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RENORMALIZED GINZBURG-LANDAU ENERGY AND LOCATION OF NEAR BOUNDARY VORTICES

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Abstract. We consider the location of near boundary vortices which arise in the study of minimizing sequences of Ginzburg-Landau functional with *degree boundary condition*. As the problem is not well-posed — minimizers do not exist, we consider a regularized problem which corresponds physically to the presence of a superconducting layer at the boundary. The study of this formulation in which minimizers now do exist, is linked to the analysis of a version of renormalized energy. As the layer width decreases to zero, we show that the vortices of any minimizer converge to a point of the boundary with maximum curvature. This appears to be the first such result for complex-valued Ginzburg-Landau type problems.

1. **Introduction.**

In this paper, we consider the Ginzburg-Landau (GL) functional (1) that has been the subject of many studies since the celebrated monograph [4] (see also references therein):

$$\mathcal{E}(u) = \int_\Omega \frac{1}{2}\,|\nabla u|^2 + \frac{1}{4\epsilon^2}(1-|u|^2)^2\,dx. \tag{1}$$

In the above, u is a complex-valued function, Ω is for simplicity, a bounded, smooth and simply connected domain in $\mathbb{R}^2$. This functional arises in the studies of superconductivity. It is a simplified version of the physical model due to the absence of magnetic field in the functional. Nevertheless, as it is noted in [4, p. xviii] that, "... it is striking to see that the degree $\deg(g, \partial\Omega)$ on the boundary condition creates the same quantized vortices as a magnetic field in type-II superconductors or as an

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angular rotation in superfluids". This motivates us to consider the minimizers of (1) with the following degree boundary condition:

$$|u| = 1 \ \text{ on } \partial\Omega \ \text{ and } \ \deg(u, \partial\Omega) = d \tag{2}$$

where d is a positive integer. The appearance of vortices and their interactions with each other and with the domain geometry are important aspects in the study of (1). For the ease of presentation, here we take the first intuitive definition of a *vortex* to be a point z such that $u(z) = 0$. More precise definition will come later (see pages 4 and 8).

The above setting, though simply stated, already leads to some interesting phenomena. It was shown in [6] that any minimizing sequence $\{u_k\}_{k\geq 1}$ of (1) in class (2) develops vortices $\{z_k\}_{k\geq 1}$ which approach the boundary $\partial\Omega$ as $k \longrightarrow \infty$. However, the sequence $\{u_k\}_{k\geq 1}$ does not converge to actual minimizers. This implies that minimizers of (1) in class (2) *do not exist*. More precisely, the degree boundary condition (2) is *not closed* under weak-H^1 convergence. Therefore the main questions in the analysis of problem (1) consist of the asymptotic locations of the vortices z_k and the behaviors of u_k near z_k – such information constitutes the asymptotic "profile" of u_k. The latter issue was resolved in [6] whereas the former remains open and it is the subject of the present work.

We note that even the formulation of the question of vortex location is not as obvious as one may think. Indeed, the usual definition of minimizing sequence does not prescribe the limiting location of the vortices. As a simple example illustrates, for any given point on the boundary $\partial\Omega$, one can construct a minimizing sequence such that whose vortices converge to that point. This example together with the asymptotic profile of any minimizing sequence and the non-existence of minimizer will be briefly explained in **Section 1.1**.

There are several ways to characterize the limiting location of the vortices. For example, one can introduce the concept of an *optimal minimizing sequence* whose vortices have the best (optimal) *rate of convergence* to the boundary. A reasonable question is to ask, "What is the best rate of convergent of the energy to the minimum value as a function of the distance of the vortex to the boundary?" This heuristic concept can be formalized as follows: for any given number $\rho > 0$, find a function u_ρ which minimizes (1) from class (2) with an additional constraint that u_ρ has d vortices but with distances of at least ρ from the boundary $\partial\Omega$. Thus given $\rho > 0$, the following are plausible formulations of minimization problems:

- minimize the functional $\mathcal{E}$ over u satisfying (2) and with all possible vortices with distances (at least) ρ from the boundary:

$$\min_{\left\{u;\ z^i_\rho:\mathrm{Dist}(z^i_\rho,\partial\Omega)\geq\rho,\ i=1,2,\ldots d\right\}} \Big\{\mathcal{E}(u):\ u \text{ has vortices at the } z^i_\rho\text{'s.}\Big\}, \tag{3}$$

  or

$$\min_{\left\{u;\ z^i_\rho:\mathrm{Dist}(z^i_\rho,\partial\Omega)\geq\rho,\ i=1,2,\ldots d\right\}} \Big\{\mathcal{E}(u):\ u \text{ has vortices } \textit{only} \text{ at the } z^i_\rho\text{'s.}\Big\} \tag{4}$$

- minimize the functional $\mathcal{E}$ over all u satisfying (2) such that u has no vortices with distances (at least) ρ from the boundary:

$$\min \Big\{\mathcal{E}(u):\ u \neq 0 \text{ for all } x \in \Omega \text{ such that } \mathrm{Dist}(x,\partial\Omega) \leq \rho.\Big\}. \tag{5}$$

However, formulations (3) and (4) are not well-defined as the problem of minimizing (1) with prescribed vortex location is ill-posed in the function space $H^1(\Omega)$. Even though formulation (5) seems reasonable, but it will lead to minimizers solving the Ginzburg-Landau with some non-trivial boundary conditions in the interior, in particular on the set $\{x : \mathrm{dist}(x, \partial\Omega) = \rho\}$. Therefore, in order to keep the problem technically tractable so as to concentrate on the key issue, we impose an additional constraint on u_ρ which corresponds physically to the presence of an ideal superconducting layer near the boundary. Namely find u_ρ which minimizes (1) in class (2) and is $\mathbb{S}^1$-valued in the boundary stripe D_ρ, where

$$D_\rho = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \le \rho\}\,. \tag{6}$$

The above constraint automatically implies that u_ρ has no vortices within distance ρ from the boundary.

Next we formally introduce the mathematical set-up of the problem. Consider the partition $\Omega = \Omega_\rho \bigcup D_\rho$ (see Figure 1) and the function space:

$$\mathcal{J}_\rho = \left\{ u \in H^1(\Omega, \mathbb{C}) \bigcap H^1(D_\rho, \mathbb{S}^1),\ \ \deg\,(u, \partial\Omega) = d \right\}. \tag{7}$$

Then the regularized problem is:

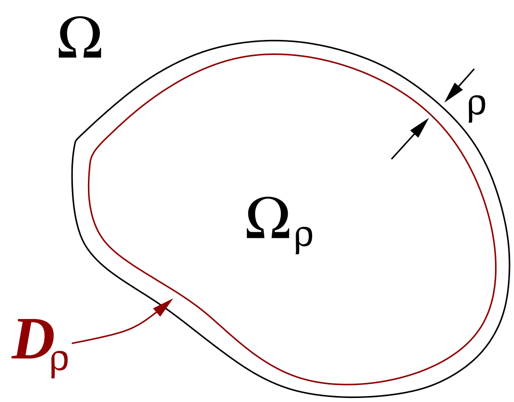

FIGURE 1. Incorporation of a "super-conducting" boundary layer.

$$\min \left\{ \mathcal{E}^\rho_\epsilon(u) = \int_\Omega \frac{1}{2}\,|\nabla u|^2 + \frac{1}{4\epsilon^2}(1 - |u|^2)^2\, d^2x, \quad u \in \mathcal{J}_\rho \right\}. \tag{8}$$

It is easy to show that for each given ϵ and ρ, minimizer *does exist* (see **Section 2**). But before stating the technical results, it is convenient to give some remarks about the notion of *degree* and *vortices* used in this paper.

Degree. Note that $\deg(u, \partial\Omega)$ is well defined for $H^1(\Omega)$ functions on smooth domain Ω due to the fact that each element of $H^{\frac{1}{2}}(\partial\Omega)$ (trace of $H^1(\Omega)$ on $\partial\Omega$) has a well-defined topological degree (or winding number) [5]. For reader's convenience, we explain how this is done as follows: every function $g \in H^{\frac{1}{2}}(\partial\Omega)$ is the strong $H^{\frac{1}{2}}$-limit of a sequence $(g_n) \subset C^\infty(\partial\Omega, S^1)$. Each g_n has a degree with respect to the counter-clockwise orientation of $\partial\Omega$ given, e.g. by the classical formula:

$$\deg(g_n, \partial\Omega) = \frac{1}{2\pi} \int_{\partial\Omega} g_n \times g_{n,\tau}.$$

Then $\lim_n \deg(g_n, \Gamma)$ can be shown to exist and the degree of g is thus defined as $\lim_n \deg(g_n, \partial\Omega)$. Note that this formula is still valid for arbitrary function in $H^{\frac{1}{2}}(\partial\Omega)$ if we interpret the integral by $H^{\frac{1}{2}}$-$H^{-\frac{1}{2}}$ duality.

Vortex. In principle, the most intuitive and physical definition of a vortex is a point such that the complex phase function u takes on the value 0. However, for $\epsilon > 0$, it is not a trivial matter to prove that the *number of zeros* of u which minimizes the functional (1) in any admissible space is the same as the degree of the boundary data for u. Such a statement does exist for the Dirchlet problem for $\epsilon \ll 1$ (see [4, Theorem IX.1] and [2]). The paper [6] which motivates this current work also proves the uniqueness of vortex (with $d = 1$) for any (suitably modified) *minimizing sequences* for (1) in class (2). We believe these statements are true in our setting. But in order not to dwell into unnecessary technicality, we interpret in this paper a "vortex a_ϵ" of u_ϵ as the center a_ϵ of a disk such that the Ginzburg-Landau energy concentrates in the following form:

$$\int_{B_\sigma(a_\epsilon)} \frac{1}{2}\left|\nabla u_\epsilon\right|^2 + \frac{1}{\epsilon^2}\left(1 - |u_\epsilon|^2\right)^2 \geq \pi \ln \frac{\sigma}{\epsilon} \tag{9}$$

for any $\sigma \ll 1$ and $\epsilon \longrightarrow 0$. Note that both a_ϵ and u_ϵ can depend on ϵ. This definition is inspired by the work of [9]. It is very intuitive and suitable for our current setting. Compared with the usual notion of "bad disk" in the sense of [4, IV p. 49] which applies to *solutions* of the Ginzburg-Landau equation, very little regularity and estimates are required of u_ϵ other than an obvious energy bound. See 8 for precise description.

With the above definition of a vortex, our ultimate goal is to study the limiting locations a_* of the vortices a_ϵ as $\rho \longrightarrow 0$ with ϵ fixed (and also the convergence of u_ϵ *away* from a_*). However, even a simpler version of the above problem corresponding to small ϵ has not been solved. We thus consider the limit problem as $\epsilon \longrightarrow 0$ *first* with ρ *fixed* and *then* study the asymptotic behavior of vortices as $\rho \longrightarrow 0$. It turns out that this simplified problem already provides non-trivial results on the limiting location of the vortices on the boundary.

We outline the contents of the paper here. The limiting vortex location will be described in terms of a *renormalized energy* $W(a_1, \ldots a_d)$ for the $\mathbb{S}^1$-problem introduced in **Section 2**. (Here a_i's are the locations of the vortices.) **Theorem 2.1** establishes a connection between the functional (1) with the finite dimensional functional W as $\epsilon \longrightarrow 0$. When $d = 1$ (single vortex case), the vortex goes to the boundary point of the *maximum curvature*. If there are several points of the maximum curvature, the vortex will simply go to one of them (**Theorem 3.1**). Our analysis does not tell us which one it will go to. We believe this information can be obtained from the higher order terms in the asymptotic expansion of W but we do not consider them in this paper. For $d > 1$ (multiple vortex case), the vortices will similarly go to boundary points with maximum curvature (**Theorem 4.1**). Furthermore, for $d = 2$, and when there are at least two points with maximum curvature, the vortices must converge to *distinct* locations (**Proposition 1**). Our results seem to be the first in the complex valued case that relates the vortex location and the curvature information of the domain boundary. For scalar valued semilinear elliptic problems, there are many works that has established such a connection (see for example [12]).

We conclude this introductory section by mentioning some works on boundary vortices in Ginzburg-Landau type problems. The work [10] studies boundary vortices in a scalar problem that models magnetic thin films. The work [1] considers (1) with Dirichlet data that vanishes at some points resulting in the appearance of boundary vortices. Our problem falls into the category of semi-stiff boundary conditions — Dirichlet and Neumann conditions for the modulus $|u|$ and phase φ of $u = |u|e^{i\varphi}$ on $\partial\Omega$ respectively. This boundary condition has been studied in [8, 6, 7].

1.1. **Near boundary vortices in minimizing sequences.** For completeness, we briefly describes here the concept and properties of minimizing sequence for (1) in class (2) for $\epsilon \ll 1$. The material is taken mainly from [6]. For simplicity, we concentrate on the case $d = 1$.

Non-existence of minimizer. First we show that the inf of $\mathcal{E}$ in class (2) is π. This follows directly by integrating by parts:

$$\frac{1}{2}\int_\Omega |\nabla u|^2 \geq \int_\Omega \mathrm{Jac}(u) = \frac{1}{2}\int_{\partial\Omega} u \wedge \frac{\partial u}{\partial \tau} = \pi \deg(u, \partial\Omega) = \pi.$$

In the next paragraph, the existence of a minimizing sequence approaching the above value will be constructed. On the other hand, there is no u in class (2) attaining the value π. For otherwise,

$$\pi = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\epsilon^2}(1 - |u|^2)^2 \geq \pi + \frac{1}{4\epsilon^2}\int_\Omega (1 - |u|^2)^2$$

so that u is a.e. $\mathbb{S}^1$-valued. But this leads to $\deg(u, \partial\Omega) = 0$ contradicting the degree boundary condition.

Arbitrary vortex limit point. Here we show that there exist minimizing sequences with energy approaching the infinmum value π and that their vortices can converge to any point on the boundary of Ω. This indicates that the usual concept of minimizing sequence is simply not sufficient to pin down the location of the limit. The construction of the minimizing sequence is done by means of a linear fractional transformation.

Let Ψ be a Riemann map from Ω to the disk of radius one, $\mathbb{D}_1$. In addition, let $\alpha \in \mathbb{D}_1$. Now consider the following transformation:

$$V_\alpha : \mathbb{D}_1 \longrightarrow \mathbb{D}_1, \quad V_\alpha(\omega) = \frac{\omega - \alpha}{1 - \bar{\alpha}\omega}.$$

Then the function u defined by $u(x) = \dfrac{\Psi(x) - \alpha}{1 - \bar{\alpha}\Psi(x)}$ satisfies $u(\Psi^{-1}(\alpha)) = 0$. Furthermore, by conformal invariant of the Dirichlet integral, it holds that

$$\int_\Omega \frac{1}{2}|\nabla u|^2 \, dx = \int_\Omega \frac{1}{2}|\nabla V_\alpha(\omega)|^2 \, d\omega = \pi.$$

(In the above, we have also used the fact that for any analytic function u, $|\nabla u|^2 = 2\mathrm{Jac}(u)$.) Now take any sequence $\alpha_k \longrightarrow \partial\mathbb{D}_1$ and define $u_k = V_{\alpha_k} \circ \Psi$. Then $\Psi^{-1}(\alpha_k) \longrightarrow \partial\Omega$. It is also easy to see that $|V_{\alpha_k}| \longrightarrow 1$ uniformly in $\mathbb{C}^0_{\mathrm{loc}}(\mathbb{D}_1)$. Hence the Dominated Convergence Theorem gives:

$$\int_\Omega \frac{1}{4\epsilon^2}(1 - |u_k|^2)^2 \, dx = \int_\Omega \frac{1}{4\epsilon^2}(1 - |V_{\alpha_k}|^2)^2 \mathrm{Jac}\Psi \, d\omega \longrightarrow 0.$$

The above construction shows that as $k \longrightarrow \infty$: (i) $\mathcal{E}_\epsilon(u_k) \longrightarrow \pi$ and (ii) the vortices of u_k can converge to any point on the boundary $\partial\Omega$.

Asymptotic profile of a minimizing sequence near its vortex. The essential content [6, Theorem 4(b)] states that any minimizing sequence $\{u_k\}_{k\geq 1}$ (suitably modified) converges to a constant value of modulus one in any compact sub-domain of Ω and for large k, u_k has only one vortex z_k and that z_k converges to the boundary. Furthermore, the behavior of u_k near z_k can be described as follows. Let the domain Ω be mapped onto the complex unit disk $\mathbb{D}_1$ by appropriate one-to-one conformal maps $\{\Phi_k\}_{k\geq 1}$. These maps have the property that $\Phi_k(z_k) = 0$ and $\Phi_k(\Pi(z_k)) = 1$ where $\Pi(z_k)$ is the closest point to z_k on $\partial\Omega$. Then the new function $v_k = u_k \circ \Phi_k^{-1} : \mathbb{D}_1 \longrightarrow \mathbb{C}$, which satisfies $v_k(0) = 0$, *converges to the identity map* of $\mathbb{D}_1$ in $C^1_{\mathrm{loc}}(\mathbb{D}_1)$. (See Figure 2.)

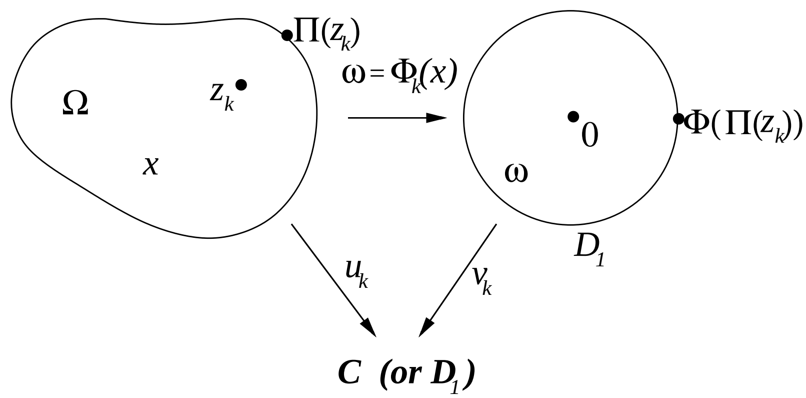

FIGURE 2. Conformal transformation to move the vortex to the origin.

Hence formally, for some fixed Riemann Map Ψ from Ω to $\mathbb{D}_1$, we have as $k \longrightarrow \infty$ that

$$u_k(x) \approx \Phi_k(x) = \frac{\Psi(x) - \alpha_k}{1 - \overline{\alpha_k}\Psi(x)} \quad \text{for } x \text{ close to } \Psi^{-1}(\alpha_k).$$

Uniqueness of Vortex. The previous statement can also be used to prove the *uniqueness* of vortex for any minimizing sequence [6, pp 96, Step 6]. As near *any* zero of u_k, the profile of u_k roughly equals to the identity map (under some Möbius transformation). Hence the energy of u_k can be explicitly computed in an appropriate neighborhood of the zero which gives a value close to π. Thus it would contradict the convergence of the energy to the infimum value of π if there are *multiple* zeros.

In principle, the last two properties just described for a minimizing sequence $\{u_k\}_{k\geq 1}$ for $\mathcal{E}(u)$ can be roughly transferred to the actual *minimizers* $u_{\epsilon,\rho}$ for $\mathcal{E}^\rho_\epsilon$ of problem (8) in the case $\rho \longrightarrow 0$ and $\epsilon \ll 1$. However, in the limit of $\epsilon \longrightarrow 0$, the behavior of $u_{\epsilon,\rho}$ can be even more precisely given by the concept of a *renormalized energy* and *canonical harmonic map*. With this in mind, the location of the vortices can be revealed through the boundary information of harmonic maps.

Now we are ready to present the results and their proofs. For the simplicity of presentation, we start from the case $d = 1$. All the proofs can be readily generalized to the case $d > 1$. Precise statements in the latter situation are given in **Section 4**. In the following, $\mathcal{E}^\rho_\epsilon$ will simply be denoted by $\mathcal{E}_\epsilon$ unless the dependence on ρ is used in the argument.

2. **Asymptotics of $\epsilon \longrightarrow 0$: Renormalized energy W.** In this section, we investigate the limit of problem (8) as $\epsilon \longrightarrow 0$ (keeping ρ fixed). We will show that the limiting problem can be described by means of a renormalized energy.

First we show that problem (8) has a minimizer. Let $\{u_n\}_{n\geq 1} \subset \mathcal{J}_\rho$ be a minimizing sequence. Then there exists $u_* \in H^1(\Omega)$ such that (up to subsequence) $u_n \rightharpoonup u_*$ in $H^1(\Omega)$. Furthermore, $u_n \longrightarrow u_*$ in $L^2(\Omega)$ so that $u_* \in H^1(D_\rho, \mathbb{S}^1)$. It remains to show that $\deg(u_*, \partial\Omega) = d$. This is due to the fact that for $\mathbb{S}^1$-valued maps, deg is preserved under weak convergence in H^1. This can be proved by either directly using the result of [13] or the concept of abdeg (approximate-bulk-degree) introduced in [7, Formula (1.10)]. For self-containedness, we explain the latter approach which is quite elementary.

Introduce the function $V : D_\rho \longrightarrow \mathbb{R}$ which solves the following problem:

$$\begin{aligned} \triangle V &= 0 \quad \text{in } D_\rho, \\ V &= 1 \quad \text{on } \partial\Omega, \\ V &= 0 \quad \text{on } \partial\Omega_\rho. \end{aligned}$$

Then we define:

$$\operatorname{abdeg}(u, D_\rho) = \frac{1}{2\pi} \int_{D_\rho} u \times (\partial_{x_1} V \partial_{x_2} u - \partial_{x_2} V \partial_{x_1} u)\, dx. \tag{10}$$

(The physical interpretation of abdeg is that it is the "average degree" of a vortex in the domain ([7, Formula (1.11)]).) Using integration by parts, we have

$$\operatorname{abdeg}(u, D_\rho) = \frac{1}{2\pi} \int_{\partial\Omega} u \times \frac{\partial u}{\partial \tau}\, ds - \frac{1}{\pi} \int_{D_\rho} V \partial_{x_1} u \times \partial_{x_2} u\, dx.$$

Now for any $\mathbb{S}^1$-valued map u, $\partial_{x_1} u \times \partial_{x_2} u = 0$ a.e. Hence we have that

$$\operatorname{abdeg}(u, D_\rho) = \deg(u, \partial\Omega). \tag{11}$$

It is clear that abdeg is preserved under weak convergence in H^1. Then (11) leads to that $\deg(u_*, \partial\Omega) = \lim_k \deg(u_k, \partial\Omega) = \lim_k \operatorname{abdeg}(u_k, D_\rho) = d$.

2.1. **Renormalized energy.** We introduce here a version of renormalized energy W. We adapt the procedure of [4, Sections I.3, I.4] which deals with Dirichlet boundary condition to our current degree boundary condition (2) with $d = 1$. Consider the following minimization problem:

$$\min \left\{ \frac{1}{2} \int_{\Omega \backslash B_\sigma(a)} |\nabla u|^2\, dx \right\} \tag{12}$$

where $u : \Omega\backslash B_\sigma(a) \longrightarrow \mathbb{S}^1$ satisfying $\deg(u, \partial\Omega) = \deg(u, \partial B_\sigma(a)) = 1$. Let u_σ be a minimizer (the existence of which was proved in [4, I.1]). As $\sigma \longrightarrow 0$, it can be shown that u_σ converges to the *canonical harmonic map with singularity at a*, $u_* : \Omega\backslash\{a\} \longrightarrow \mathbb{S}^1$. In addition, the energy of u_σ, $\int_{\Omega\backslash B_\sigma(a)} |\nabla u_\sigma|^2$ will converge to that of u_* in some renormalized sense. More precisely, let Φ_0, R, W be three functions defined by the following procedure:

$$\triangle \Phi_0 = 2\pi \delta_a \ \text{ in } \Omega \text{ and } \left.\Phi_0\right|_{\partial\Omega} = 0, \tag{13}$$

$$R(x) := \Phi_0(x) - \log|x - a|, \quad \text{in } \Omega, \tag{14}$$

$$W(a) := -\pi R(a). \tag{15}$$

Then we have

$$\frac{1}{2} \int_{\Omega\backslash B_\sigma(a)} |\nabla u_\sigma|^2 = \pi \log \frac{1}{\sigma} + W(a) + O(\sigma), \tag{16}$$

and

$$\frac{1}{2} \int_{\Omega\backslash B_\sigma(a)} |\nabla u_*|^2 = \pi \log \frac{1}{\sigma} + W(a) + O(\sigma^2). \tag{17}$$

The map u_* is constructed by treating Φ_0 as its conjugate phase function [4, I.3].

For comparison and also later use, we write down the renormalized energy for the corresponding variational problem with Dirichlet boundary condition (see [4, Section 4, p. 21] for detail). Given any smooth function $g : \partial\Omega \longrightarrow \mathbb{S}^1$ with $\deg(g, \partial\Omega) = 1$, consider

$$\min\left\{\frac{1}{2}\int_{\Omega\backslash B_\sigma(a)} |\nabla u|^2\, dx\right\} \tag{18}$$

where $u : \Omega\backslash B_\sigma(a) \longrightarrow \mathbb{S}^1$ satisfying $\deg(u, \partial B_\sigma(a)) = 1$ and $u\Big|_{\partial\Omega} = g$. Then the W in (16) and (17) is replaced by W_g which is defined by:

$$\triangle\Phi_0 = 2\pi\delta_a \ \text{ in } \Omega \text{ and } \left.\frac{\partial\Phi_0}{\partial\nu}\right|_{\partial\Omega} = g \times g_\tau, \tag{19}$$

$$R(x) := \Phi_0(x) - \log|x - a|, \quad \text{in } \Omega, \tag{20}$$

$$W_g(a) := -\pi R(a) + \frac{1}{2}\int_{\partial\Omega} \Phi_0\,(g \times g_\tau)\,. \tag{21}$$

In principle, the limiting behavior of the energy functional (1) can also be related to W, in the sense that the "minimizer" u^ϵ and its energy $\mathcal{E}(u^\epsilon)$ will converge to u_* and $\mathcal{E}(u_*)$ (in the renormalized sense). This is rigorously established for the Dirichlet boundary condition [4]. However, as mentioned in **Section 1.1**, the main difficulty here is that the minimizer for $\mathcal{E}$ with (2) *does not* exists. The introduction of the boundary stripe is exactly used to confine the the vortex in the interior of Ω so that it does not escape to the boundary. By means of this procedure, we can then link the limit of $\mathcal{E}$ (with (2)) to (15). But even this seems not completely trivial. Hence we find it worthwhile to formulate and prove it rigorously.

Before stating our main theorem, we briefly describe the key result (for $n = 2$) of [9, Theorem 1.2] which we rely to define the vortices of u_ϵ:

Let $\{u_\epsilon\}_{\epsilon>0} \subset H^1(\Omega, \mathbb{C})$ from class (2) such that for some $r > 0$, $|u| > \frac{1}{2}$ on the set $\{x : Dist(x, \partial\Omega) \leq r\}$. It further satisfies for ϵ small enough that:

$$\mathcal{E}(u_\epsilon) \leq d\pi \ln\frac{1}{\epsilon} + C \tag{22}$$

where C is some fixed constant. Then there exists an σ_0 (depending only on Ω) so that for any $\sigma \leq \sigma_0$ and $\epsilon \leq \epsilon_0(\sigma_0, C)$, there are points $a_{\epsilon,1}, a_{\epsilon,2}, \ldots, a_{\epsilon,m}$ and positive integers $d_1, d_2, \ldots, d_m$ satisfying

$$\sum_i d_i = d, \tag{23}$$

and for all i,

$$\int_{B_\sigma(a_{\epsilon,i})} \frac{1}{2}|\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2}(1 - |u_\epsilon|^2)^2 \geq d_i\pi \ln\frac{\sigma}{\epsilon} - C, \tag{24}$$

$$deg\big(u_\epsilon, \partial B_\sigma(a_{\epsilon,i})\big) = d_i. \tag{25}$$

The points $a_{\epsilon,i}$'s are defined as the "vortices" of u_ϵ. The non-uniqueness of the definition is not a concern in this paper as we are mainly interested in the *limiting location of the $a_{\epsilon,i}$'s* as $\epsilon \longrightarrow 0$. We remark that in [9, Theorem 1.2], a fixed Dirichlet condition is imposed for all u_ϵ. The fact that it can be generalized to functions u_ϵ satisfying $|u_\epsilon| \geq \frac{1}{2}$ on the set $\{x : \mathrm{Dist}(x, \partial\Omega) \leq r\}$ for some fixed $r > 0$ is stated in [9, Eqn. (4.1)]. This is useful to our case as $|u| = 1$ on D_ρ.

With the energy bound (22) and concentration on the balls (24), we have that the energy is *uniformly bounded away from the vortices*:

$$\int_{\Omega\setminus\bigcup_i B_\sigma(a_i)} \frac{1}{2}|\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2}(1-|u_\epsilon|^2)^2 \leq C. \tag{26}$$

This would allow us to derive strong convergence results away from the vortices.

Now we are ready to state our main theorem.

Theorem 2.1 (Renormalized Energy)**.** *Let u_ϵ be a minimizer of $\mathcal{E}_\epsilon^\rho$. Then we have:*

$$\lim_{\epsilon\to 0}\left[\mathcal{E}_\epsilon^\rho(u_\epsilon) - \pi\log\frac{1}{\epsilon}\right] = \min_{a\in\overline{\Omega_\rho}} W(a) + \gamma \tag{27}$$

where γ is some universal constant (see (28)).

Furthermore, the vortex a_ϵ of u_ϵ (up to a subsequence) converges to a point a_* that minimizes the expression in the right hand side of (27). If there is only one such a_* then the stated convergence holds for the whole sequence $\{u_\epsilon\}_{\epsilon>0}$.

Remark 1.

- From **Theorem 3.1**, for $\rho \ll 1$, we can in fact replace the right hand side by $\min_{a\in\partial\Omega_\rho} W(a)+\gamma$.
- The constant γ is given by
$$\lim_{\epsilon\longrightarrow 0}\left[I(\epsilon) - \pi\log(\frac{1}{\epsilon})\right] \tag{28}$$
where $I(\epsilon)$ is the minimum value of $\mathcal{E}$ on the unit ball B_1 with Dirichlet boundary condition $u(x) = x$ on ∂B_1 [4, IX.2].
- With the technique and result used here, we can actually show that the energy $\mathcal{E}_\epsilon^\rho(\cdot) - \pi\log\frac{1}{\epsilon}$ Γ-converges to $W(\cdot)+\gamma$. But for simplicity of presentation, we do not formulate such a statement.

The Theorem is proved by establishing matching upper and lower bounds.

2.2. **Upper bound:** $\limsup_{\epsilon\to 0}\left[\mathcal{E}_\epsilon^\rho(u_\epsilon) - \pi\log\frac{1}{\epsilon}\right] \leq \min_{a\in\overline{\Omega_\rho}} W(a)+\gamma$**.** This is done the by direct construction as in [4, VIII.1]. Let $0<\delta\ll 1$ and $a\in\Omega$ with $\mathrm{Dist}(a,\partial\Omega)\geq \rho+\delta$. Then there is an $\mathbb{S}^1$-valued map v_1 on $\Omega\backslash B_\delta(a)$ such that it solves the problem (12) with boundary conditions:

$$v_1\Big|_{\partial B_\delta(a)}(x) = \frac{x-a}{|x-a|} \quad \text{and} \quad \deg(v_1,\partial\Omega) = 1$$

Such a minimizer exists and and its Dirichlet energy satisfies:

$$\frac{1}{2}\int_{\Omega\backslash B_\delta(a)} |\nabla v_1|^2 = \pi\log\frac{1}{\delta} + W(a) + O(\delta).$$

Inside the ball $B_\delta(a)$, consider v_2 which minimize (1) with Dirichlet boundary condition: $v_2\Big|_{\partial B_\delta(a)}(x) = v_1$. Now set $v = v_1$ on $\Omega\backslash B_\delta(a)$ and $v = v_2$ on $B_\delta(a)$. Then

$$\mathcal{E}(v) = \pi\log\frac{1}{\epsilon} + I\left(\frac{\epsilon}{\delta}\right) + \pi\log\left(\frac{\epsilon}{\delta}\right) + W(a) + O(\delta)$$

Letting $\epsilon\longrightarrow 0$ and making use of (28) and then letting $\delta\longrightarrow 0$ lead to the result.

2.3. **Lower bound:** $\liminf_{\epsilon\to 0}\left[\mathcal{E}^{\rho}_{\epsilon}(u_\epsilon)-\pi\log\frac{1}{\epsilon}\right]\geq \min_{a\in\overline{\Omega_\rho}} W(a)+\gamma$**.**

Step I. Convergence. For all k, we have $u_\epsilon \longrightarrow u_*$ in $C^k_{\text{loc}}(D_\rho)$. This follows from the asymptotic matching upper and lower bounds of the energy. From the previous explicit construction of test function, we have that

$$\mathcal{E}_\epsilon(u_\epsilon)\leq \pi\log\frac{1}{\epsilon}+C \quad \text{for some fixed constant } C. \tag{29}$$

Recall that $|u|=1$ on D_ρ. Then for $\sigma\ll\rho$, the vortex (defined through (24)—(25)) must lie inside $\Omega_{\rho-\sigma}$. Now choose δ such that $\sigma\ll\rho-\delta$. By (26), there is a constant C such that as $\epsilon\longrightarrow 0$, it holds

$$\int_{D_\delta}|\nabla u_\epsilon|^2\,dx\leq C.$$

Since u_ϵ can be expressed as $e^{i\varphi_\epsilon}$ with some multi-valued phase function which solves the Laplace equation, standard regularity results of harmonic function gives that $\varphi_\epsilon\longrightarrow\varphi_*$ in $C^k_{\text{loc}}(D_\delta)$ and hence the desired convergence of u_ϵ holds.

Step II. Intermediate Lower Bound. Let $0<\delta<\rho$. Note that $D_\delta\subset D_\rho$ and $\Omega_\rho\subset\Omega_\delta$.

Let $g^\delta_\epsilon = u_\epsilon\Big|_{\partial\Omega_\delta}$. Consider the following functional with Dirichlet boundary condition:

$$\tilde{\mathcal{E}}^{\delta}_{\epsilon}(v)=\int_{\Omega_\delta}\frac{1}{2}|\nabla v|^2+\frac{1}{4\epsilon^2}(1-|v|^2)^2\,d^2x,\quad v\in H^1(\Omega_\delta,\mathbb{C}),\ \ v\Big|_{\partial\Omega_\delta}=g^\delta_\epsilon. \tag{30}$$

Let $\tilde v^\delta_\epsilon$ be a minimizer of $\tilde{\mathcal{E}}^\delta_\epsilon$. Define:

$$u^\delta_\epsilon=\begin{cases} u_\epsilon & \text{for } x\in D_\delta\\ \tilde v^\delta_\epsilon & \text{for } x\in\Omega_\delta\end{cases} \tag{31}$$

Then automatically, we have

$$\mathcal{E}^\rho_\epsilon(u_\epsilon)\geq\mathcal{E}^\delta_\epsilon(u^\delta_\epsilon).$$

Now by **Step I**, as $\epsilon\longrightarrow 0$, we have $g^\delta_\epsilon\longrightarrow g^\delta$ in C^k for any k. Hence

$$\begin{aligned}\lim_{\epsilon\to 0}\left[\mathcal{E}^\rho_\epsilon(u_\epsilon)-\pi\log\frac{1}{\epsilon}\right] &\geq \liminf_{\epsilon\to 0}\left[\mathcal{E}^\delta_\epsilon(u^\delta_\epsilon)-\pi\log\frac{1}{\epsilon}\right]\\ &\geq \frac{1}{2}\int_{D^\delta}|\nabla u_*|^2+\min_{a\in\overline{\Omega_\delta}}W_{g^\delta}(a)+\gamma\end{aligned} \tag{32}$$

where W_g is the renormalized energy with Dirichlet boundary condition g on $\partial\Omega_\delta$ (21). In the last step of the above, we have invoked the result [4, Lemma VIII.2] for the Dirichlet boundary condition.

Step III. Optimal Lower Bound. However, *obviously*,

$$\frac{1}{2}\int_{D^\delta}|\nabla u_*|^2+\min_{a\in\overline{\Omega^\delta}}W_{g^\delta}(a)\geq\min_{a\in\overline{\Omega^\delta}}W(a). \tag{33}$$

This is because $W(a)$ is the *minimum* of the renormalized energy in the space of $\mathbb{S}^1$-valued map with vortex located at a (see (12), (16), (17)). Hence

$$\liminf_{\epsilon\to 0}\left[\mathcal{E}^\rho_\epsilon(u_\epsilon)-\pi\log\frac{1}{\epsilon}\right]\geq\min_{a\in\overline{\Omega_\delta}}W(a)+\gamma.$$

By letting $\delta \longrightarrow \rho$ in the above, we get:

$$\begin{aligned}\lim_{\epsilon\to 0}\left[\mathcal{E}_\epsilon^\rho(u_\epsilon)-\pi\log\frac{1}{\epsilon}\right] &\geq \lim_{\delta\to\rho}\left[\mathcal{E}_\epsilon^\delta(u_\epsilon^\delta)-\pi\log\frac{1}{\epsilon}\right]\\ &\geq \lim_{\delta\to\rho}\min_{a\in\overline{\Omega_\delta}} W(a)+\gamma=\min_{a\in\overline{\Omega_\rho}} W(a)+\gamma.\end{aligned}$$

where in the above we have used the continuity property of W.

2.4. **Convergence of the vortex of u_ϵ.** Let a_ϵ be the vortex of u_ϵ. Suppose a_ϵ converges to $\tilde{a}$ which is not equal to any of the minimizers a_* of the right hand side of (27). Let $\sigma=\sigma(\rho)$ be some fixed number which is much smaller than $|\tilde{a}-a_*|$ and ρ.

Then Proposition 4.1 from [9] leads to that for ϵ small, we have

$$\int_{B_\sigma(\tilde{a})}\frac{1}{2}|\nabla u|^2+\frac{1}{4\epsilon^2}(1-|u|^2)^2\,dx\geq\pi\log\frac{\sigma}{\epsilon}-C\quad\text{and}\quad\deg(u_\epsilon,\partial B_\sigma(\tilde{a}))=1.$$

Together with the upper bound from **Section 2.2**, we have that

$$\int_{\Omega\setminus B_\sigma(\tilde{a})}\frac{1}{2}|\nabla u|^2+\frac{1}{4\epsilon^2}(1-|u|^2)^2\,dx\leq C(\sigma).$$

From [9, Theorem 1.2], it follows that $u_\epsilon\rightharpoonup u_*$ in $H^{1,2}_{\rm loc}(\overline{\Omega\setminus B_\sigma(\tilde{a})})$ for some $u_*:\Omega\setminus\tilde{a}\longrightarrow\mathbb{S}^1$.

We now improve the above convergence in a neighborhood of $\partial B_\sigma(\tilde{a})$. For this, we apply the same argument for $\frac{\sigma}{2}$ and obtain that

$$\int_{B_\sigma(\tilde{a})\setminus B_{\frac{\sigma}{2}}(\tilde{a})}\frac{1}{2}|\nabla u|^2+\frac{1}{4\epsilon^2}(1-|u|^2)^2\,dx\leq C(\sigma). \tag{34}$$

Now let $\tilde{v}_\epsilon^\sigma$ be a minimizer of (1) on $B_\sigma(\tilde{a})\setminus B_{\frac{\sigma}{2}}(\tilde{a})$ with the following *Dirichlet* boundary conditions:

$$\tilde{v}_\epsilon^\sigma=u_\epsilon\quad\text{on}\quad\partial B_\sigma(\tilde{a})\quad\text{and}\quad\partial B_{\frac{\sigma}{2}}(\tilde{a})$$

and define:

$$\tilde{u}_\epsilon=\begin{cases}u_\epsilon & \text{on}\quad\Omega\setminus\left(B_\sigma(\tilde{a})\setminus B_{\frac{\sigma}{2}}(\tilde{a})\right)\\ \tilde{v}_\epsilon^\sigma & \text{on}\quad B_\sigma(\tilde{a})\setminus B_{\frac{\sigma}{2}}(\tilde{a})\end{cases}$$

It is clear that $\mathcal{E}(u_\epsilon)\geq\mathcal{E}(\tilde{u}_\epsilon)$ on Ω.

By (34), using [11, Theorem 1], the function $\tilde{u}_\epsilon$ is uniformly bounded in $C^k_{\rm loc}\Big(B_\sigma(\tilde{a})\setminus B_{\frac{\sigma}{2}}(\tilde{a})\Big)$ for all k. Combining with [3, Theorem 1], we have as $\epsilon\longrightarrow 0$, $\tilde{u}_\epsilon$ (up to a subsequence) converges to some $\tilde{u}_*$ in C^k_K for any $K\subset\subset B_\sigma(\tilde{a})\setminus B_{\frac{\sigma}{2}}(\tilde{a})$. In particular, $g_\sigma=\lim_\epsilon u_\epsilon\big|_{\partial B_{\frac{3\sigma}{4}}(\tilde{a})}$ exists and belongs to $C^\infty\Big(\partial B_{\frac{3\sigma}{4}}(\tilde{a})\Big)$. Hence,

$$\begin{aligned}&\lim_{\epsilon\to 0}\left[\int_\Omega\frac{1}{2}|\nabla u_\epsilon|^2+\frac{1}{4\epsilon^2}(1-|u_\epsilon|^2)^2\,dx-\pi\log\frac{1}{\epsilon}\right]\\ \geq\ &\lim_{\epsilon\to 0}\left[\int_\Omega\frac{1}{2}|\nabla\tilde{u}_\epsilon|^2+\frac{1}{4\epsilon^2}(1-|\tilde{u}_\epsilon|^2)^2\,dx-\pi\log\frac{1}{\epsilon}\right]\\ \geq\ &\int_{\Omega\setminus B_{\frac{3\sigma}{4}}(\tilde{a})}\frac{1}{2}|\nabla\tilde{u}_*|^2\,dx+\min_{a\in\overline{B_{\frac{3\sigma}{4}}(\tilde{a})}}W_{g_\sigma}(a)+\gamma\end{aligned}$$

(where we have used [4, Lemma VIII.2] again). Similar to (33), we have that

$$\int_{\Omega \setminus B_{\frac{3\sigma}{4}}(\tilde{a})} \frac{1}{2} |\nabla u_*|^2 \, dx + \min_{a \in B_{\frac{3\sigma}{4}}(\tilde{a})} W_{g_\sigma}(a) \geq \min_{a \in B_{\frac{3\sigma}{4}}(\tilde{a})} W(a)$$

which again by the continuity of W, is strictly bigger than $W(a_*)$ for σ small enough. This leads to a contradiction. Hence we must have $a_\epsilon \longrightarrow a_*$.

3. Asymptotics of $W(a)$ as $a \longrightarrow \partial\Omega$.

Here we consider the asymptotic behavior of W as $a \longrightarrow \partial\Omega$. The following is our main result.

Theorem 3.1. *For all $a \in \Omega$, let $\Pi(a)$ be the closest point on the boundary $\partial\Omega$ to a. Then as $a \longrightarrow \partial\Omega$,*

$$W(a) = -\pi \log \frac{1}{2\rho} - \pi \frac{\rho}{2\mathcal{R}(\Pi(a))} + O(\rho^2) \tag{35}$$

where $\rho = dist(a, \partial\Omega)$ and $\mathcal{R}(p)$ is the radius of curvature of a point $p \in \partial\Omega$. (The convention is that $\mathcal{R}$ is positive for the convex part of $\partial\Omega$).

The above shows that as $\rho \longrightarrow 0$, the point a_ρ which minimizes renormalized energy will converge to a boundary point with the highest curvature.

Proof. The Theorem is proved by means of an explicit formula for the renormalized energy. This is done through the construction of a Riemann map.

From the definition of Φ_0 in (13), we see that it is given by $\log|F|$ where F is any one-to-one conformal map which maps Ω onto the unit disk $\mathbb{D}_1$ such that $F(a) = 0$. This map can be constructed from any Riemann map Ψ which maps Ω onto $\mathbb{D}_1$ as follows:

$$F(z) = \frac{\Psi(z) - \Psi(a)}{1 - \overline{\Psi(a)}\Psi(z)}. \tag{36}$$

Hence

$$\Phi_0(z) = \log \left| \frac{\Psi(z) - \Psi(a)}{1 - \overline{\Psi(a)}\Psi(z)} \right| \tag{37}$$

$$\text{and} \quad R(z) = \Phi_0(z) - \log|z - a| = \log \left| \frac{\frac{\Psi(z)-\Psi(a)}{z-a}}{1 - \overline{\Psi(a)}\Psi(z)} \right| \tag{38}$$

so that

$$R(a) = \log \left| \frac{\Psi'(a)}{1 - |\Psi(a)|^2} \right|. \tag{39}$$

To continue, it is well known that the Riemann map can be given by by the Green's function of the Laplacian as follows. Take any $z_* \in \Omega$. Let φ be the solution of the following equation:

$$\triangle \varphi = 2\pi \delta_{z_*} \quad \text{and} \quad \varphi \Big|_{\partial\Omega} = 0.$$

Let ψ be the harmonic conjugate of φ. Then $\Psi(z) = e^{\varphi(z) + i\psi(z)}$. (The Ψ so constructed satisfies $\Psi(z_*) = 0$.) Note the following

$$|\Psi(z)| = e^{\varphi}, \quad \Psi'(z) = e^{\varphi + i\psi}(\varphi_x + i\varphi_x), \quad |\Psi'(z)| = e^{\varphi} |\nabla\varphi| \quad (\text{recall: } \varphi_x = -\psi_y)$$

The above leads to the following representation of $R(a)$:

$$R(a) = \log \left| \frac{e^{\varphi(a)} |\nabla\varphi(a)|}{1 - e^{2\varphi(a)}} \right|. \tag{40}$$

Next we expand the above expression for $\rho \ll 1$. Let ν be the unit outward normal at $\partial\Omega$ at $a_* = \Pi(a)$. Then,

$$
\begin{aligned}
e^{\varphi(a)} &= 1 - \frac{\partial \varphi}{\partial \nu}(a_*)\rho + O(\rho^2);\\
|\nabla\varphi(a)| &= \sqrt{\left(\frac{\partial\varphi(a_*)}{\partial\nu} - \frac{\partial^2\varphi(a_*)}{\partial\nu^2}\rho + O(\rho^2)\right)^2}\\
&= \sqrt{\left(\frac{\partial\varphi(a_*)}{\partial\nu}\right)^2 - 2\frac{\partial\varphi(a_*)}{\partial\nu}\frac{\partial^2\varphi(a_*)}{\partial\nu^2}\rho + O(\rho^2)}\\
&= \frac{\partial\varphi(a_*)}{\partial\nu}\left(1 - \rho\frac{\frac{\partial^2\varphi(a_*)}{\partial\nu^2}}{\frac{\partial\varphi(a_*)}{\partial\nu}}\right) + O(\rho^2) \quad \left(\text{by Hopf Lemma, } \frac{\partial\varphi(a_*)}{\partial\nu} > 0\right);\\
1 - e^{2\varphi(a)} &= -2\varphi(a) - 2\varphi^2(a) + O(\rho^3)\\
&= 2\frac{\partial\varphi(a_*)}{\partial\nu}\rho - \frac{\partial^2\varphi(a_*)}{\partial\nu^2}\rho^2 - 2\left(\frac{\partial\varphi(a_*)}{\partial\nu}\right)^2\rho^2 + O(\rho^3)\\
&= 2\frac{\partial\varphi(a_*)}{\partial\nu}\rho\left(1 - \frac{\rho}{2}\frac{\frac{\partial^2\varphi(a_*)}{\partial\nu^2}}{\frac{\partial\varphi(a_*)}{\partial\nu}} - \frac{\partial\varphi(a_*)}{\partial\nu}\rho + O(\rho^2)\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\log\frac{e^{\varphi(a)}\,|\nabla\varphi(a)|}{(1-e^{2\varphi(a)})} &= \log\frac{1}{2\rho} + \log\left(\frac{\left(1 - \frac{\partial\varphi}{\partial\nu}(a_*)\rho\right)\left(1 - \rho\frac{\frac{\partial^2\varphi(a_*)}{\partial\nu^2}}{\frac{\partial\varphi(a_*)}{\partial\nu}}\right)}{1 - \frac{\rho}{2}\frac{\frac{\partial^2\varphi(a_*)}{\partial\nu^2}}{\frac{\partial\varphi(a_*)}{\partial\nu}} - \frac{\partial\varphi(a_*)}{\partial\nu}\rho} + O(\rho^2)\right)\\
&= \log\frac{1}{2\rho} - \frac{\rho}{2}\frac{\frac{\partial^2\varphi(a_*)}{\partial\nu^2}}{\frac{\partial\varphi(a_*)}{\partial\nu}} + O(\rho^2).
\end{aligned}
$$

To proceed further, we will make use of the harmonicity of φ. We introducing the coordinate system (x, y) such that the origin is the center of the osculating circle of $\partial\Omega$ at a_* and a_* lies on the x-axis so that $a_* = (\mathcal{R}(a_*), 0)$. Let also (r, θ) be the polar coordinates with respect to (x, y) (see Figure 3). By the second order approximation property of the osculating circle, we have near a_* that

$$\varphi(r\cos\theta, r\sin\theta) = O(\theta^3)$$

$$\text{so that} \quad \frac{\partial^2}{\partial\theta^2}\varphi(r\cos\theta, r\sin\theta)\Big|_{(\mathcal{R}(a_*),0)} = 0.$$

Elementary computations give

$$
\begin{aligned}
\frac{\partial}{\partial\theta}\varphi(r\cos\theta, r\sin\theta) &= -\frac{\partial\varphi}{\partial x}r\sin\theta + \frac{\partial\varphi}{\partial y}r\cos\theta\\
\frac{\partial^2}{\partial\theta^2}\varphi(r\cos\theta, r\sin\theta) &= \varphi_{xx}r^2\sin^2\theta + \varphi_{yy}r^2\cos\theta - 2\varphi_{xy}r^2\cos\theta\sin\theta\\
&\quad -\varphi_x r\cos\theta - \varphi_y r\sin\theta.
\end{aligned}
$$

So

$$\frac{\partial^2}{\partial\theta^2}\varphi(r\cos\theta, r\sin\theta)\Big|_{(\mathcal{R}(a_*),0)} = -\varphi_x(a_*)\mathcal{R}(a_*) + \varphi_{yy}(a_*)\mathcal{R}^2(a_*) = 0.$$

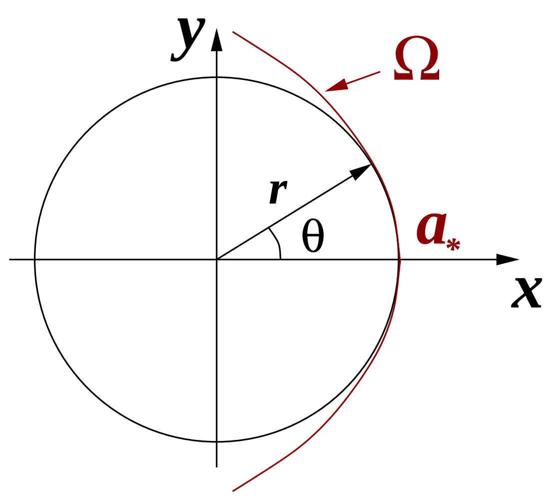

FIGURE 3. Geometry of osculating circle.

The harmonicity of φ gives $\varphi_{yy} = -\varphi_{xx}$ and hence

$$\frac{\varphi_{xx}(a_*)}{\varphi_x(a_*)} = -\frac{1}{\mathcal{R}(a_*)}$$

concluding the proof. □

4. **Multiple degree case.** Here we investigate problem (8) for the higher degree case, $d > 1$. By the similar argument as the previous section, we can obtain the following

$$\lim_{\epsilon\to 0}\left[\mathcal{E}_\epsilon(u_\epsilon) - \pi d\log\frac{1}{\epsilon}\right] = \min_{a_1,a_2,\dots a_d\in\Omega_\rho} W(a_1,a_2,\dots a_d) + d\gamma \tag{41}$$

where the renormalized energy W now takes the following form:

$$W(a_1,a_2,\dots a_d) = -\pi\sum_{i\neq j}\log|a_i - a_j| - \pi\sum_{i=1}^{d} R(a_i) \tag{42}$$

in which

$$\triangle R(x) = 0, \quad R\Big|_{\partial\Omega} = -\sum_{i=1}^{d}\log|x - a_i|\,. \tag{43}$$

Note that for the minimization problem, it is easy to eliminate the case of vortices with multiple degree. Hence the above formula is only written for degree one vortices. In order not to be side-tracked too much, we simply explain heuristically how this is done. By [9, Theorem 1.2] (see also page 8), there are points $a_{\epsilon,i}$'s such that u_ϵ converges strongly to u_* outside $\bigcup_i B_\sigma(a_{\epsilon,i})$. As in several steps in **Sections 2.2** and **2.3**, we replace u_ϵ *inside* each of $B_\sigma(a_{\epsilon,i})$ by a minimizer $\tilde v_\epsilon$ of the Ginzburg-Landau functional with *Dirichlet data* $u_\epsilon\Big|_{\partial B_\sigma(a_{\epsilon,i})}$. In this case, we can employ the technique of [4]. In particular, for $\epsilon \ll 1$, the function $\tilde v_\epsilon$ will only have *degree one* vortices [4, Theorem VI.2]. Then [4, Lemma VIII.2] can be applied to give the lower bound of the energy in terms of the renormalized energy W using only degree one singularity points.

With the above explanation, we will simply start from the expressions (42) and (43). In this case, R is simply the *sum* of the previous R for the degree one case, i.e.

$$R(x) = \textstyle\sum_{j=1}^{d} \tilde{R}_j(x)$$
$$\text{where } \triangle \tilde{R}_j(x) = 0, \text{ and } \tilde{R}_j\Big|_{\partial\Omega} = -\log|x - a_j| \quad \text{(see (38))}$$

Hence we have

$$\begin{aligned} W(a_1, a_2, \dots a_d) &= -\pi \sum_{i\neq j} \log|a_i - a_j| - \pi \sum_{i=1}^{d} R(a_i) \\ &= -\pi \sum_{i\neq j} \log|a_i - a_j| - \pi \sum_{i,j=1}^{d} \tilde{R}_j(a_i) \\ &= -\pi \sum_{i\neq j} \log|a_i - a_j| - \pi \sum_{i\neq j} \tilde{R}_j(a_i) - \pi \sum_{i}^{d} \tilde{R}_i(a_i). \quad (44) \end{aligned}$$

With the above formula, we have the following theorem.

Theorem 4.1. *Let $\{a_{\rho,i}\}_{i=1}^{d}$ be a minimizer of the right hand side of (41). Then as $\rho \longrightarrow 0$ all $a_{\rho,i}$'s converge to points on $\partial\Omega$ with the maximum curvature. In particular, if there is only one point a_* with the maximal curvature, then all the points $a_{\rho,i}$ will go to a_*.*

Remark 2. Note that in the case $\partial\Omega$ has multiple points with the maximum curvature, the above result only states that a_i's must go to any one of them. It currently does *not* state that they will all go to *only one* of the points. In fact, **Proposition 1** later indicates that in general they will go to different points.

Proof. First note that by (38), $R(z) + \log(z - a) = \Phi_0(z)$. We can then rewrite the expression (44) as

$$W(a_1, a_2, \dots a_d) = -\pi \sum_{i\neq j} \log \left| \frac{\Psi(a_i) - \Psi(a_j)}{1 - \Psi(a_i)\overline{\Psi(a_j)}} \right| - \pi \sum_{i=1}^{d} \tilde{R}_i(a_i), \quad (45)$$

where $\Psi : \Omega \to \mathbb{D}_1$ is a fixed Riemann map. By (35), we have for $\rho \ll 1$,

$$W(a_1, a_2, \dots a_d) = -\pi d \log \frac{1}{\rho} - \pi \sum_{i=1}^{d} \frac{\rho}{2\mathcal{R}(\Pi(a_i))} - \pi \sum_{i\neq j} \log \left| \frac{\Psi(a_i) - \Psi(a_j)}{1 - \Psi(a_i)\overline{\Psi(a_j)}} \right| + O(\rho^2)$$

We need two key properties of the following function for $i \neq j$:

$$f(a_i, a_j) = -\log \left| \frac{\Psi(a_i) - \Psi(a_j)}{1 - \Psi(a_i)\overline{\Psi(a_j)}} \right|. \quad (46)$$

First, the function f is *positive*. This follows from the fact that for all a, the Riemann map F (36) maps onto a disk of radius one $\mathbb{D}_1$ and hence $-\log|F| > 0$.

Second, for any fixed positive constant γ, there exists a constant C_γ such that for any two points $a_i,\ a_j \in \Omega$ such that $|a_i - a_j| \geq \gamma$, then $f(a_i, a_j) \leq C_\gamma \rho^2$. This follows from the fact that $f(a_i, a_j)$ equals zero if either one of $\mathrm{Dist}(a_i, \partial\Omega)$ and $\mathrm{Dist}(a_j, \partial\Omega)$ equals zero. Then the statement is a consequence of Taylor expansion.

This also follows from an explicit formula for f. Let $\Psi(a_i) = r_1 e^{i\theta_1}$ and $\Psi(a_j) = r_2 e^{i\theta_2}$. Elementary computation (see (48) also) gives

$$f(a_i, a_j) = -\frac{\pi}{2}\log\left[1 - \frac{(1-r_1^2)(1-r_2^2)}{(1-r_1r_2)^2 + 2r_1r_2(1-\cos(\theta_2-\theta_2))}\right].$$

Since for $\rho \ll 1$, the minimization of $\mathcal{E}$ gives $(1-r_1)$ and $(1-r_2)$ of order ρ. Hence $f \leq C\rho^2$. Note that this property implies that when a_i and a_j are far away from each other, $f(a_i, a_j)$ is of smaller order than the term with the curvature information.

With the above information, we can proceed easily by contradiction. (We simply denote $a_{\rho,i}$ by a_i.) Suppose for some i, a_i does not converge to a maximum curvature point $a_* \in \partial\Omega$. Then there exists $\gamma_0 > 0$ such that $|a_i - a_*| \geq \gamma_0$ for all ρ small. Now choose $\gamma \ll \gamma_0$ and another point $\tilde{a}_i \in \partial\Omega$ such that $|\tilde{a}_i - a_*| \leq \gamma$ and $|\tilde{a}_i - a_j| \geq \gamma$ for all $j \neq i$. We claim that the renormalized energy with a_i re-positioned to $\tilde{a}_i$ is strictly lowered:

$$\begin{aligned}
&-\pi\frac{\rho}{2\mathcal{R}(\Pi(a_i))} - \pi\sum_{j\neq i}\log\left|\frac{\Psi(a_i)-\Psi(a_j)}{1-\Psi(a_i)\overline{\Psi(a_j)}}\right| + O(\rho^2)\\
&\qquad\qquad > -\pi\frac{\rho}{2\mathcal{R}(\Pi(\tilde{a}_i))} - \pi\sum_{j\neq i}\log\left|\frac{\Psi(\tilde{a}_i)-\Psi(a_j)}{1-\Psi(\tilde{a}_i)\overline{\Psi(a_j)}}\right| + O(\rho^2)
\end{aligned}$$

(In the above, we only write down the terms that matter in the argument.) If not, we obtain:

$$-\pi\sum_{j\neq i}\log\left|\frac{\Psi(a_i)-\Psi(a_j)}{1-\Psi(a_i)\overline{\Psi(a_j)}}\right| \leq \pi\frac{\rho}{2\mathcal{R}(\Pi(a_i))} - \pi\frac{\rho}{2\mathcal{R}(\Pi(\tilde{a}_i))} + O(\rho^2)$$

As $\mathcal{R}(\Pi(a_i)) > \mathcal{R}(\Pi(\tilde{a}_i))$, the above will contradict the positivity of $f(a_i, a_j)$ as $\rho \longrightarrow 0$. □

The next result states more clearly the fact that vortices will like to repel each other as much as possible. For concreteness, we consider the case of $d = 2$.

Proposition 1. *Let $d = 2$, and suppose $\partial\Omega$ achieves its maximum curvature at two distinct points P and Q (for example an ellipse). Consider the following problem*

$$\min\left\{W(a_1, a_2) : a_i \in \Omega_\rho\right\}.$$

If $(a_1(\rho), a_2(\rho))$ is a solution, then (up to subsequence and relabeling of the points)

$$\lim_{\rho\to 0} a_1(\rho) = P \quad \textit{and} \quad \lim_{\rho\to 0} a_2(\rho) = Q.$$

Proof. The previous Theorem states that $a_1(\rho)$ and $a_2(\rho)$ must approach either P or Q. Furthermore, the analysis in proof implies that if a_1 and a_2 go to *different* points, then

$$W(a_1(\rho), a_2(\rho)) = -2\pi\log\frac{1}{\rho} - \pi\frac{\rho}{\mathcal{R}_*} + f(a_1, a_2) + O(\rho^2)$$

where $\mathcal{R}_* = \mathcal{R}(P) = \mathcal{R}(Q)$ and $0 \leq f(a_1(\rho), a_2(\rho)) \leq C\rho^2$.

Now suppose both $a_1(\rho)$ and $a_2(\rho)$ approach P. For convenience, we assume that the Riemann map Ψ maps P to $(1, 0)$. Next we write $\Psi(a_1(\rho)) = r_1e^{i\theta_1}$ and $\Psi(a_2(\rho)) = r_2e^{i\theta_2}$. Then it holds that

$$\lim_{\rho\to 0} r_1,\, r_2 = 1 \quad \text{and} \quad \lim_{\rho\to 0}\theta_1,\, \theta_2 = 0. \tag{47}$$

Next we compute:

$$
\begin{aligned}
f(a_1,a_2) &= -\pi \log\left|\frac{\Psi(a_1)-\Psi(a_2)}{1-\overline{\Psi(a_1)}\Psi(a_2)}\right| = -\frac{\pi}{2}\log\left|\frac{r_1e^{i\theta_1}-r_2e^{i\theta_2}}{1-r_1r_2e^{i(\theta_2-\theta_1)}}\right|^2 \\
&= -\frac{\pi}{2}\log\left|\frac{r_1-r_2e^{i(\theta_2-\theta_1)}}{1-r_1r_2e^{i(\theta_2-\theta_1)}}\right|^2 \\
&= -\frac{\pi}{2}\log\left|\frac{r_1-r_2\cos(\theta_2-\theta_1)-ir_2\sin(\theta_2-\theta_1)}{1-r_1r_2\cos(\theta_2-\theta_1)-ir_1r_2\sin(\theta_2-\theta_1)}\right|^2 \\
&= -\frac{\pi}{2}\log\left[\frac{r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1)}{1-2r_1r_2\cos(\theta_2-\theta_1)+r_1^2r_2^2}\right] \\
&= -\frac{\pi}{2}\log\left[1-\frac{(1-r_1^2)(1-r_2^2)}{(1-r_1r_2)^2+2r_1r_2(1-\cos(\theta_2-\theta_2))}\right]. \qquad (48)
\end{aligned}
$$

In order for this configuration to be a minimizer, we need $f(a_1,a_2)\le C\rho^2$ which leads to

$$\frac{(1-r_1^2)(1-r_2^2)}{(1-r_1r_2)^2+2r_1r_2(1-\cos(\theta_2-\theta_2))}\le C\rho^2$$

On the other hand, by the smoothness and invertibility of the Riemann map Ψ, we have as $\rho\longrightarrow 0$ that

$$(1-r_1),\ \ (1-r_2)\ge C\rho.$$

so that

$$C\le (1-r_1r_2)^2+2r_1r_2(1-\cos(\theta_2-\theta_2))$$

which clearly contradicts statement (47) as $\rho\longrightarrow 0$. Hence $a_1(\rho)$ and $a_2(\rho)$ cannot converge to the same point. □

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REFERENCES

[1] N. André and I. Shafrir, *On the minimizers of a Ginzburg-Landau-type energy when the boundary condition has zeros*, Adv. Differential Equations, **9** (2004), 891–960.

[2] P. Bauman, N. N. Carlson and D. Phillips, *On the zeros of solutions to Ginzburg-Landau type systems*, SIAM J. Math. Anal., **24** (1993), 1283–1293.

[3] F. Bethuel, H. Brezis and F. Hélein, *Asymptotics for the minimization of a Ginzburg-Landau functional*, Calculus of Variations and PDEs, **1** (1993), 123–148.

[4] F. Bethuel, H. Brezis and F. Hélein, "Ginzburg-Landau Vortices," Progress in Nonlinear Differential Equations and their Applications, **13**, Birkhäuser Boston, Inc., Boston, MA, 1994.

[5] A. Boutet de Monvel-Berthier, V. Georgescu and R. Purice, *A boundary value problem related to the Ginzburg-Landau model*, Comm. Math. Phys., **142** (1991), 1–23.

[6] L. Berlyand and P. Mironescu, *Ginzburg-Landau minimizers with prescribed degrees. Capacity of the domain and emergence of vortices*, J. Funct. Anal., **239** (2006), 76–99.

[7] L. Berlyand and V. Rybalko, *Solutions with vortices of a semi-stiff boundary value problem for the Ginzburg-Landau equation*, J. Eur. Math. Soc., **12** (2010), 1497–1531.

[8] L. Berlyand and K. Voss, *Symmetry breaking in annular domains for a Ginzburg-Landau superconductivity model*, in "Proceedings of IUTAM 99/4 Symposium," Sydney, Australia, Kluwer Acad. Publ., (2001), 189–200.

[9] R. L. Jerrard, *Lower bounds for generalized Ginzburg-Landau functionals*, SIAM J. Math. Anal., **30** (1999), 721–746.

[10] M. Kurzke, *Boundary vortices in thin magnetic films*, Calc. Var. Partial Differential Equations, **26** (2006), 1–28.

[11] P. Mironescu, *Explicit bounds for solutions to a Ginzburg-Landau type equations*, Rev. Roumain Math. Pure Appl., **41** (1996), 263–271.

[12] W.-M. Ni and I. Takagi, *Locating the peaks of least-energy solutions to a semilinear Neumann problem*, Duke Math. J., **70** (1993), 247–281.

[13] B. White, *Homotopy classes in Sobolev spaces and the existence of energy minimizing maps*, Acta Math., **160** (1988), 1–17.

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