(\alpha)G_j(\alpha) + \frac{1}{2}\sum_{(s,j)\in I_2} G_s(\alpha)G_j(\alpha) + o\left(\frac{e}{3}\right)^{\frac{1}{h}}. \quad (95)$$

Now for a fixed $n \geq 1$, let $s(n) > 0$ be the integer satisfying

$$\left[\frac{\varphi(s(n)h)}{h}\right] = -n.$$

Since

$$\varphi(sh) = \varphi(0) + sh\varphi^{'}(0) + \frac{(sh)^2}{2}\varphi^{''}(0) + O(h^3) = -\frac{\kappa}{2}(sh)^2 + O(h^3),$$

we have

$$s(n) = \left[\sqrt{\frac{2n}{\kappa h}}\right]. \quad (96)$$

Changing the order of summation in (95) and using the notation (96), we have that if $w^{0,-n_0}(\tau) \geq \frac{1}{2}$, then

$$\sum_{j=1}^{n_0} G_j(\alpha) \sum_{|s|\leq s(n_0-j)} G_s(\alpha) + \frac{1}{2}G_0(\alpha) \sum_{|s|\leq s(-n_0)} G_s(\alpha) \geq \frac{1}{2}G_0(\alpha) \sum_{|s|\geq s(-n_0)+1} G_s(\alpha) + \sum_{j=-\infty}^{-1} G_j(\alpha) \sum_{|s|>s(n_0-j+1)} G_s(\alpha) + o\left(\frac{e}{3}\right)^{\frac{1}{h}}. \quad (97)$$

As of the second case, $w^{0,-n_0}(\tau) \leq \frac{1}{2}$, the same reasoning gives

$$\sum_{j=1}^{n_0} G_j(\alpha) \sum_{|s|\leq s(n_0-j)} G_s(\alpha) + \frac{1}{2}G_0(\alpha) \sum_{|s|\leq s(-n_0)} G_s(\alpha) \leq \frac{1}{2}G_0(\alpha) \sum_{|s|\geq s(-n_0)+1} G_s(\alpha) + \sum_{j=-\infty}^{-1} G_j(\alpha) \sum_{|s|>s(n_0-j+1)} G_s(\alpha) + o\left(\frac{e}{3}\right)^{\frac{1}{h}}. \quad (98)$$

We point out here that the main technical difficulty in establishing the asymptotic behavior of the terms in (98) is the term

$$\sum_{j=-\infty}^{-1} G_j(\alpha) \sum_{|s|>s(n_0-j+1)} G_s(\alpha)$$

which involves an infinite sum of the products of Bessel functions and Riemann sums. We will estimate this term from below using finitely many such products, thus obtaining the lower bound (85). On the other hand, by means of a priori bounds for Bessel functions, we will obtain the upper bound (86). Once we verify that those bounds match, we will then obtain the asymptotic expansion of all the terms in (98).

4.2. **The lower bound** (85)**.** Let $w^{0,-n_0}(\tau) \geq \frac{1}{2}$. Fix an arbitrary $N \geq 1$. It follows from (98) that

$$\sum_{j=1}^{n_0} G_j(\alpha) \sum_{|s|\leq s(n_0-j)} G_s(\alpha) + \frac{1}{2}G_0(\alpha) \sum_{|s|\leq s(-n_0)} G_s(\alpha)$$
$$\geq \frac{1}{2}G_0(\alpha) \sum_{|s|\geq s(-n_0)+1} G_s(\alpha) + \sum_{j=-N}^{-1} G_j(\alpha) \sum_{|s|>s(n_0-j+1)} G_s(\alpha). \quad (99)$$

For every $j = 1, \ldots, n_0$, by (96), we have that

$$\sum_{|s|\leq s(n_0-j)} G_s(\alpha) = 1 - \sum_{|s|>s(n_0-j)} G_s(\alpha) = 1 - \sum_{|s|>\sqrt{\frac{2(n_0-j)}{\kappa h}}} G_s(\alpha).$$

Similar to the proof of Theorem 1, the re-scaling $t = \mu h$ leads to:

$$\sum_{|s|>\sqrt{\frac{n_0-j}{\kappa h}}} G_s(\alpha) = \int_{|s|>\sqrt{\frac{2(n_0-j)}{\kappa h}}} G_{[s]}(\alpha)ds = 2\frac{\sqrt{\tau}}{h}\int_{\sqrt{\frac{2(n_0-j)}{\mu\kappa}}}^{\infty} G_{[\frac{\sqrt{\tau}x}{h}]}(\alpha)dx. \quad (100)$$

Using Proposition 1, the above can be written as

$$\frac{\sqrt{\tau}}{h}\int_{\sqrt{\frac{2(n_0-j)}{\mu\kappa}}}^{\infty} G_{[\frac{\sqrt{\tau}x}{h}]}(\alpha)dx = \frac{1}{\sqrt{4\pi}} RS_{\frac{h}{\sqrt{\tau}}}\left[e^{-\frac{x^2}{4}}, \left(\sqrt{\frac{2(n_0-j)}{\mu\kappa}}, +\infty\right)\right]$$
$$+ C\frac{h}{\sqrt{\tau}} RS_{\frac{h}{\sqrt{\tau}}}\left[\frac{1}{x}e^{-\frac{x^2}{4}}, \left(\sqrt{\frac{2(n_0-j)}{\mu\kappa}}, +\infty\right)\right]. \quad (101)$$

By the uniform Riemann sum estimate (75), we thus have

$$\sum_{|s|>\sqrt{\frac{n_0-j}{\kappa h}}} G_s(\alpha) = \frac{2}{\sqrt{4\pi}}\int_{\sqrt{\frac{2(n_0-j)}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}dx + O(\sqrt{h}).$$

By property (46),

$$\sum_{|s|\leq s(n_0-j)} G_s(\alpha) = 1 - \sum_{|s|>\sqrt{\frac{n_0-j}{\kappa h}}} G_s(\alpha) = \frac{2}{\sqrt{4\pi}}\int_0^{\sqrt{\frac{2(n_0-j)}{\mu\kappa}}} e^{-\frac{x^2}{4}}dx + O(\sqrt{h}).$$

Next, combining the above expansion with (54), as $h \longrightarrow 0$, we have

$$\sum_{j=1}^{n_0} G_j(\alpha) \sum_{|s|\leq s(n_0-j)} G_s(\alpha) = 2\frac{\sqrt{h}}{4\pi\sqrt{\mu}} \sum_{k=1}^{n_0-1} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}dx + O(h), \quad (102)$$

and

$$\frac{1}{2}G_0(\alpha)\sum_{|s|\leq S_0}G_s(\alpha)=\frac{\sqrt{h}}{4\pi\sqrt{\mu}}\int_0^{\sqrt{\frac{2n_0}{\mu\kappa}}}e^{-\frac{x^2}{4}}\,dx+O(h), \tag{103}$$

$$\frac{1}{2}G_0(\alpha)\sum_{|s|\geq S_0+1}G_s(\alpha)=\frac{\sqrt{h}}{4\pi\sqrt{\mu}}\int_{\sqrt{\frac{2n_0}{\mu\kappa}}}^{\infty}e^{-\frac{x^2}{4}}\,dx+O(h), \tag{104}$$

$$\sum_{j=-N}^{-1}G_j(\alpha)\sum_{|s|>s(n_0-j+1)}G_s(\alpha)=2\frac{\sqrt{h}}{4\pi\sqrt{\mu}}\sum_{k=n_0+1}^{n_0+N}\int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty}e^{-\frac{x^2}{4}}\,dx+O(h). \tag{105}$$

In addition, combining (102)–(105), the inequality (99) implies

$$\sum_{k=1}^{n_0-1}\int_0^{\sqrt{\frac{2k}{\mu\kappa}}}e^{-\frac{x^2}{4}}\,dx+\frac{1}{2}\int_0^{\sqrt{\frac{2n_0}{\mu\kappa}}}e^{-\frac{x^2}{4}}\,dx\geq\frac{1}{2}\int_{\sqrt{\frac{2n_0}{\mu\kappa}}}^{\infty}e^{-\frac{x^2}{4}}\,dx+\sum_{k=n_0+1}^{n_0+N}\int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty}e^{-\frac{x^2}{4}}\,dx.$$

Since $N\geq 1$ was chosen arbitrarily, letting $N\longrightarrow\infty$ gives

$$\sum_{k=1}^{n_0-1}\int_0^{\sqrt{\frac{2k}{\mu\kappa}}}e^{-\frac{x^2}{4}}\,dx+\frac{1}{2}\int_0^{\sqrt{\frac{2n_0}{\mu\kappa}}}e^{-\frac{x^2}{4}}\,dx\geq\frac{1}{2}\int_{\sqrt{\frac{2n_0}{\mu\kappa}}}^{\infty}e^{-\frac{x^2}{4}}\,dx+\sum_{k=n_0+1}^{\infty}\int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty}e^{-\frac{x^2}{4}}\,dx$$

which leads to (i).

4.3. **The matching upper bound.** Now let $w^{0,-n_0}(\tau)\leq\frac{1}{2}$. It follows from (54), as well as the monotonicity of $G_n(\alpha)$ in n, that the inequality

$$G_n(\alpha)\leq\sqrt{\frac{h}{4\pi\mu}}$$

holds for all $n\geq 1$. Thus, from (98) we have

$$\sum_{j=1}^{n_0}G_j(\alpha)\sum_{|s|\leq s(n_0-j)}G_s(\alpha)+\frac{1}{2}G_0(\alpha)\sum_{|s|\leq s(-n_0)}G_s(\alpha)$$
$$\leq\frac{1}{2}G_0(\alpha)\sum_{|s|\geq s(-n_0)}G_s(\alpha)+\sqrt{\frac{h}{4\pi}}\sum_{j=-\infty}^{-1}\sum_{|s|>s(n_0-j+1)}G_s(\alpha). \tag{106}$$

Now fix $j\leq -1$, and denote $m:=n_0-j+1$. Using (100) and (101), we have

$$\sum_{|s|>s(m)}G_s(\alpha)=\frac{2}{\sqrt{4\pi}}RS_{\frac{h}{\sqrt{\tau}}}\left[e^{-\frac{x^2}{4}},\left(\sqrt{\frac{2m}{\mu\kappa}},+\infty\right)\right]$$
$$+C\frac{h}{\sqrt{\tau}}RS_{\frac{h}{\sqrt{\tau}}}\left[\frac{1}{x}e^{-\frac{x^2}{4}},\left(\sqrt{\frac{2m}{\mu\kappa}},+\infty\right)\right]. \tag{107}$$

Note that

$$\left|RS_{\frac{h}{\sqrt{\tau}}}\left[e^{-\frac{x^2}{4}},\left(\sqrt{\frac{2m}{\mu\kappa}},+\infty\right)\right]-\int_{\sqrt{\frac{2m}{\mu\kappa}}}^{\infty}e^{-\frac{x^2}{4}}\,dx\right|\leq\frac{h}{\sqrt{\tau}}e^{-\frac{m}{2\mu\kappa}}, \tag{108}$$

and

$$\left|RS_{\frac{h}{\sqrt{\tau}}}\left[\frac{1}{x}e^{-\frac{x^2}{4}},\left(\sqrt{\frac{2m}{\mu\kappa}},+\infty\right)\right]-\int_{\sqrt{\frac{2m}{\mu\kappa}}}^{\infty}\frac{1}{x}e^{-\frac{x^2}{4}}\,dx\right|\leq\frac{h}{\sqrt{\tau}}e^{-\frac{m}{2\mu\kappa}}. \tag{109}$$

Moreover, we have

$$\int_{\sqrt{\frac{2m}{\mu\kappa}}}^{\infty} \frac{1}{x} e^{-\frac{x^2}{4}}\, dx \le C \int_{\sqrt{\frac{2m}{\mu\kappa}}}^{\infty} \frac{x}{2} e^{-\frac{x^2}{4}}\, dx = Ce^{-\frac{m}{2\mu\kappa}}. \tag{110}$$

Combining (108), (109) and (110), the equation (107) reads as

$$\sum_{|s|>s(m)} G_s(\alpha) = \frac{2}{\sqrt{4\pi}} \int_{\sqrt{\frac{2m}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx + \frac{2}{\sqrt{4\pi\mu}} e^{-\frac{m}{2\mu\kappa}} \left(\sqrt{h} + o(\sqrt{h})\right).$$

Therefore,

$$\sum_{j=-\infty}^{-1} \sum_{|s|>s(n_0-j+1)} G_s(\alpha) = \frac{1}{\sqrt{4\pi}} \sum_{k=n_0+1}^{\infty} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx + O(\sqrt{h}).$$

Using (102), (103) and (104), the inequality (106) yields the upper bound

$$\sum_{k=1}^{n_0-1} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx + \frac{1}{2} \int_0^{\sqrt{\frac{2n_0}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx \le \frac{1}{2} \int_{\sqrt{\frac{2n_0}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx + \sum_{k=n_0+1}^{\infty} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx \tag{111}$$

which gives (ii).

4.4. **Proof of Theorem 3: the limiting case $\mu \to \infty$.** Recall that n_0 satisfies

$$\sum_{k=1}^{n_0-1} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx + \frac{1}{2} \int_0^{\sqrt{\frac{2n_0}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx \le \frac{1}{2} \int_{\sqrt{\frac{2n_0}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx + \sum_{k=n_0+1}^{\infty} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx \tag{112}$$

and

$$\sum_{k=1}^{n_0} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx + \frac{1}{2} \int_0^{\sqrt{\frac{2n_0+2}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx \ge \frac{1}{2} \int_{\sqrt{\frac{2n_0+2}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx + \sum_{k=n_0+2}^{\infty} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx. \tag{113}$$

Note again that $n_0 = n_0(\mu, \kappa)$ is a function of μ and κ. We first claim that as $\mu \to \infty$, up to a subsequence, we have

$$\lim_{\mu\to\infty} \frac{n_0}{\mu\kappa} = a, \text{ with } 0 < a < \infty. \tag{114}$$

We will argue by contradiction.

Suppose $a = \infty$, i.e. $n_0 \gg \mu$. Then, for sufficiently large μ, we have $\frac{2k}{\mu\kappa} \ge 1$ for all $k \ge n_0 + 1$. Hence, as $\mu \longrightarrow \infty$, we can estimate the right hand side of (112) in the following way:

$$\begin{aligned}\sum_{k=n_0+1}^{\infty} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx \le \sum_{k=n_0+1}^{\infty} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} xe^{-\frac{x^2}{4}}\, dx = 2\frac{e^{\frac{-2(n_0+1)}{\mu\kappa}}}{1 - e^{-\frac{1}{2\mu\kappa}}} &= \\ = 2e^{\frac{-2(n_0+1)}{\mu\kappa}} (2\mu\kappa + o(\mu)) &= o(\mu).\end{aligned}$$

On the other hand, for sufficiently large μ, since

$$\sum_{k=1}^{n_0-1} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx \ge \sum_{k=1}^{\mu\kappa+1} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx \ge \sum_{k=1}^{\mu\kappa+1} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} xe^{-\frac{x^2}{4}}\, dx = \frac{4}{e}\mu\kappa + o(\mu),$$

we arrive at a contradiction with (112).

Next, suppose $a = 0$, i.e. $n_0 \ll \mu$. In this case, the contradiction with (113) follows from the fact that

$$\sum_{k=1}^{n_0} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx \leq n_0 \int_0^{\sqrt{\frac{2n_0}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx = o(n_0)$$

while

$$\sum_{k=n_0+2}^{\infty} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx \geq \sum_{k=n_0+2}^{2n_0+1} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx \geq n_0 \int_1^{\infty} e^{-\frac{x^2}{4}}\, dx.$$

Thus the claim (114) holds.

It remains to show that $a = 1$. Suppose $a > 1$. Omitting the integer part notation, assume that $n_0 = a\mu\kappa$, then

$$\sum_{k=n_0+1}^{\infty} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx = \sum_{k=\mu\kappa}^{\infty} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx - \sum_{k=\mu\kappa+1}^{a\mu\kappa+1} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx$$

and

$$\sum_{k=1}^{n_0-1} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx = \sum_{k=1}^{\mu\kappa} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx + \sum_{k=\mu\kappa+1}^{a\mu\kappa-1} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx.$$

Hence as $\mu \longrightarrow +\infty$, the equation (112) reads as

$$\begin{aligned}\sum_{k=1}^{\mu\kappa} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx + \sum_{k=\mu\kappa+1}^{a\mu\kappa-1} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx + \sum_{k=\mu\kappa+1}^{a\mu\kappa+1} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}} \\ \leq \sum_{k=\mu\kappa}^{\infty} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx + O(1),\end{aligned}$$

or equivalently,

$$\sum_{k=1}^{\mu\kappa} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\, dx + 2\pi\mu\kappa(a-1) \leq \sum_{k=\mu\kappa}^{\infty} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\, dx + O(1). \tag{115}$$

Thus, in order to obtain a contradiction, it suffices to show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \int_0^{\sqrt{\frac{2k}{n}}} e^{-\frac{x^2}{4}}\, dx = \lim_{n\to\infty} \frac{1}{n} \sum_{k=n}^{\infty} \int_{\sqrt{\frac{2k}{n}}}^{\infty} e^{-\frac{x^2}{4}}\, dx. \tag{116}$$

For this, note that as $n \longrightarrow +\infty$, we have that

$$\frac{1}{n} \sum_{k=1}^{n} \int_0^{\sqrt{\frac{2k}{n}}} e^{-\frac{x^2}{4}}\, dx = RS_{\frac{1}{n}} \left[\int_0^{\sqrt{2x}} e^{-\frac{y^2}{4}}\, dy, [0,1] \right] = \int_0^1 \int_0^{\sqrt{2x}} e^{-\frac{y^2}{4}}\, dy dx + o(1),$$

and

$$\frac{1}{n} \sum_{k=n}^{\infty} \int_{\sqrt{\frac{2k}{n}}}^{\infty} e^{-\frac{x^2}{4}}\, dx = RS_{\frac{1}{n}} \left[\int_{\sqrt{2x}}^{\infty} e^{-\frac{y^2}{4}}\, dy, [1,\infty) \right] = \int_1^{\infty} \int_{\sqrt{2x}}^{\infty} e^{-\frac{y^2}{4}}\, dy dx + o(1).$$

Hence to establish (116), it remains to show that

$$\int_0^1 \int_0^{\sqrt{2x}} e^{-\frac{y^2}{4}}\, dy dx = \int_1^{\infty} \int_{\sqrt{2x}}^{\infty} e^{-\frac{y^2}{4}}\, dy dx. \tag{117}$$

By $\int_0^\infty e^{-\frac{y^2}{4}}dy = \sqrt{\pi}$, we have

$$\int_0^1 \int_0^{\sqrt{2x}} e^{-\frac{y^2}{4}} dydx = \sqrt{\pi} - \int_0^1 \int_{\sqrt{2x}}^\infty e^{-\frac{y^2}{4}} dydx.$$

Thus (117) is equivalent to

$$\int_0^\infty \int_{\sqrt{2x}}^\infty e^{-\frac{y^2}{4}} dydx = \sqrt{\pi} \tag{118}$$

whose validity is verified by,

$$\int_0^\infty \int_{\sqrt{2x}}^\infty e^{-\frac{y^2}{4}} dydx = \int_0^\infty \int_0^{\frac{y^2}{2}} e^{-\frac{y^2}{4}} dxdy = \int_0^\infty \frac{y^2}{2} e^{-\frac{y^2}{4}} dy = \int_0^\infty e^{-\frac{y^2}{4}} dy = \sqrt{\pi}.$$

Consequently, (116) holds, yielding a contradiction with (115) if $a > 1$. The case $a < 1$ is excluded analogously.

Proof of corollary 1. The proof follows by observing that when $\mu\kappa$ is small enough, the left hand side of (35) dominates the right hand side even for $n_0 = 1$. Precisely, if $\mu\kappa \to 0$ and $n_0 = 1$,

$$\int_0^{\sqrt{\frac{2}{\mu\kappa}}} e^{-\frac{x^2}{4}} dx \to \sqrt{\pi}$$

while

$$\sum_{k=1}^\infty \int_{\sqrt{\frac{2k}{\mu\kappa}}}^\infty e^{-\frac{x^2}{4}} dx \le \sum_{k=1}^\infty \int_{\sqrt{\frac{2k}{\mu\kappa}}}^\infty xe^{-\frac{x^2}{4}} dx = \frac{2e^{-\frac{2}{\mu\kappa}}}{1-e^{-\frac{1}{2\mu\kappa}}} \to 0, \mu\kappa \to 0.$$

□

4.5. **Anisotropic curvature-driven motions.** This subsection proves Theorem 4. Without loss of generality, we will assume that $w_{s_{n_0},j_{n_0}}(\tau) = \frac{1}{2}$. The corresponding inequalities for the case $w_{s_{n_0},j_{n_0}}(\tau) < \frac{1}{2}$ or $w_{s_{n_0},j_{n_0}}(\tau) > \frac{1}{2}$ can be obtained analogously. As before, we start by noting that

$$\frac{1}{2} = w_{s_{n_0},j_{n_0}}(\tau) = \sum_{(s'h,j'h)\in\Omega} G_{s'-s_{n_0}}(\alpha)G_{j'-j_{n_0}}(\alpha) = \sum_{(sh+s_{n_0}h,jh+j_{n_0}h)\in\Omega} G_s(\alpha)G_j(\alpha). \tag{119}$$

Similar to (91)-(92), we introduce the following sets:

$$\begin{array}{rcl} I_1^\theta &:=& \{(s,j) : \frac{p}{q}sh < jh < g(sh - s_{n_0}h) - j_{n_0}h\}, \\ I_2^\theta &:=& \{(s,j) : g(sh - s_{n_0}h) - j_{n_0}h < jh < \frac{p}{q}sh\}, \\ I_3^\theta &:=& \{(s,j) : jh < \min\{g(sh - s_{n_0}h) - j_{n_0}h, \frac{p}{q}sh\}\}, \end{array} \tag{120}$$

and

$$\begin{array}{rcl} bI_1^\theta &:=& \{(s,j) : s = \frac{p}{q}j, jh < g(sh - s_{n_0}h) - j_{n_0}h\}, \\ bI_2^\theta &:=& \{(s,j) : s = \frac{p}{q}j, jh \ge g(sh - s_{n_0}h) - j_{n_0}h\}. \end{array} \tag{121}$$

(See Figure 5 for the illustration of the above sets.)

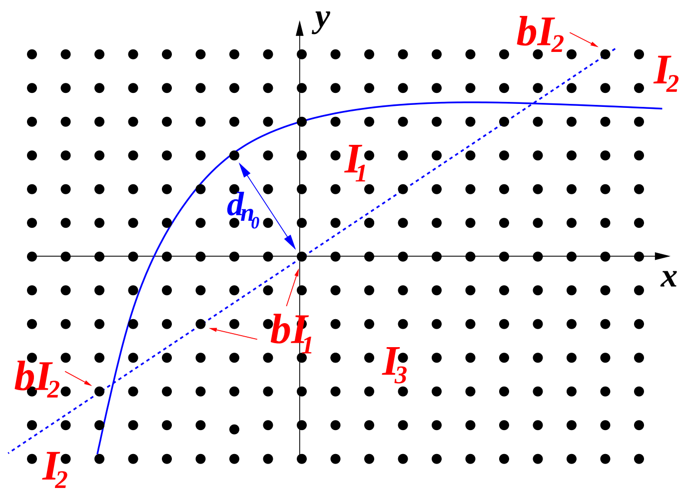

Figure 5. The sets $I_1^\theta,\ I_2^\theta,\ I_3^\theta$ and $bI_1^\theta,\ bI_2^\theta$ for $\tan\theta = \frac{2}{3}$.

With this notation, (119) reads as

$$\sum_{(s,j)\in I_1^\theta\cup I_3^\theta} G_s(\alpha)G_j(\alpha) + \sum_{(s,j)\in bI_1^\theta} G_s(\alpha)G_j(\alpha) = \frac{1}{2}.$$

On the other hand, the properties (46) and (47) of the heat kernel yield that

$$\sum_{(s,j)\in I_2^\theta\cup I_3^\theta} G_s(\alpha)G_j(\alpha) + \frac{1}{2}\sum_{(s,j)\in bI_1^\theta\cup bI_2^\theta} G_s(\alpha)G_j(\alpha) = \frac{1}{2}.$$

Thus

$$\begin{aligned}\sum_{(s,j)\in I_1^\theta} G_s(\alpha)G_j(\alpha) + \frac{1}{2}\sum_{(s,j)\in bI_1^\theta} G_s(\alpha)G_j(\alpha)&\\ = \frac{1}{2}\sum_{(s,j)\in bI_2^\theta} G_s(\alpha)G_j(\alpha) + \sum_{(s,j)\in I_2^\theta} G_s(\alpha)G_j(\alpha).&\end{aligned} \quad (122)$$

A particular technical difficulty associated with the analysis of (122) stems from the fact that, for given $h > 0$, an arbitrary line whose tangent is different from $\tan\theta$, intersects I_1^θ at most at a bounded (i.e. $O(1)$) number of points. Therefore, if we choose a natural summation order along the vertical and horizontal grid lines, when describing the sum $\sum_{(s,j)\in I_1^\theta} G_s(\alpha)G_j(\alpha)$, we cannot apply the asymptotic analysis described above and used in the proof of Theorem 2, i.e. (55). However, there are finitely many lines at angle θ, which intersect I_1^θ at $O\left(\frac{1}{\sqrt{h}}\right)$ number of points. Specifically, those are the lines at angle θ that pass through the points (s_i, j_i) for $1 \le i \le n_0 - 1$. Therefore, we will perform the asymptotic analysis only along those lines.

We now proceed with asymptotic expansions of the terms in (122).

Step I: analysis of $\sum_{(s,j)\in bI_1^\theta}$ and $\sum_{(s,j)\in bI_2^\theta}$. First note that

$$\sum_{(s,j)\in bI_1^\theta} G_s(\alpha)G_j(\alpha) = \sum_{s=s_{n_0}^-}^{s_{n_0}^+} G_{sq}(\alpha)G_{sp}(\alpha)$$

where, omitting the integer part notation,

$$s_{n_0}^{\pm} = \pm\sqrt{\frac{2}{h\kappa}}\frac{\sqrt{ps_{n_0}-qj_{n_0}}}{(p^2+q^2)^{3/4}} - \frac{s_{n_0}}{q}.$$

We next perform the same rescaling as in (100):

$$\sum_{s=s_{n_0}^-}^{s_{n_0}^+} G_{sq}(\alpha)G_{sp}(\alpha) = \int_{s_{n_0}^-}^{s_{n_0}^+} G_{[s]q}(\alpha)G_{[s]p}(\alpha)ds$$

$$= \frac{\sqrt{\tau}}{h}\int_{\frac{h}{\sqrt{\tau}}s_{n_0}^-}^{\frac{h}{\sqrt{\tau}}s_{n_0}^+} G_{\left[\frac{\sqrt{\tau}}{h}x\right]q}(\alpha)G_{\left[\frac{\sqrt{\tau}}{h}x\right]p}(\alpha)dx.$$

Note that for $t=\mu h$

$$s_{n_0}^{\pm}\frac{h}{\sqrt{\tau}} = \pm\sqrt{\frac{2}{\mu\kappa}}\frac{\sqrt{ps_{n_0}-qj_{n_0}}}{(p^2+q^2)^{3/4}} + \frac{s_{n_0}\sqrt{h}}{q\sqrt{\mu}} = \pm\sqrt{\frac{2d_{n_0}}{\mu\kappa(p^2+q^2)}} + \frac{s_{n_0}\sqrt{h}}{q\sqrt{\mu}}.$$

Thus using the same reasoning as in the proof of Theorem 2, in particular equation (101), we have,

$$\sum_{s=s_{n_0}^-}^{s_{n_0}^+} G_{sq}(\alpha)G_{sp}(\alpha) = \frac{2}{4\pi}\frac{\sqrt{h}}{\sqrt{\mu}}\int_0^{\sqrt{\frac{2d_{n_0}}{\mu\kappa(p^2+q^2)}}} e^{-\frac{(p^2+q^2)x^2}{4}}dx + O(h\ln h)$$

$$= \frac{2}{4\pi}\frac{\sqrt{h}}{\sqrt{\mu}}\frac{1}{\sqrt{p^2+q^2}}\int_0^{\sqrt{\frac{2d_{n_0}}{\mu\kappa}}} e^{-\frac{x^2}{4}}dx + O(h\ln h). \tag{123}$$

Similarly, we also have

$$\sum_{(s,j)\in bI_2^\theta} G_s(\alpha)G_j(\alpha) = \sum_{s=-\infty}^{s_{n_0}^- -1} G_{sq}(\alpha)G_{sp}(\alpha) + \sum_{s=s_{n_0}^+ +1}^{\infty} G_{sq}(\alpha)G_{sp}(\alpha)$$

$$= \frac{2}{4\pi}\frac{\sqrt{h}}{\sqrt{\mu}}\frac{1}{\sqrt{p^2+q^2}}\int_{\sqrt{\frac{2d_{n_0}}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}dx + O(h). \tag{124}$$

Step II: analysis of $\sum_{(s,j)\in I_1^\theta}$. Note that

$$\sum_{(s,j)\in I_1^\theta} G_s(\alpha)G_j(\alpha) = \sum_{i=1}^{n_0-1}\sum_{s=s_i^-}^{s_i^+} G_{s_i-s_{n_0}+sq}(\alpha)G_{j_i-j_{n_0}+sp}(\alpha),$$

where

$$s_i^{\pm} = \pm\sqrt{\frac{2}{h\kappa}}\frac{\sqrt{ps_i-qj_i}}{(p^2+q^2)^{3/4}} - \frac{s_i}{q}.$$

Proceeding as in **Step I**, for every i, we obtain,

$$\sum_{s=s_i^-}^{s_i^+} G_{s_i-s_{n_0}+sq}(\alpha)G_{j_i-j_{n_0}+sp}(\alpha)$$
$$= \frac{\sqrt{\tau}}{h}\int_{\frac{h}{\sqrt{\tau}}s_i^-}^{\frac{h}{\sqrt{\tau}}s_i^+} G_{\left[\frac{\sqrt{\tau}}{h}x\right]q+s_i-s_{n_0}}(\alpha)G_{\left[\frac{\sqrt{\tau}}{h}x\right]p+j_i-j_{n_0}}(\alpha)dx.$$

Since for $\tau = \mu h$, we have

$$\begin{aligned}
\frac{\sqrt{\tau}}{h}xp + j_i - j_{n_0} &= \frac{\sqrt{\tau}}{h}\left(xp - \frac{\sqrt{h}}{\sqrt{\mu}}(j_i - j_{n_0})\right),\\
\frac{\sqrt{\tau}}{h}xq + s_i - s_{n_0} &= \frac{\sqrt{\tau}}{h}\left(xq - \frac{\sqrt{h}}{\sqrt{\mu}}(s_i - s_{n_0})\right),\\
\text{and}\quad s_i^{\pm}\frac{h}{\sqrt{\tau}} &= \pm\sqrt{\frac{2}{\mu\kappa}}\frac{\sqrt{ps_i - qj_i}}{(p^2+q^2)^{3/4}} + \frac{s_i\sqrt{h}}{q\sqrt{\mu}} = \pm\sqrt{\frac{2d_i}{\mu\kappa(p^2+q^2)}} + \frac{s_i\sqrt{h}}{q\sqrt{\mu}}.
\end{aligned}$$

Hence

$$\sum_{s=s_i^-}^{s_i^+} G_{s_i-s_{n_0}+sq}(\alpha)G_{j_i-j_{n_0}+sp}(\alpha)$$
$$= \frac{2}{4\pi}\frac{\sqrt{h}}{\sqrt{\mu}}\int_0^{\sqrt{\frac{2d_i}{\mu\kappa(p^2+q^2)}}} e^{-\frac{p^2\left(x-\frac{\sqrt{h}}{\sqrt{\mu}}(j_i-j_{n_0})\right)^2}{4}} e^{-\frac{q^2\left(x-\frac{\sqrt{h}}{\sqrt{\mu}}(s_i-s_{n_0})\right)^2}{4}} dx + O(h\ln h). \tag{125}$$

Finally, using the expansion $e^{-(x+a\sqrt{h})^2} = e^{-x^2} - 2ax\sqrt{h}e^{-x^2} + o(\sqrt{h})$, and the change of variable $y \to \sqrt{p^2+q^2}x$, (125) yields

$$\sum_{s=s_i^-}^{s_i^+} G_{s_i-s_{n_0}+sq}(\alpha)G_{j_i-j_{n_0}+sp}(\alpha) = \frac{2}{4\pi}\frac{\sqrt{h}}{\sqrt{\mu}}\frac{1}{\sqrt{p^2+q^2}}\int_0^{\sqrt{\frac{2d_i}{\mu\kappa}}} e^{-\frac{x^2}{4}}dx. \tag{126}$$

Therefore

$$\sum_{(s,j)\in I_1^\theta} G_s(\alpha)G_j(\alpha) = \frac{2}{4\pi}\frac{\sqrt{h}}{\sqrt{\mu}}\frac{1}{\sqrt{p^2+q^2}}\sum_{i=1}^{n_0-1}\int_0^{\sqrt{\frac{2d_i}{\mu\kappa}}} e^{-\frac{x^2}{4}}dx + o(\sqrt{h}). \tag{127}$$

Step III: analysis of $\sum_{(s,j)\in I_2^\theta}$. We use the same notation and reasoning as in **Step II**. Note that if $p \neq 0$, the asymptotic expansion of this term can be obtained somewhat more easily than the one in Theorem 2 (where $p = 0$) since there is no need to establish the upper and lower bounds in this case. Hence,

$$\sum_{(s,j)\in I_2^\theta} G_s(\alpha)G_j(\alpha) = 2\sum_{i=n_0+1}^{\infty}\sum_{s=s_i^+}^{\infty} G_{s_i-s_{n_0}+sq}(\alpha)G_{j_i-j_{n_0}+sp}(\alpha) \tag{128}$$
$$= \frac{2}{4\pi}\frac{\sqrt{h}}{\sqrt{\mu}}\frac{1}{\sqrt{p^2+q^2}}\sum_{i=n_0+1}^{\infty}\int_{\sqrt{\frac{2d_i}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}dx + o(\sqrt{h}).$$

The above steps imply that, for sufficiently small h, the equality (122) can be true if and only if the coefficients in front of the leading order terms (i.e. $\sqrt{h}$ terms) match. This directly implies (40) and thus completes the proof of Theorem 4.

5. **Numerical experiments.**

We have implemented the scheme (7) numerically for various initial data Ω_0 and parameters $\tau = \triangle t$ and $h = \triangle x$. The simulation is performed using Matlab and the discrete heat equation is solved using Fast Fourier Transform. Our results are presented in this section.

We start with the evolution of a circle and a dumpbell-shaped domain. In Figures 6 and 7, the blue and red contours indicate the initial and final configurations. Note that the evolution picture for the circle does not have the red contour since the circle vanishes before the end of the simulation period. Note also that for the dumpbell evolution, for small time, the flat part of the connecting neck does not move as the curvature there is zero which is below the threshold value given by Corollary 1. Thus the strong comparison principle does not hold in the critical case.

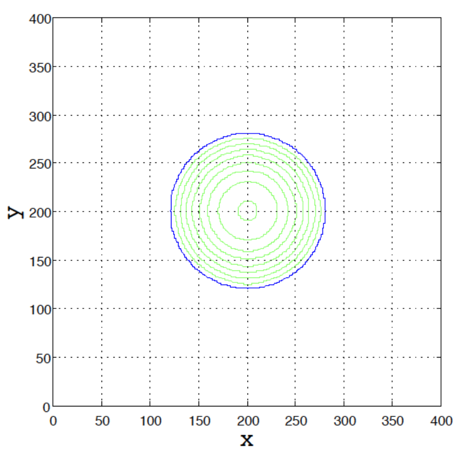

Figure 6: evolution of a circle.

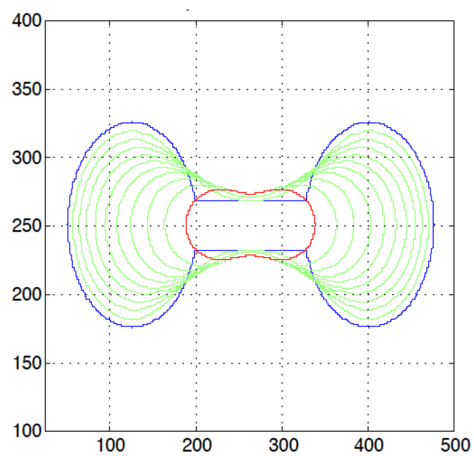

Figure 7: evolution of a dumpbell.

Next we present in more detail the motion of a circle, in particular the dependence on the underlying parameters. The figures below show how the radius of a circle changes for different values of the parameter h and μ.

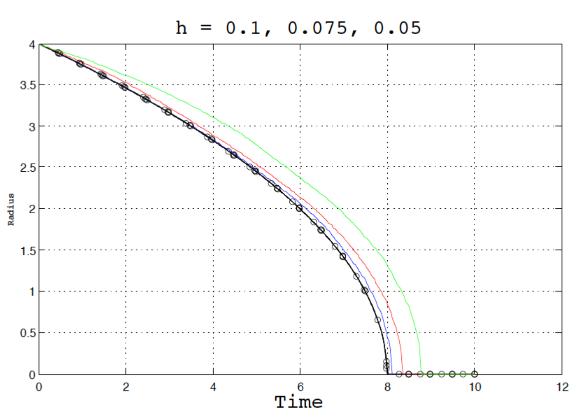

Figure 8: evolution of the circle radius with different values of h ($h = 0.1$ (green); $h = 0.075$ (red); $h = 0.05$ (blue)). They indicate that upon decreasing the grid size h, the evolution converges to the exact solution (given by the black curve).

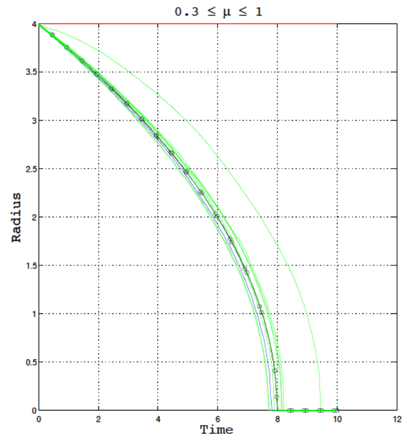

Figure 9: evolution of the circle radius with different values of μ. The black curve is the exact mean curature motion. For μ large (close to 1), the evolution is close to the exact solution. The red curve shows that the radius does not change indicating that $\mu = 0.3$ is small enough that the whole evolution is completely pinned (Corollary 1).

We next investigate the anisotropy of the velocity function. We first present the result for the evolution of a long rectangle tilted at 45°.

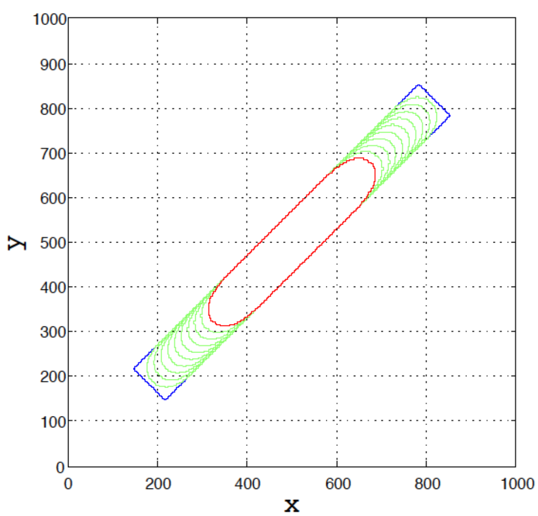

Figure 10: snap shots of the motion of a rectangle tilted at 45°. Note that after some initial transient, the motion of the short sides are roughly described by a translational invariant solution (which is the exact solution for an infinitely long rectangle).

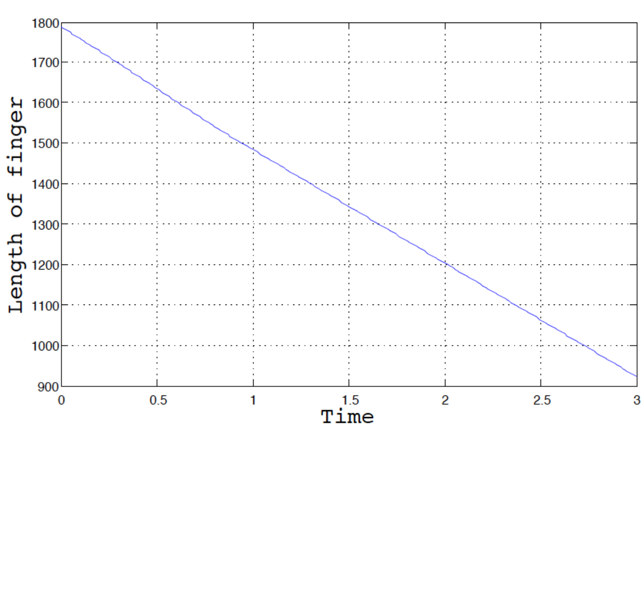

Figure 11: length of the rectangle vs time. It clearly indicates a linear relationship with time confirming the translational invariant motion.

We conclude the section by computing the velocity of short sides of the long rectangle domain for a range of tilt angles. For each tilt angle, five equally spaced values of μ between 0.5 and 1 are performed. The results are illustrated below.

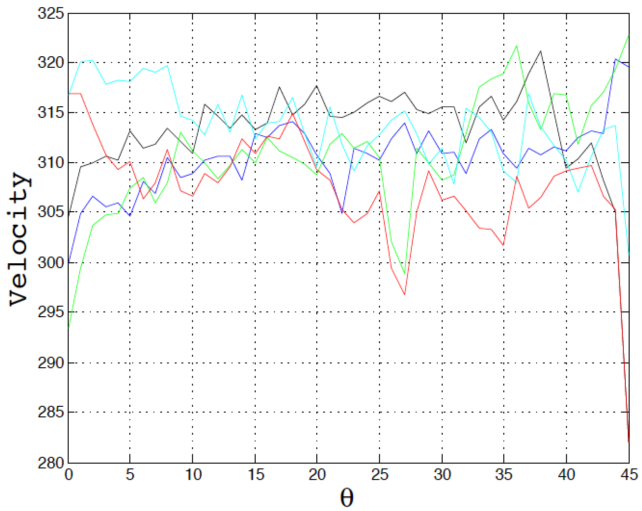

Figure 12: Velocities for tilt angles $\theta = 0°, 1°, \ldots, 45°$ and $\mu = 1$ (black), $\mu = 0.875$ (blue), $\mu = 0.75$ (green), $\mu = 0.625$ (cyan), $\mu = 0.5$ (red).

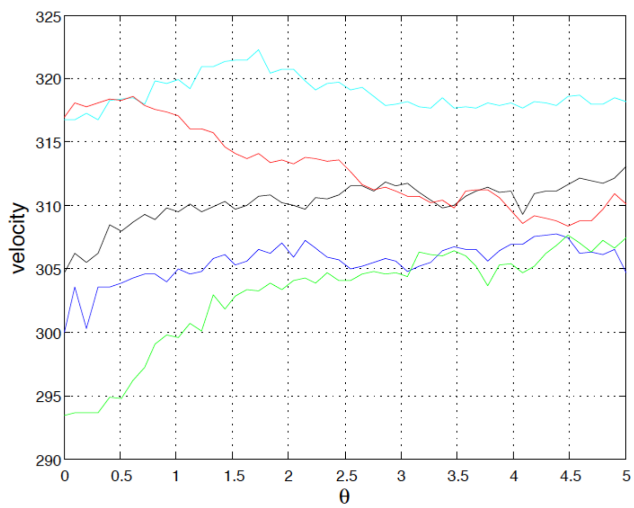

Figure 13: Velocities for tilt angles θ ranging from $0°$ to $5°$. $\mu = 1$ (black), $\mu = 0.875$ (blue), $\mu = 0.75$ (green), $\mu = 0.625$ (cyan), $\mu = 0.5$ (red). The increment in θ is $0.1°$.

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