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Stochastic Motion by Mean Curvature

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Abstract

We prove the existence of a continuously time-varying subset $K(t)$ of R^n such that its boundary $\partial K(t)$, which is a hypersurface, has normal velocity formally equal to the (weighted) mean curvature plus a random driving force. This is the first result in such generality combining curvature motion and stochastic perturbations. Our result holds for any C^2 convex surface energy. The $K(t)$ can have topological changes. The randomness is introduced by means of stochastic flows of diffeomorphisms generated by Brownian vector fields which are white in time but smooth in space.

We work in the context of geometric measure theory, using sets of finite perimeter to represent $K(t)$. The evolution is obtained as a limit of a time-stepping scheme. Variational minimizations are employed to approximate the curvature motion. Stochastic calculus is used to prove global energy estimates, which in turn give a tightness statement of the approximating evolutions.

1. Introduction

In this paper, we introduce stochastic perturbations into motion by mean curvature. This motion is but one example of the general notion of curvature driven flows for interfaces, which have been widely used in the modeling of solidification and coarsening processes [AC, Lan, KKL]. By an *interface*, we mean a hypersurface separating two domains in R^n of different physical properties. The interfacial velocity is some prescribed function of curvature or other quantities defined in the bulk region. The motion law is such that the sum of the surface and bulk energies of the system is decreased in time. However, the physical environment is seldom deterministic – there are perturbations coming from thermal fluctuations and impurities. Hence, it is important to consider the effects of randomness.

Motion by mean curvature, in which the normal velocity of the interface simply equals its mean curvature, has attracted much attention. It involves interesting aspects of the topology of curves and surfaces and also poses challenging questions for partial differential equations. The motion can lead to singularities and topological changes of the interface. Several machineries have been developed to tackle these difficulties. [Bra] first used varifold theory to prove an existence result in arbitrary codimension. The technique so far only applies to the isotropic surface energy. [CGG] and [ES] established global unique viscosity solution to the level set formulation of the curvature motion. But there might be ambiguities of the interfacial locations due to the flattening of the level sets. The phase-field approach, in particular the Allen-Cahn equation [AC] in which the interface is diffused, was shown to produce motion by mean curvature in the sharp interfacial limit [BK] (spherical case), [MS, Che] (when a smooth solution of the interfacial motion exists). [ES] extended this result to the viscosity setting. [Ilm, Son] further showed that BRAKKE'S varifold solution can be obtained from the phase-field approach.

Recently, [ATW] used a variational method to prove a general existence result global in time.[1] The surfaces can have topological changes. The advantages of this approach compared with the previous ones are that it works directly in the sharp-interfacial regime and can handle very general (even non-smooth) anisotropic surface energy integrand. [ATW] called their evolution *flat flow*[2]. The evolution is approximated by a time-stepping scheme which involves variational minimizations. We extend this approach to include stochastic noise.

Questions about stochastic dynamics of interfaces have been widely discussed in the physics literature. The works touch upon lattice models, derivations of continuum (macroscopic) equations, the study of interfacial structures, scaling limits, etc. We refer to [BS, KrSp, Zan] for good introductions to these topics. [KO1] and [KO2] derived a stochastic version of curvature flows in the spirit of the phase-field equation. [Fun1] and [Fun2] studied random sharp-interfacial limits in the case of one spatial dimension and a convex curve in the plane. However, many questions remain unsettled in general situations.

1.1. Mathematical Approach

In this paper, we tread stochastic motion by mean curvature in the sharp-interfacial regime by using techniques of geometric measure theory [Fed]. We establish a continuously time-varying subset $\{K(t)\}_{t \geqq 0}$ of R^n such that its boundary $\partial K(t)$, which is a hypersurface, has velocity formally equal to the (weighted) mean curvature plus some random vector.

If $\partial K(t)$ undergoes pure motion by mean curvature without perturbations, its surface area decreases according to

$$\frac{d}{dt}\mathscr{H}^{n-1}(\partial K(t)) = -\int_{x \in \partial K(t)} |h(x)|^2 \, d\mathscr{H}^{n-1}x \tag{1}$$

[1] [LS] also proved a similar result in the isotropic case.

[2] The surface is shown to evolve continuously in time in terms of the flat norm for integral currents of geometric measure theory.

where $\mathscr{H}^{n-1}$ denotes the Hausdorff $(n-1)$-dimensional measure and h is the mean curvature of $\partial K(t)$. This motion law can be extended to more general surface energy integrands Φ. The corresponding concept of curvature is then Φ-weighted mean curvature h_Φ. Now the surface energy of $\partial K(t)$ decreases according to

$$\frac{d}{dt}\Phi(\partial K(t)) = -\int_{\partial K(t)} |h_\Phi|^2 \, d\mathscr{H}^{n-1}. \tag{2}$$

We omit the word "weighted" for simplicity.

In [ATW], such an evolution is recast as a negative gradient flow for the surface energy functional:

$$\frac{d}{dt}\partial K(t) = -\nabla\Phi(\partial K(t)). \tag{3}$$

An implicit time stepping scheme is used to solve this equation. During each time interval, the set $K(t)$ is changed to a new shape which is a minimizer of an appropriate functional. This procedure approximates (3) in discrete time.

We incorporate randomness into the above approach. Stochastic noises are introduced by means of a random flow of diffeomorphisms of the underlying space. This flow is generated by a Brownian vector field which is white in time but smooth in space. More precisely, we want $\partial K(t)$ to evolve according to the equation

$$v_n = h_\Phi + \langle F, \hat{n}\rangle \tag{4}$$

where v_n is the normal velocity of $\partial K(t)$, $\hat{n}$ is the outward normal, h_Φ is the Φ-weighted mean curvature[3]; F is a white noise vector field defined on the whole background domain. (We only consider the normal component of F with respect to $\partial K(t)$ since the tangential part does not change the shape of $K(t)$.) In essence, we have in mind that $K(t)$ is evolving so as to reduce its surface energy $\Phi(\partial K(t)$ but this motion is constantly perturbed by F which acts by deforming the space.

One of the main difficulties in introducing white noise into such a geometric motion is how to combine the nonlinearity of the evolution and the statistical cancellation property of the noise. We achieve this by a time-splitting scheme. Within each time interval of $\triangle t$, we perform two operations. First, we change the set $K(t)$ by minimizing a functional which is the same as the one in [ATW]. This approximates $v_n = h_\Phi$. Then we transport the set by the flow of diffeomorphisms generated by F. We repeat this process for each interval. Using tools from stochastic calculus, we prove that the previous construction produces a tight sequence (as $\triangle t \longrightarrow 0$) of probability measures on an appropriate space of stochastic processes. Any weak limit of the measures concentrates on the space of continuous evolutions of $K(t)$.

The main theorem is stated in Section 2.7 after the introduction of some terminology and notations. An outline of the proof is given in Section 2.8.

Some remarks are in order: The method we employ is quite general. It produces a sharp-interfacial evolution, allowing topological changes. In the deterministic case, the surface energy Φ can be any convex integrand. In our random case, we need it

[3] The sign of h_Φ is chosen so that the equation is well-posed: a sphere wants to shrink.

to be in C^2 due to our use of Ito's formula from stochastic calculus which involves second derivatives.

On the other hand, so far we can only show that our construction gives (4) formally. The motion law and regularity of our evolving set $\partial K(t)$ are not quite clear. For the unperturbed deterministic problem, it is shown in [ATW 7.4] that the variational approach gives the same evolution as the classical solution as long as the latter exists. It seems a challenging problem to prove similar results beyond the appearance of interfacial singularities.

1.2. Related Models

As mentioned earlier, curvature driven flows have wide applications in modeling solidification processes. We refer to [Lan] and [KKL] for an introduction to these physical phenomena. A more general form of interfacial velocity is given by

$$v_n = \mu(h_\Phi + \Lambda). \tag{5}$$

μ is the mobility function which measures the kinetics — how fast the interface can react to driving forces. Λ denotes bulk quantities which might depend on the temperature field, concentration of solutes, impurities, etc. We refer to [Gur] for a derivation of (5) from a thermodynamical point of view. [TCH] gives a review of several mathematical approaches to tackle such interfacial evolutions. In a more complete model, the bulk variables are subject to diffusion equations. This has been considered in [AW, Luc, Son]. Stochastic perturbation has also been incorporated in this case [Yip].

Many other phenomenological continuum equations have been developed to study similar growth processes, which are discussed in [BS, KrSp, Zan] and references therein. Typical equations considered in these works include

$$\frac{\partial f}{\partial t} = A_1 \nabla^2 f + A_2 (\nabla f)^2 + \eta, \qquad \text{KPZ equation}, \tag{6}$$

$$\frac{\partial f}{\partial t} = -A_1 \nabla^4 f + A_2 \nabla^2 (\nabla f)^2 + \eta, \qquad \text{diffusion-dominated growth}, \tag{7}$$

where A_1 and A_2 are positive constants. η is commonly taken to be the *space-time white noise*: $\langle \eta(x,t),\ \eta(y,s)\rangle = \delta(x-y)\delta(t-s)$. Questions of particular interests concerning these equations include the interfacial structures and scaling exponents.

In addition, [KO1] and [KO2] derived random interfacial dynamics starting from the time-dependent stochastic Ginzburg-Landau equation. Assuming that the interfacial curvature is small compared with the diffused interfacial thickness, they came up with the equation[4]

$$\frac{\partial}{\partial t} f = \sqrt{1 + |\nabla f|^2}\ \operatorname{div} \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} + \eta(x,t) \tag{8}$$

[4] Here the interface is represented by the graph of f. The term $\sqrt{1 + |\nabla f|^2}$ accounts for the tilt of the interface.

where the noise term satisfies the fluctuation-dissipation relation

$$\langle \eta(x,t),\ \eta(y,s)\rangle = C\sqrt{1+|\nabla f|^2}\ \delta(x-y)\delta(t-s). \tag{9}$$

This is similar to our equation (4) where we replace η by $\langle F,\ (-\nabla f, 1)\rangle$ with F being a vector field white in time but smooth in space.

2. Statement of Theorem and Outline of Proof

We introduce here some notations for our theorem. In the whole paper, we work in a fixed domain $\mathscr{O}$ of R^n with compact closure and nice boundary. (We can also regard $\mathscr{O}$ as an n-torus.) All random variables and stochastic processes (such as the Brownian motion and flows set forth later) are defined on a common probability space $(\Omega, \mathscr{F}, P)$ where $\mathscr{F}$ is a σ-field of Ω and P is a probability measure on Ω. E always means the expectation taken with respect to P.

2.1. Crystal Shape ($\mathscr{K}$)

These are described by subsets of $\mathscr{O}$ with *finite perimeters*. K is called such a set if

$$|\partial K| = \sup\left\{\int_K \operatorname{div} g\, d\mathscr{L}^n : g \in C_0^1(\mathscr{O}, R^n),\ \|g\|_\infty \leqq 1\right\} < \infty \tag{10}$$

$|\partial K|$ is called the *perimeter* of K. $\mathscr{K}$ is metrized by the L^1 norm:

$$\|K - L\|_{L^1} = \int_{x\in\mathscr{O}} |K(x) - L(x)|\, d\mathscr{L}^n x = \mathscr{L}^n(K \triangle L). \tag{11}$$

(By abuse of notation, K can mean both the set K or its characteristic function χ_K.) Each $K \in \mathscr{K}$ can also be considered as an n-dimensional integral current in the context of geometric measure theory [Fed]. It is denoted by $[\![K]\!]$. $\partial[\![K]\!]$ refers to its current boundary.

The main properties we need for this kind of sets are compactness under L^1 of the collection $\{K \in \mathscr{K} : |\partial K| \leqq M < \infty\}$ and the existence of a well defined notion of normal and boundary, namely, approximate normal (n_K) and reduced boundary ($\partial^* K$). These concepts are described in detail in [EG] and [Giu].

2.2. Surface Energy (Φ)

This notion is used to describe interfacial surface energy. A *surface-senergy integrand* Φ is a function from S^{n-1} to R_+. It is usually extended to a map from R^n to R_+ by positive homogeneity of degree 1: $\Phi(\lambda v) = \lambda\Phi(v)$ $(\lambda \geqq 0,\ v \in S^{n-1})$. Φ is called *isotropic* if $\Phi(v) = c\,|v|$ for some positive constant c.

The Φ *surface energy* of $K \in \mathscr{K}$ is defined as

$$\Phi(\partial K) = \int_{\partial K} \Phi(n_K)\, d\mathscr{H}^{n-1} \tag{12}$$

where n_K is the outward normal vector of ∂K.[5]

[5] In this paper, ∂K always denotes the reduced boundary of K.

In this paper, we assume that Φ is in C^2 and is convex as a function from R^n to R_+.

2.3. Φ-Weighted Mean Curvature h_Φ

The h_Φ of a hypersurface ∂K can be defined as a weighted sum of the mean curvatures of ∂K or more generally as the first variation of the Φ-surface energy. A nice account of such a concept is given in [Tay]. Here we give the formula in the graph case.

Suppose a surface in R^n is represented by a graph: $x_n = f(x_1, \ldots x_{n-1})$ and the surface energy integrand (assumed to be positively homogenous of degree 1) is given by $\Phi : (p_1, \ldots p_n) \in R^n \longrightarrow R_+$. Then the Φ-weighted mean curvature of f is

$$h_\Phi = \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} \left(\frac{\partial \Phi}{\partial p_i}(-\nabla f, 1) \right). \tag{13}$$

2.4. Minimization Step – Approximation of Motion by Mean Curvature

Given $\triangle t > 0$ and a $K \in \mathscr{K}$, we replace K by a new set K' which is a $(\Phi, \triangle t, K)$-minimizer, i.e., K' minimizes the functional

$$\Phi(\partial L) + \frac{1}{\triangle t} \int_{x \in L \triangle K} \operatorname{Dist}(x, \partial K)\, d\mathscr{L}^n x \tag{14}$$

over all $L \in \mathscr{K}$ where $L \triangle K = (L \backslash K) \cup (K \backslash L)$ and $\operatorname{Dist}(\cdot, \partial K)$ is the distance function from a point to the (topological or reduced) boundary of K. Such a change from K to K' is an approximation of motion by mean curvature of K as explained in [ATW, 2.12].

2.5. Perturbation Step – Stochastic Flows

We stochastically perturb the set K by deforming the domain $\mathscr{O}$ by using a Brownian flow. These and other related terminologies are described in Appendix B. We also refer to [KS] for basic concepts in probability theory such as random variables, stochastic processes, martingales, etc.

Let $(\Omega, \mathscr{F}, P)$ be a probability space equipped with a filtration $\{\mathscr{F}_t\}_{t \geqq 0}$. Let $\{F,\ \mathscr{F}_t;\ 0 \leqq t < \infty\}$ be a Brownian Motion in the space of vector fields defined on $\mathscr{O}$. The support of F is contained in $\mathscr{O}$ for all $t \geqq 0$. The local characteristics of F are denoted by $(a(x, y, t), b(x, t))_{x, y \in \mathscr{O}, t \geqq 0}$, which belongs to the class $\mathscr{B}_{ub}^{2,\delta}$ $(\delta > 0)$ (Section B.2). Under this assumption, F can be written as

$$F(x, t) = M(x, t) + \int_0^t b(x, r)\, dr \tag{15}$$

where $M(x, t)$ is a continuous $C^2(\mathscr{O}, R^n)$-valued martingale with cross variation given by

$$\left\langle M^i(x,t),\ M^j(x,t)\right\rangle = \int_0^t a^{ij}(x,y,r)\,dr. \tag{16}$$

The *Brownian flow* $\{\varphi_{s,t}\}_{t\geqq s\geqq 0}$ *generated by* F is the solution of the stochastic differential equation

$$\varphi_{s,t}(x) = x + \int_s^t F(\varphi_{s,r}(x), dr), \quad x \in \mathscr{O},\ 0 \leqq s \leqq t \leqq 1. \tag{17}$$

Under the stated assumptions of F, a unique solution for this equation exists. $\{\varphi_{s,t}\}_{t\geqq s\geqq 0}$ is a 2-parameter family of C^2-diffeomorphisms of $\mathscr{O}$. It equals the identity map outside $\mathscr{O}$ for all $t \geqq s \geqq 0$.

We use $\varphi_{s,t}$ to perturb K:

$$\varphi_{s,t}K = \text{(diffeomorphic) image of } K \text{ under } \varphi_{s,t}. \tag{18}$$

This gives the effect that the boundary of K is transported by F.

2.6. Construction of Discretized Stochastic Motion by Mean Curvature

Now we combine the previous ingredients and define our approximate evolutions.

Let $\triangle t = 1/N$ be the time discretization interval. We denote $t_i = i\triangle t$. Starting from a fixed initial shape K_0 at 0^-, we construct the discretized evolution $\{K^N(t)\}_{t\in[0,1]}$ as follows. For $0 \leqq i \leqq N-1$,

$$\begin{aligned} K^N(t_i^+) &= \text{a } (\Phi, \triangle t, K^N(t_i^-))\text{-minimizer},\\ K^N(t) &= \varphi_{t_i,t}K^N(t_i^+) \quad \text{for } t_i \leqq t < t_{i+1}, \end{aligned}$$

i.e., at each t_i, we change the set by minimization so as to approximate motion by mean curvature; in between any two t_i's, the set is perturbed by the Brownian flow φ. The $\{K^N(t)\}_{t\in[0,1]}$ thus defined is a piecewise continuous time-varying set with discontinuities at the $\{t_i\}$'s ($0 \leqq t_i \leqq 1$). Sometimes we will use K^N_{i+} to denote $K^N(t_i^+)$. A similar remark holds for K^N_{i-}. Now $K^N(\cdot)$ is a stochastic process taking values in $\mathscr{K}$ with its sample paths being right continuous with left-hand limits. Such a space is denoted by $\mathscr{D}([0,1],\mathscr{K})$ and is endowed with the Skorokhod topology[6].

[6] For details of the Skorokhod topology, see [Bil]. Actually we can formulate our result without using this topology (see Theorem 10.1). We introduce this terminology here just because it is a very common space used in the study of piecewise continuous stochastic processes.

2.7. Theorem – Tightness of Stochastic Motion by Mean Curvature

Let Φ be a C^2 and convex surface-energy intergrand and let F be a Brownian vector field in the class $\mathscr{B}_{ub}^{2,\delta}$ ($\delta > 0$) with support contained in $\mathscr{O}$. Let Π^N be the law of $\left\{K^N(t)\right\}_{t\in[0,1]}$ on $\mathscr{D}([0,1],\mathscr{K})$ induced by the construction described in Section 2.6.

Then $\left\{\Pi^N\right\}_{N\geqq 1}$ is tight[7]. Furthermore, any weak limit Π_* of Π^N satisfies the following statements.

1. $\Pi_*(C([0,1],\mathscr{K})) = 1$, i.e., Π_* is supported on the space of *continuous evolutions* of K.
2. *Uniform Surface Energy Estimate.* For any positive integer m, there is a constant C_m such that

$$\Pi_*\left(\sup_{t\in[0,1]}\left\{\Phi(\partial K(t))^m : K \in C([0,1],\mathscr{K})\right\}\right) \leqq C_m. \tag{19}$$

3. *Weak Continuity Estimate.* For any positive integer m and $f \in C^2(\mathscr{O})$, there is a constant $C(f,m)$ such that for $0 \leqq s \leqq t \leqq 1$,

$$\int_{K\in C([0,1],\mathscr{K})}\left|\int_{x\in K(t)} f(x)\,d\mathscr{L}^n x - \int_{x\in K(s)} f(x)\,d\mathscr{L}^n x\right|^{2m} d\Pi_* K$$
$$\leqq C(f,m)\,|t-s|^m\,. \tag{20}$$

2.8. Outline of Proof

As described in the previous main statement, we are establishing a compactness result. We will prove that our set evolution heuristically satisfies $E\,\|K(t)-K(s)\|^{2m} \leqq C_m\,|t-s|^m$ in some weak sense. Then the Kolmogorov-Čentsov Theorem A.1, stated in Appendix A, says that $K(t)$ varies continuously in time.

In our discrete scheme, the set is changed by two procedures: minimization and stochastic perturbation. The estimate during the first step is essentially the same as in [ATW]. We show that formally (Corollary 3.2),

$$\|\text{change of set}\|_{L^1} \leqq C\sqrt{\triangle t}.$$

The proof relies heavily on the regularity of the minimizers – the boundary of any minimizer enjoys a lower density ratio bound (Proposition 3.1).

For stochastic perturbations, we establish the following weaker form of the continuity statement:

$$\left|\int_{K^N(t+\triangle t)} f\,d\mathscr{L}^n x - \int_{K^N(t)} f\,d\mathscr{L}^n x\right| \leqq C(f)\sqrt{\triangle t}$$

where f is any smooth function defined on $\mathscr{O}$.

[7] See the appendix for the definitions of tightness and weak limit of probability measures.

Combining these two estimates, we have (see (31) and (32))

$$E\left|\int_{K^N(t)} f\, d\mathscr{L}^n x - \int_{K^N(s)} f\, d\mathscr{L}^n x\right|^{2m} \leqq C(m, f)\,|t-s|^m .$$

This proves that the set evolves weakly continuously in time. Here we treat the sets as random measures. However, our sets are much better than arbitrary measures. They have a boundary notion which acts like a spatial distributional derivative. In this regard, we prove the uniform energy estimate (see (33))

$$E\left\{\sup_{\lambda\in[0,1]} \Phi(\partial K^N(\lambda))^m\right\} \leqq C_m,$$

which implies that the sets have finite perimeters. With this extra ingredient, the weak continuity statement can be improved to strong L^1 continuity.

The proofs of (31)–(33) make use of the techniques from stochastic calculus, mainly Ito's formula and Martingale's inequality. For their statements we refer to [KS]. A particularly useful inequality is the Burkholder-Davis-Gundy Inequalities (BDG) [KS, 3.3.28] which is used in several places in this paper. We state it here for later reference:

Let M_t be a continuous (local) martingale such that $M_0 = 0$. Then for all $m > 0$, there are universal positive constants k_m and K_m (not depending on M) such that

$$k_m E\left(\langle M\rangle_\tau^m\right) \leqq E\left((M_\tau^*)^{2m}\right) \leqq K_m E\left(\langle M\rangle_\tau^m\right) \tag{21}$$

where τ is any stopping time, $\langle M\rangle_t$ is the quadratic variation of M and $M_t^* = \sup_{0\leqq s\leqq t}|M_s|$.

3. Approximation of Motion by Mean Curvature

In this section, we describe the estimates related to the minimization step. Recall the set-up in Section 2.4. Given a set K_0, we find a $(\Phi, \triangle t, K_0)$-minimizer. In our actual application, K_0 is the shape at time t_i^-, and any minimizer K can be chosen to be the shape at t_i^+.

The regularity of the $(\Phi, \triangle t, K_0)$-minimizers is very important in establishing the continuity statement of the overall evolution. The starting point is a lower bound for the $(n-1)$-dimensional density ratio, from which we can control the volume change of the set.

The existence of $(\Phi, \triangle t, K_0)$-minimizers is easily deduced from the compactness property of integral currents or functions of bounded variations. Furthermore, any minimizer lies in the convex hull of K_0.

The following is a collection of results from [AWT, 3.4, 5.3]. We set forth the following notations which are standard in geometric measure theory:

$B^n(p,r) = \{x : |p-x| \leqq r\}$, $U^n(p,r) = \{x : |p-x| < r\}$,
$\Phi^0 = \sup \Phi(n)$, $\Phi_0 = \inf \Phi(n)$,
γ_k $(2 \leqq k \leqq n)$ is the isoperimetric constant,
$\alpha(n)$ is the volume of unit n-ball in R^n,
$\beta(n)$ is the Besicovitch-Federer covering constant for R^n.

Proposition 3.1 **(**(n−1)-Dimensional Density Bound for Minimizers [ATW, 3.4]**).** *Let K be a $(\Phi, \triangle t, K_0)$-minimizer. Then for all $p \in \operatorname{spt} \partial[\![K]\!]$,*

$$\frac{\mathscr{H}^{n-1}(\partial K \cap B^n(p,r))}{r^{n-1}} \geqq \theta \quad \textit{for all } 0 < r \leqq \sqrt{\triangle t} \tag{22}$$

where

$$\theta = \frac{1}{(n-1)^{n-1}} \left(\frac{\Phi_0}{2\gamma_{n-1}\Phi_0} \right)^{1/(n-2)} \inf \left\{ 1, \left(\frac{n\Phi_0}{3D} \right)^{n-1} \right\}$$

and D is the number from Proposition 3.3.[8]

Proof. Denote $T = \partial[\![K]\!]$. Let $p \in \operatorname{spt} \partial[\![K]\!]$. Define $\rho(x) = |x - p|$. For all $r > 0$, consider

$$T_r = T \llcorner \{x : \rho(x) < r\}, \quad m(r) = \boldsymbol{M}(T_r) = \mathscr{H}^{n-1}(\partial K \cap B^n(p,r)).$$

For almost every $r > 0$, the slice $\langle T, \rho, r\rangle = \partial(T \llcorner \{x : \rho(x) \leqq r\}) = \partial T_r$ is an integral $(n-2)$-current and $\boldsymbol{M}(\partial T_r) = \boldsymbol{M}\langle T, \rho, r\rangle \leqq m'(r)$ (since $\operatorname{Lip} \rho = 1$). By the isoperimetric inequality, there is an integral $(n-1)$-current R supported in $B^n(p,r)$ such that $\partial R = \partial T_r = \langle T, \rho, r\rangle$ and $\boldsymbol{M}(R) \leqq \gamma_{n-1}\boldsymbol{M}(\partial T_r)^{(n-1)/(n-2)} \leqq \gamma_{n-1} m'(r)^{(n-1)/(n-2)}$. Consider the cone $Q = [\![p]\!] \times\!\!\!\!\times (R - T_r)$. ($Q$ is the n-dimensional current formed by joining p to all the points on $R - T_r$. For a precise definition, see [Fed, 4.1.11].) Since $\partial(R - T_r) = 0$, we have $\partial Q = R - T_r$ and

$$\boldsymbol{M}(Q) \leqq \frac{r}{n}\boldsymbol{M}(R - T_r) \leqq \frac{r}{n}\left[\gamma_{n-1} m'(r)^{(n-1)/(n-2)} + m(r)\right].$$

Note that

$$\Phi(T + \partial Q) - \Phi(T) = \Phi(R + (T - T_r)) - \Phi(T_r + (T - T_r)) = \Phi(R) - \Phi(T_r).$$

Since K is a $(\Phi, \triangle t, K_0)$-minimizer, we have

$$\begin{aligned} &\Phi(\partial K) + \frac{1}{\triangle t}\int_{K \triangle K_0} \operatorname{Dist}(x, \partial K_0)\, d\mathscr{L}^n x \\ &\leqq \Phi(\partial(K + Q)) + \frac{1}{\triangle t}\int_{L \triangle K_0} \operatorname{Dist}(x, \partial K_0)\, d\mathscr{L}^n x \end{aligned}$$

where L is the set corresponding to the current $K + Q$. Since $K_0 \triangle L \backslash K_0 \triangle K \subset K \triangle L$, we deduce that

$$\Phi(T_r) \leqq \Phi(R) + \mathscr{L}^n(K \triangle L) \sup\left\{ \frac{\operatorname{Dist}(x, \partial K_0)}{\triangle t} : x \in B^n(p,r) \right\}.$$

[8] This result implies that $\mathscr{H}^{n-1}(\operatorname{spt} \partial[\![K]\!] \triangle \partial[\![K]\!]) = 0$.

Since $\mathscr{L}^n(K \triangle L) \leqq \boldsymbol{M}(Q)$, the last inequality gives

$$\begin{aligned} \Phi_0 m(r) &\leqq \Phi^0 \gamma_{n-1} m'(r)^{(n-1)/(n-2)} \\ &\quad + \frac{r}{n}\left[\gamma_{n-1} m'(r)^{(n-1)/(n-2)} + m(r)\right] \sup\left\{\frac{\operatorname{Dist}(x, \partial K_0)}{\triangle t} : x \in B^n(p, r)\right\}. \end{aligned}$$

By Proposition 3.3, $\operatorname{Dist}(x, \partial K_0) \leqq D\sqrt{\triangle t}$. Hence

$$\begin{aligned} m(r) &\leqq \frac{\Phi^0}{\Phi_0}\gamma_{n-1} m'(r)^{(n-1)/(n-2)} \\ &\quad + \frac{rD}{n\sqrt{\triangle t}\Phi_0}\left[\gamma_{n-1} m'(r)^{(n-1)/(n-2)} + m(r)\right], \\ m(r)\left(1 - \frac{rD}{n\sqrt{\triangle t}\Phi_0}\right) &\leqq \left(\gamma_{n-1}\frac{\Phi^0}{\Phi_0}\right)\left(1 + \frac{rD}{n\sqrt{\triangle t}\Phi^0}\right) m'(r)^{(n-1)/(n-2)}. \end{aligned}$$

Now restrict $r \leqq r_0 = n\Phi_0\sqrt{\triangle t}/3D$ and set $C^{(n-1)/(n-2)} = 2\gamma_{n-1}\Phi^0/\Phi_0$. Then

$$\begin{aligned} & m(r) \leqq C^{(n-1)/(n-2)} m'(r)^{(n-1)/(n-2)} \\ & \Longrightarrow \left((n-1)m(r)^{1/(n-1)}\right)' = \frac{m(r)'}{m(r)^{(n-2)/(n-1)}} \geqq \frac{1}{C} \\ & \Longrightarrow m(r) \geqq \frac{r^{n-1}}{((n-1)C)^{n-1}}. \end{aligned}$$

If $r_0 \leqq r \leqq \sqrt{\triangle t}$, then

$$\frac{m(r)}{r^{n-1}} \geqq \frac{m(r_0)}{r_0^{n-1}}\left(\frac{r_0}{r}\right)^{n-1} \geqq \left(\frac{1}{(n-1)C}\right)^{n-1}\left(\frac{n\Phi_0}{3D}\right)^{n-1}.$$

The whole proposition follows if we set

$$\theta = \frac{1}{(n-1)^{n-1}}\left(\frac{\Phi_0}{2\gamma_{n-1}\Phi_0}\right)^{1/(n-2)} \inf\left\{1, \left(\frac{n\Phi_0}{3D}\right)^{n-1}\right\}. \qquad \square$$

Corollary 3.2 (Volume Difference Estimate). *Let K_0 be a set with a lower bound θ for the $(n-1)$-dimensional density ratio in the sense of (22). Let K be a $(\Phi, \triangle t, K_0)$-minimizer. Then* [9]

$$\mathscr{L}^n(K \triangle K_0) \leqq A(\Phi, n)\frac{R}{\theta}\mathscr{H}^{n-1}(\partial K_0) + \frac{\triangle t}{R}(\Phi(\partial K_0) - \Phi(\partial K)) \quad \text{for all } R \leqq \tfrac{1}{2}\sqrt{\triangle t},$$

where $A(\Phi, n)$ is a number depending only on Φ and n.

[9] Note that we are making use of the lower density ratio bound of K_0, *NOT* K.

Proof[10]. By the fact that K is a $(\Phi, \triangle t, K_0)$-minimizer, we have, using K_0 as a comparison shape, that

$$\int_{K\triangle K_0} \mathrm{Dist}(x, \partial K_0)\, d\mathscr{L}^n \leqq \triangle t(\Phi(\partial K_0) - \Phi(\partial K)). \tag{23}$$

Now,

$$\begin{aligned}\mathscr{L}^n(K\triangle K_0) &= \mathscr{L}^n(K\triangle K_0 \cap \{\mathrm{Dist}(x, \partial K_0) \geqq R\}) \\ &\quad + \mathscr{L}^n(K\triangle K_0 \cap \{\mathrm{Dist}(x, \partial K_0) \leqq R\}).\end{aligned} \tag{24}$$

For the first term on the right of (24), we have by (23), that

$$\mathscr{L}^n(K\triangle K_0 \cap \{\mathrm{Dist}(x, \partial K_0) \geqq R\}) \leqq \frac{\triangle t}{R}(\Phi(\partial K_0) - \Phi(\partial K)). \tag{25}$$

For the second term on the right of (24), by Besicovitch-Federer Covering Theorem, we can cover ∂K_0 by balls of radius $2R$ such that they do not overlap more than $\beta(n)$ times. Hence,

$$\begin{aligned}\mathscr{L}^n\ &(K\triangle K_0 \cap \{\mathrm{Dist}(x, \partial K_0) \leqq R\}) \\ &\leqq \sum_{B(p_i,2R)} \mathscr{L}^n(B(p_i, 2R)) = \alpha(n) \sum_{B(p_i,2R)} (2R)^n \\ &= \alpha(n)2^n R \sum_{p_i} R^{n-1} \\ &\leqq \alpha(n)2^n\theta^{-1} R \sum_{p_i} \mathscr{H}^{n-1}(\partial K_0 \cap B(p_i, 2R)) \\ &\qquad \text{(by the lower density ratio bound for } K_0) \\ &\leqq \alpha(n)\beta(n)2^n\theta^{-1} R\mathscr{H}^{n-1}(\partial K_0).\end{aligned} \tag{26}$$

The corollary follows by adding (25) and (26). □

Proposition 3.3 (ATW, 5.3). *Let K be a $(\Phi, \triangle t, K_0)$-minimizer. Then*

$$\mathrm{Dist}(\partial K, \partial K_0) \leqq D(\Phi, n)\sqrt{\triangle t}$$

where $D(\Phi, n)$ depends only on Φ and the dimension.

Proof. Suppose that there is a point $p \in \partial K$ such that $B(p, R) \subset K_0$. (The proof for the case $B(p, R) \subset K_0^c$ is similar.) As a comparison set, let $K' = K \cup B(p, \frac{1}{2}R)$. Then,

$$\begin{aligned}&\Phi(\partial K) + \frac{1}{\triangle t}\int_{K\triangle K_0} \mathrm{Dist}(x, \partial K_0)\, d\mathscr{L}^n \\ &\quad\leqq \Phi(\partial K') + \frac{1}{\triangle t}\int_{K'\triangle K_0} \mathrm{Dist}(x, \partial K_0)\, d\mathscr{L}^n \\ &\Longrightarrow \int_{B(p,R/2)\setminus K} \mathrm{Dist}(x, \partial K_0)\, d\mathscr{L}^n \leqq \triangle t\left(\Phi(\partial K') - \Phi(\partial K)\right).\end{aligned} \tag{27}$$

Note that $\Phi(\partial K') - \Phi(\partial K) = \Phi(\partial B(p, \frac{1}{2}R)) - \Phi(\partial(B(p, \frac{1}{2}R) \cap K))$.

[10] This proof follows [LS, 1.5]. It is much simplier than the original argument in [ATW, 4.2].

To simplify the notation, assume that the Wulff shape of Φ is a ball[11]. Then for all $r > 0$ and $U \subset R^n$,

$$\frac{\Phi(\partial B(p,r))^{\frac{1}{n-1}}}{\mathscr{L}^n(B(p,r))^{\frac{1}{n}}} \leqq \frac{\Phi(\partial U)^{\frac{1}{n-1}}}{\mathscr{L}^n(U)^{\frac{1}{n}}}$$

$$\Longrightarrow \quad \Phi(\partial B(p,r)) - \Phi(\partial U) \leqq \Phi(\partial B(p,r)) \left[1 - \left(\frac{\mathscr{L}^n(U)}{\mathscr{L}^n(B(p,r))}\right)^{\frac{n-1}{n}}\right].$$

Assuming further that $U \subset B(p,r)$, we obtain

$$\begin{aligned}\Phi(\partial B(p,r)) - \Phi(\partial U) &\leqq \Phi(\partial B(p,r)) \left[1 - \left(1 - \frac{\mathscr{L}^n(B(p,r)\backslash U)}{\mathscr{L}^n(B(p,r))}\right)^{\frac{n-1}{n}}\right] \\ &\leqq \Phi(\partial B(p,r)) \frac{\mathscr{L}^n(B(p,r)\backslash U)}{\mathscr{L}^n(B(p,r))}\end{aligned}$$

where in the last inequality we have used $1 - (1-x)^{\frac{n-1}{n}} \leqq x$ for $0 \leqq x \leqq 1$.

Now set $U = B(p, \frac{1}{2}R) \cap K$ in the above. Then

$$\begin{aligned}\tfrac{1}{2}R\mathscr{L}^n(B(p,\tfrac{1}{2}R)\backslash K) &\leqq \quad \text{left-hand side of (27)} \\ &\leqq \quad \text{right-hand side of (27)} \\ &\leqq \triangle t\, \Phi(\partial(p, R/2)) \frac{\mathscr{L}^n(B(p,\frac{1}{2}R)\backslash K)}{\mathscr{L}^n(B(p,\frac{1}{2}R))}.\end{aligned}$$

The extreme inequalities yield

$$\begin{aligned}\tfrac{1}{2}R\mathscr{L}^n(B(p,\tfrac{1}{2}R)\backslash K) &\leqq \triangle t\, \Phi(\partial B(p,\tfrac{1}{2}R)) \frac{\mathscr{L}^n(B(p,\frac{1}{2}R)\backslash K)}{\mathscr{L}^n(B(p,\frac{1}{2}R))}, \\ \tfrac{1}{2}R &\leqq D(\Phi,n)\triangle t R^{n-1}\frac{1}{R^n}, \\ R &\leqq D(\Phi,n)\sqrt{\triangle t}\end{aligned}$$

where $D(\Phi, n)$ depends only on Φ and the dimension. □

Proposition 3.4 (n-Dimensional Density Bound for Minimizers)**.** *Let K be a $(\Phi, \triangle t, K_0)$-minimizer. Then for all $p \in \partial K$,*

$$\mathscr{L}^n(K \cap B(p,r)), \quad \mathscr{L}^n(K^c \cap B(p,r)) \geqq C(\theta,n)r^n \quad \text{for all} \quad 0 < r \leqq \sqrt{\triangle t}$$

where $C(\theta, n)$ is a constant depending only on the lower bound θ for the $(n-1)$-dimensional density ratio bound (Proposition 3.1) and on the dimension.

[11] The Wulff shape of Φ is the unique shape having the smallest Φ energy among all solids with unit volume. When Φ is isotropic, the Wulff shape is just the ball.

Proof (by contradiction)**.** Suppose that there exist a point $p_0 \in \partial K_0$ and $r_0 \leqq \triangle t$ such that

$$\mathscr{L}^n(K \cap B(p_0, r_0)) \leqq C r_0^n \tag{28}$$

where C will be chosen below. (The proof for $\mathscr{L}^n(K^c \cap B(p_0, r_0))$ is similar.) Then

$$\mathscr{L}^n(K \cap B(p_0, r_0)) = \int_0^{r_0} \mathscr{H}^{n-1}(K \cap \partial B(p_0, s))\, ds \leqq C r_0^n.$$

Thus there exists s_0 with $\frac{1}{2} r_0 \leqq s_0 \leqq r_0$ such that

$$\mathscr{H}^{n-1}(K \cap \partial B(p_0, s_0)) \leqq 2C r_0^{n-1}. \tag{29}$$

Considering the comparison set $K' = K \backslash B(p_0, s_0)$, we have

$$\begin{aligned} \Phi(\partial K) &+ \frac{1}{\triangle t} \int_{K \triangle K_0} \operatorname{Dist}(x, \partial K_0)\, d\mathscr{L}^n \\ &\leqq \Phi(\partial K') + \frac{1}{\triangle t} \int_{K' \triangle K_0} \operatorname{Dist}(x, \partial K_0)\, d\mathscr{L}^n, \\ \Phi(\partial K) &\leqq \Phi(\partial K') + \int_{K' \triangle K} \operatorname{Dist}(x, \partial K_0)\, d\mathscr{L}^n. \end{aligned}$$

Now $\Phi(\partial K) - \Phi(\partial K') = \Phi(\partial K \cap B(p, s_0)) - \Phi(K \cap \partial B(p_0, s_0))$. Invoking Propositions 3.1, 3.3, (29) and (28), we have

$$\begin{aligned} \theta s_0^{n-1} &\leqq 2C\Phi^0 C r_0^{n-1} + \frac{D\sqrt{\triangle t}}{\triangle t} \mathscr{L}^n(K \triangle K'), \\ \theta \frac{r_0^{n-1}}{2^{n-1}} &\leqq 2C\Phi^0 C r_0^{n-1} + \frac{D}{\sqrt{\triangle t}} C r_0^n, \\ \frac{\theta - 2^n C \Phi^0}{2^{n-1} C D} \sqrt{\triangle t} &\leqq r_0. \end{aligned}$$

Choosing C small enough leads to a contradiction to the hypothesis that $r_0 \leqq \sqrt{\triangle t}$. □

In this paper, we just make use of the lower bound for the $(n-1)$-dimensional density ratio and the volume-difference estimate. However, any $(\Phi, \triangle t, K_0)$-minimizer also enjoys other regularity properties: $\partial[\![K]\!]$ is Bomberi (Φ, ω, δ)-minimal; $\operatorname{spt} \partial[\![K]\!]$ is Almgren (γ, δ)-restricted with respect to the empty set; $\operatorname{spt} \partial K(t)$ is $\mathscr{H}^{n-1}$ almost everywhere a twice differentiable hypersurface (when Φ is smooth and elliptic). These are all stated in [ATW, Section 3].

4. Continuity and Energy Estimates

We now prove the continuity and energy estimates which are crucial in showing the tightness of the probability measures induced by our time-discretization scheme.

In the following, we use the notations introduced in Section 2.6. Let m be any positive integer, $0 \leqq s \leqq t \leqq 1$ and f be any C^2 function on $\mathscr{O}$. We use $C(f, m)$, C_m to denote positive constants depending only on f, m and the size of $\mathscr{O}$.

Theorem 4.1. *The processes* $\left\{K^N(t)\right\}_{N \geqq 1}$ *constructed in Section 2.6 satisfy the following statements.*

Weak Hölder Continuity. *For any* $f \in C^2(\mathscr{O})$, *let* $K_f^N(t) = \int_{x \in K^N(t)} f(x)\, d\mathscr{L}^n x$. *It can be decomposed as*

$$K_f^N(t) = S_f^N(t) + M_f^N(t). \tag{30}$$

(i) S_f^N *has the estimate*

$$E\left|S_f^N(t) - S_f^N(s)\right|^{2m} \leqq C(f, m)\,|t - s|^m\,. \tag{31}$$

(ii) $M_f^N(t)$ *is a piecewise constant function with jumps at the* t_i*'s. Moreover, for any* $t_p, t_q \in [0, 1]$,

$$E\left|M_f^N(t_q^+) - M_f^N(t_p^-)\right|^{2m} \leqq C(f, m)\left|t_q - t_p\right|^m\,. \tag{32}$$

Uniform Energy Bound.

$$E\left\{\sup_{\lambda \in [0,1]} \Phi(\partial K^N(\lambda))^m\right\} \leqq C_m. \tag{33}$$

We start the proof by first defining the decomposition (30). For simplicity, we assume that $t = t_q^+$. Then set

$$S_f^N(t) = \sum_{0 < t_i \leqq t} K_f^N(t_i^-) - K_f^N(t_{i-1}^+) + K_f^N(t_0^+), \tag{34}$$

$$M_f^N(t) = \sum_{0 < t_i \leqq t} K_f^N(t_i^+) - K_f^N(t_i^-). \tag{35}$$

Essentially S_f^N measures the changes of the sets due to the deformations by stochastic flows. M_f^N describes the changes (or jumps) during the minimization steps to approximate motion by mean curvature.

The proof of Theorem 4.1 relies on the use of Ito's formula to estimate various quantities. It is divided into four parts.

- Proof of (31) (Section 5).
- Proof of (33) (Section 6).
- Proof of (32) (Section 7).

- Proof of the lower bound for the density ratio under stochastic perturbations (Section 8).

To prove Theorem 4.1, we write the whole evolution as a stochastic integral, and patch together the estimates of each discretized interval by making use of the statistical cancellation property of the Brownian flow. Hence we need to make sure that the evolution is *adapted* to the filtration upon which the Brownian flow is defined.[12] To achieve this, it suffices to show the existence of a *Borel measurable map* Γ from $\mathscr{K}$ to $\mathscr{K}$ such that $\Gamma(K)$ gives a $(\Phi, \triangle t, K)$-minimizer. The technicality underlying this is treated in [SV, Chapter 12.1]. It is applied to our present situation in [Yip]. We do not repeat here.

Next, we rewrite (86) here for later reference.

$$\begin{aligned} d\left\langle F^\alpha(x,t),\ F^\beta(y,t)\right\rangle &= a^{\alpha,\beta}(x,y),\\ d\left\langle \partial_\gamma F^\alpha(x,t),\ \partial_\delta F^\beta(y,t)\right\rangle &= \partial_\gamma\partial'_\delta a^{\alpha,\beta}(x,y),\\ d\left\langle F^\alpha(x,t),\ \partial_\gamma F^\beta(y,t)\right\rangle &= \partial'_\gamma a^{\alpha,\beta}(x,y) \end{aligned}$$

where the differentiation ∂ on $a(\cdot,\cdot)$ is with respect to the first variable and ∂' to the second variable. Note that we use $\langle\ ,\ \rangle$ to denote the cross variation process between two semi-martingales.

Recall also the relationship (17) between φ and F.

5. Proof of (31) – Perturbations by Stochastic Flows

Without loss of generality, assume that $t = t_q \geqq s = t_p$. Then,

$$\begin{aligned} &S_f^N(t_q) - S_f^N(t_p)\\ &\quad= \sum_{i=p+1}^{q} K_f^N(t_i^-) - K_f^N(t_{i-1}^+)\\ &\quad= \sum_{i=p+1}^{q}\left(\int_{x\in K^N(t_i^-)} f(x)\,d\mathscr{L}^n x - \int_{x\in K^N(t_{i-1}^+)} f(x)\,d\mathscr{L}^n x\right)\\ &\quad= \sum_{i=p+1}^{q}\left(\int_{x\in\varphi_{t_{i-1},t_i}(K^N(t_{i-1}^+))} f(x)\,d\mathscr{L}^n x - \int_{x\in K^N(t_{i-1}^+)} f(x)\,d\mathscr{L}^n x\right)\\ &\quad= \sum_{i=p+1}^{q}\left(\int_{x\in K^N(t_{i-1}^+)} f(\varphi_{t_{i-1},t_i}(x))\det(D\varphi_{t_{i-1},t_i}(x))\,d\mathscr{L}^n x\right.\\ &\qquad\left. - \int_{x\in K^N(t_{i-1}^+)} f(x)\,d\mathscr{L}^n x\right). \end{aligned} \tag{36}$$

[12] See [KS, Chapter 1] for the definition and the need of adaptedness.

We apply Ito's Formula to rewrite the quantities in the parentheses and establish the following expression (see (40)),

$$S_f^N(t_q) - S_f^N(t_p) = \int_{t_p}^{t_q} A(K^N(r), f, dr) + B(K^N(r), f, dr)$$

where $A(K^N(t), f, t)$ is a function of bounded variations and $B(K^N(t), f, t)$ is a semi-martingale.

5.1. Ito's Formula for $f(\varphi_{t_{i-1},t}(x)),\ t_{i-1} \leqq t$

$$\begin{aligned}
& df(\varphi_{t_{i-1},t}(x)) \\
&= \sum_{\alpha=1}^{n} \frac{\partial f}{\partial x_\alpha}(\varphi_{t_{i-1},t}(x))\, d\varphi^\alpha_{t_{i-1},t}(x) \\
&\quad + \frac{1}{2}\sum_{\beta,\gamma=1}^{n} \frac{\partial^2 f}{\partial x_\beta \partial x_\gamma}(\varphi_{t_{i-1},t}(x)) d\left\langle \varphi^\beta_{t_{i-1},t}(x),\ \varphi^\gamma_{t_{i-1},t}(x)\right\rangle \\
&= \sum_{\alpha=1}^{n} \frac{\partial f}{\partial x_\alpha}(\varphi_{t_{i-1},t}(x))\, dF^\alpha(\varphi_{t_{i-1},t}(x), dt) \\
&\quad + \frac{1}{2}\sum_{\beta,\gamma=1}^{n} \frac{\partial^2 f}{\partial x_\beta \partial x_\gamma}(\varphi_{t_{i-1},t}(x)) a^{\beta,\gamma}(\varphi_{t_{i-1},t}(x), \varphi_{t_{i-1},t}(x), t)\, dt. \qquad (37)
\end{aligned}$$

5.2. Ito's Formula for $\det(D\varphi_{t_{i-1},t}(x)),\ t_{i-1} \leqq t$

We write $\det(D(\varphi_{t_{i-1},t}(x)) = \sum_\sigma \operatorname{sgn}(\sigma) \frac{\partial \varphi^1_{t_{i-1},t}(x)}{\partial x_{\sigma(1)}} \cdots \frac{\partial \varphi^n_{t_{i-1},t}(x)}{\partial x_{\sigma(n)}}$ where $\{\sigma\}$ is the collection of all permutations of $(1, 2, \cdots, n)$ and $\operatorname{sgn}(\sigma)$ is the sign of σ. After routine calculations[13], we obtain

$$\begin{aligned}
& d\left[\det(D\varphi_{t_{i-1},t}(x))\right] \\
&= \operatorname{div} F(\varphi_{t_{i-1},t}(x), dt) \det(D\varphi_{t_{i-1},t}(x)) \\
&\quad + \frac{1}{2}\left\{ \sum_{\beta,\gamma=1}^{n} \left(\partial_\beta \partial'_\gamma - \partial_\beta \partial'_\gamma\right) a^{\beta,\gamma}(\varphi_{t_{i-1},t}(x), \varphi_{t_{i-1},t}(x), t) \right\} \\
&\quad \times \det(D\varphi_{t_{i-1},t}(x))\, dt. \qquad (38)
\end{aligned}$$

[13] Similar calculation has been done in [Kun, 4.3.1].

5.3. *Combination of* (37) *and* (38)

$$
\begin{aligned}
&d\left[f(\varphi_{t_{i-1},t}(x))\det(D\varphi_{t_{i-1},t}(x))\right]\\
&\quad= f(\varphi_{t_{i-1},t}(x))d\left[\det(D\varphi_{t_{i-1},t}(x))\right] + \det(D\varphi_{t_{i-1},t}(x))df(\varphi_{t_{i-1},t}(x))\\
&\qquad + d\left\langle f(\varphi_{t_{i-1},t}(x)),\ \det(D\varphi_{t_{i-1},t}(x))\right\rangle\\
&\quad= f(\varphi_{t_{i-1},t}(x))\,\mathrm{div}\,F(\varphi_{t_{i-1},t}(x),dt)\det(D\varphi_{t_{i-1},t}(x))\\
&\qquad+\frac{f(\varphi_{t_{i-1},t}(x))}{2}\\
&\qquad\quad\times\left\{\sum_{\beta,\gamma=1}^{n}\left(\partial_\beta\partial'_\gamma-\partial_\beta\partial'_\gamma\right)a^{\beta,\gamma}(\varphi_{t_{i-1},t}(x),\varphi_{t_{i-1},t}(x),t)\right\}\\
&\qquad\quad\times\det(D\varphi_{t_{i-1},t}(x))\,dt\\
&\qquad+\left\{\sum_{\alpha=1}^{n}\frac{\partial f}{\partial x_\alpha}(\varphi_{t_{i-1},t}(x))F^\alpha(\varphi_{t_{i-1},t}(x),dt)\right.\\
&\qquad\quad\left.+\frac{1}{2}\sum_{\beta,\gamma=1}^{n}\frac{\partial^2 f}{\partial x_\beta\partial x_\gamma}(\varphi_{t_{i-1},t}(x))a^{\beta,\gamma}(\varphi_{t_{i-1},t}(x),\varphi_{t_{i-1},t}(x),t)\,dt\right\}\\
&\qquad\quad\times\det(D\varphi_{t_{i-1},t}(x))\\
&\qquad+\left\{\sum_{\alpha=1}^{n}\frac{\partial f}{\partial x_\alpha}(\varphi_{t_{i-1},t}(x))d\left\langle F^\alpha(\varphi_{t_{i-1},t}(x),t),\ \mathrm{div}\,F(\varphi_{t_{i-1},t}(x),t)\right\rangle\right\}\\
&\qquad\quad\times\det(D\varphi_{t_{i-1},t}(x)).
\end{aligned}
$$

We can also write

$$
d\left\langle F^\alpha(\varphi_{t_{i-1},t}(x),t),\ \mathrm{div}\,F(\varphi_{t_{i-1},t}(x),t)\right\rangle=\sum_{\delta=1}^{n}\partial'_\delta a^{\alpha,\delta}(\varphi_{t_{i-1},t}(x),\varphi_{t_{i-1},t}(x),t)\,dt.
$$

The final formula we arrive at is

$$
\begin{aligned}
&\int_K f(\varphi_{t_{t-1},t}(x))\det(D\varphi_{t_{t-1},t}(x))\,d\mathscr{L}^n x-\int_K f(x)\,d\mathscr{L}^n x\\
&\qquad=\int_{t_{i-1}}^{t}\int_K A_i(f,x,r)\,d\mathscr{L}^n x\,dr+\int_{t_{i-1}}^{t}\int_K B_i(f,x,dr)\,d\mathscr{L}^n x \qquad (39)
\end{aligned}
$$

where $A_i(f,x,r)$ denotes

$$
\begin{aligned}
&\left\{\frac{f(\varphi_{t_{i-1},r}(x))}{2}\left(\sum_{\beta,\gamma=1}^{n}\left(\partial_\beta\partial'_\gamma-\partial_\beta\partial'_\gamma\right)a^{\beta,\gamma}(\varphi_{t_{i-1},r}(x),\varphi_{t_{i-1},r}(x))\right)\right.\\
&\quad+\frac{1}{2}\sum_{\beta,\gamma=1}^{n}\frac{\partial^2 f}{\partial x_\beta\partial x_\gamma}(\varphi_{t_{i-1},r}(x))a^{\beta,\gamma}(\varphi_{t_{i-1},r}(x),\varphi_{t_{i-1},r}(x))\\
&\quad\left.+\left(\sum_{\alpha,\delta=1}^{n}\frac{\partial f}{\partial x_\alpha}(\varphi_{t_{i-1},r}(x))\partial'_\delta a^{\alpha,\delta}(\varphi_{t_{i-1},r}(x),\varphi_{t_{i-1},r}(x))\right)\right\}\det(D\varphi_{t_{i-1},r}(x))
\end{aligned}
$$

and $B_i(f, x, dr)$ denotes

$$\Big\{ f(\varphi_{t_{i-1},r}(x)) \operatorname{div} F(\varphi_{t_{i-1},r}(x), dr) + \sum_{\alpha=1}^{n} \frac{\partial f}{\partial x_\alpha}(\varphi_{t_{i-1},r}(x)) F^\alpha(\varphi_{t_{i-1},r}(x), dr) \Big\} \det(D\varphi_{t_{i-1},r}(x)).$$

In addition, we define

$$A^N(f, x, t) = \sum_{i=0} \int_{t_{i-1}\wedge t}^{t_i \wedge t} A_i(f, x, r)\, dr, \quad B^N(f, x, t) = \sum_{i=0} \int_{t_{i-1}\wedge t}^{t_i \wedge t} B_i(f, x, dr).$$

Hence from (36), we get

$$\begin{aligned} S_f^N(t_q) - S_f^N(t_p) &= \sum_{i=p+1}^{q} \int_{t_{i-1}}^{t_i} \int_{K_{i-1+}^N} A_i(f, x, r)\, dr + dB_i(f, x, r)\, d\mathscr{L}^n x \\ &= \int_{t_p}^{t_p} \int_{K^N(r)} dA^N(f, x, r) + dB^N(f, x, r)\, d\mathscr{L}^n x. \qquad (40) \end{aligned}$$

Now $A^N(f, x, t)$ is a process of bounded variation and $B^N(f, x, t)$ is a semi-martingale. Since the local characteristic (a, b) of F belongs to the class of $.\mathscr{B}_{ub}^{1,\delta}(\mathscr{O})$, we conclude from (113) that

$$\begin{aligned} \left| dA^N(f, x, t) \right| &\leqq C(f)\, dt, \\ \left| d\left\langle B^N(f, x, t),\ B^N(f, y, t) \right\rangle \right| &\leqq C(f)\, dt \end{aligned}$$

where $C(f)$ is a constant depending only on f and its derivatives up to second order.

By the BDG Inequality (21), for any positive integer m, we have

$$\begin{aligned} &E \Big| \int_{t_p}^{t_q} \int_{K^N(r)} dA^N(f, x, r) + dB^N(f, x, r)\, d\mathscr{L}^n x \Big|^{2m} \\ &\leqq C_m E \Big| \int_{t_p}^{t_q} \int_{K^N(r)} dA^N(f, x, r)\, d\mathscr{L}^n x \Big|^{2m} \\ &\quad + C_m E \Big| \int_{t_p}^{t_q} \int_{K^N(r)} dB^N(f, x, r)\, d\mathscr{L}^n x \Big|^{2m} \\ &\leqq C_m(f) \left| t_q - t_p \right|^{2m} + C_m(f) E \\ &\quad \times \Big| \int_{t_p}^{t_q} \iint_{(x,y)\in (K^N(r), K^N(r))} d\left\langle B^N(f, x, r),\ B^N(f, y, r) \right\rangle d\mathscr{L}^n x\, d\mathscr{L}^n y \Big|^m \\ &\leqq C_m(f) \left| t_q - t_p \right|^m \qquad (41) \end{aligned}$$

Hence the whole (31) is proved[14].

[14] There should also be a term for the bounded-variation part of the semi-martingale B^N. But its estimate can be absorbed into that for A^N.

6. Proof of (33) – Uniform Boundary Estimates

Statement (33) implies that almost surely the random measures associated with our sets have finite perimeters for all time.

By the definition of minimizations, we always have $\Phi(\partial K^N_{i+}) \leqq \Phi(\partial K^N_{i-})$. In between the minimizations, the set K^N_{i+} is deformed by the Brownian flow φ, i.e., for $t_i \leqq t < t_{i+1}$, $\Phi(\partial K^N(t)) = \Phi(\varphi_{t_i,t}\partial K^N_{i+})$. Combining the two steps (assuming that $t = t_q$ for simplicity), we obtain

$$\begin{aligned}\Phi(\partial K^N_{q^+}) &\leqq \Phi(\partial K^N_{q^-}) \leqq \Phi(\partial K^N_{q^-}) - \Phi(\partial K^N_{q-1^+}) + \Phi(\partial K^N_{q-1^+})\\ &\vdots\\ &\leqq \Phi(\partial K^N_{0^+}) + \sum_{i=0}^{q-1} \Phi(\partial K^N_{i+1^-}) - \Phi(\partial K^N_{i^+})\\ &= \Phi(\partial K^N_{0^+}) + \sum_{i=0}^{q-1} \Phi(\varphi_{t_i,t_{i+1}}\partial K^N_{i^+}) - \Phi(\partial K^N_{i^+}). \qquad (42)\end{aligned}$$

Now we look at the term $\Phi(\varphi_{t_i,t_{i+1}}\partial K^N_{i^+}) - \Phi(\partial K^N_{i^+})$ in detail.

6.1. Ito's Formula for $\Phi(\varphi_{s,t}\partial K)$, $0 \leqq s \leqq t \leqq 1$

Let $K \in \mathscr{K}$. We borrow the notations and formulas from geometric measure theory, especially the change-of-variables formula for $(n-1)$-dimensional integration. We write

$$\partial[\![K]\!] = \boldsymbol{t}(\partial K, 1, \sigma), \quad \varphi_{s,t}\partial[\![K]\!] = \boldsymbol{t}(\varphi_{s,t}\partial K, 1, \sigma_t).$$

Then

$$\Phi(\varphi_{s,t}\partial[\![K]\!]) = \int_{\varphi_{s,t}\partial K} \Phi(\sigma_t)\, d\mathscr{H}^{n-1} = \int_{\partial K} \Phi(\left[\textstyle\bigwedge_{n-1} D\varphi_{s,t}\right]\sigma)\, d\mathscr{H}^{n-1}.$$

These notations can be found in [ATW, 3.1] and [Fed, Chapter 1]. We briefly describe them here. $\partial[\![K]\!]$ is an $(n-1)$-integral current. σ denotes the approximate tangent plane of $\partial[\![K]\!]$. It is a simple unit $(n-1)$-vector in the Grassmann vector space $\bigwedge_{n-1} R^n$. $\bigwedge_{n-1} D\varphi_{s,t}$ is a linear map on $\bigwedge_{n-1} R^n$ such that

$$\left[\textstyle\bigwedge_{n-1} D\varphi_{s,t}\right](v_1 \wedge \cdots \wedge v_{n-1}) = (D\varphi_{s,t}v_1) \wedge \cdots \wedge (D\varphi_{s,t}v_{n-1}),$$

$$v_1, \ldots, v_{n-1} \in R^n.$$

Let $\pi_t = \left[D\varphi_{s,t}\right]\sigma$. (It can also be treated as a vector in R^n.) Since Φ is in C^2, by Ito's Formula, we have,

$$d\Phi(\pi_t) = \sum_{i=1}^n \partial_i \Phi(\pi_t) d\pi_t^i + \frac{1}{2}\sum_{ij}^n \partial^2_{ij}\Phi(\pi_t) d\langle \pi_t^i, \pi_t^j\rangle.$$

For $0 \leqq s \leqq t \leqq 1$, define[15]

$$A_s(x,\sigma,t) = \int_s^t \frac{1}{2}\sum_{ij}^{n} \partial_{ij}^2 \Phi(\pi_r) d\left\langle \pi_r^i,\ \pi_r^j \right\rangle,$$

$$B_s(x,\sigma,t) = \int_s^t \sum_{i=1}^{n} \partial_i \Phi(\pi_r) d\pi_r^i.$$

Then,

$$\Phi(\varphi_{s,t}\partial K) = \Phi(\partial K) + \int_s^t \int_{\partial K} dA_s(x,\sigma,r)\, d\mathscr{H}^{n-1}x + \int_s^t \int_{\partial K} dB_s(x,\sigma,r)\, d\mathscr{H}^{n-1}x. \tag{43}$$

Of course A_s and B_s can be written in terms of the local characteristics of F. It is then easy to see that A_s is a process of bounded variation and B_s is a semi-martingale. Since $\Phi \in C^2$ and since the local characteristic (a, b) of F belongs to $\mathscr{B}_{ub}^{1,\delta}(\mathscr{O})$, we conclude from (113) that

$$|dA_s(x,\sigma,t)| \leqq C dt,$$
$$|d\left\langle B_s(x,\sigma,t),\ B_s(y,\sigma,t)\right\rangle| \leqq C dt.$$

6.2. Derivation of the Uniform Energy Estimates

Set

$$A^N(x,\sigma,t) = \sum_{i=0} \int_{t_i\wedge t}^{t_{i+1}\wedge t} dA_{t_i}(x,\sigma,r),$$

$$B^N(x,\sigma,t) = \sum_{i=0} \int_{t_i\wedge t}^{t_{i+1}\wedge t} B_{t_i}(x,\sigma,dr).$$

Substitute (43) into (42):

$$\Phi(\partial K^N(t)) \leqq \Phi(\partial K^N(0)) + \int_0^t \int_{\partial K^N(r)} dA^N(x,\sigma,r)\, d\mathscr{H}^{n-1}x + \int_0^t \int_{\partial K^N(r)} dB^N(x,\sigma,r)\, d\mathscr{H}^{n-1}x. \tag{44}$$

Denote $M_t^N = \int_0^t \int_{\partial K^N(r)} dB^N(x,\sigma,r)\, d\mathscr{H}^{n-1}x$. Then M_t^N is a (local)-martingale with

[15] Note the similarities between the definitions of $A_{s,\partial K}(x,\sigma,t)$, $B_{s,\partial K}(x,\sigma,t)$ and those of $A_i(f,x,t)$, $B_i(f,x,t)$ on page 330. The arguments followed are also quite parallel.

$$\left\langle M_t^N \right\rangle = \int_0^t \iint_{(x,y)\in\partial K^N(r)\times\partial K^N(r)} d\left\langle B^N(x,\sigma,r),\ B^N(y,\sigma,r)\right\rangle \times d\mathscr{H}^{n-1}x\, d\mathscr{H}^{n-1}y \leqq \int_0^t \Phi(\partial K^N(r))^2\, dr.$$

Hence, for any positive integer m,

$$\Phi(\partial K^N(t))^{2m} \leqq C_m \Phi(\partial K(0))^{2m} + C_m \left(\int_0^t \int_{\partial K^n(r)} dA^N(x,\sigma,r)\, d\mathscr{H}^{n-1}x\right)^{2m} + C_m (M_t^N)^{2m}.$$

Upon taking expectation and using the BDG Inequality (21) again, we get

$$\begin{aligned} &E\left\{\sup_{\lambda\in[0,t]} \Phi(\partial K^N(\lambda))^{2m}\right\} \\ &\leqq C_m + C_m \int_0^t E\Phi(\partial K^N(r))^{2m}\, dr + C_m E\left\langle M_t^N\right\rangle^m \\ &\leqq C_m + C_m \int_0^t E\Phi(\partial K^N(r))^{2m}\, dr + C_m E\left(\int_0^t \Phi(\partial K^N(r))^2\, dr\right)^m \\ &\leqq C_m + D_m \int_0^t E\left\{\sup_{\lambda\in[0,r]} \Phi(\partial K^N(\lambda))^{2m}\right\} dr. \end{aligned}$$

By Gronwall's Inequality, we finally get (33):

$$E\left\{\sup_{\lambda\in[0,1]} \Phi(\partial K^N(\lambda))^{2m}\right\} \leqq C_m.$$

7. Proof of (32) – Approximation of Motion by Mean Curvature

This section treads the changes of the sets during the minimization steps to approximate motion by mean curvature.

Recall that $M_f^N(t) = \sum_{0<t_i\leqq t} K_f^N(t_i^+) - K_f^N(t_i^-)$. Assume that $t = t_q > t_p = s$. Then

$$\left|M_f^N(t_q) - M_f^N(t_p)\right| \leqq \|f\|_\infty \sum_{p\leqq t_i\leqq q} \mathscr{L}^n(K^N(t_i^+)\triangle K^N(t_i^-)). \tag{45}$$

Now $\partial K_{i^-}^N = \varphi_{t_{i-1},t_i}(\partial K_{i-1^+}^N)$. Let θ_i' be the lower bound on the density ratio of $\partial K_{i^-}^N$, i.e., for any $p' \in \operatorname{spt} \partial[\![K_{i^-}^N]\!]$ and $0 < r \leqq \sqrt{\triangle t}$,

$$\frac{\mathscr{H}^{n-1}(\partial K_{i^-}^N \cap B^n(p',r))}{r^{n-1}} \geqq \theta_i'. \tag{46}$$

According to Corollary 3.2, upon choosing $R = \triangle t \left(t_q - t_p\right)^{-1/2}$, we obtain

$$\mathscr{L}^n(K_{i^+}^N \triangle K_{i^-}^N) \leqq A \frac{\triangle t}{\sqrt{t_q - t_p}} \frac{\mathscr{H}^{n-1}\left(\partial K_{i^-}^N\right)}{\theta_i'} + \sqrt{t_q - t_p}\left(\Phi(\partial K_{i^-}^N) - \Phi(\partial K_{i^+}^N)\right). \tag{47}$$

Hence,

$$\begin{aligned}
&\sum_{i=p}^{q} \mathscr{L}^n(K_{i^+}^N \triangle K_{i^-}^N) \\
&\quad \leqq A \left\{ \sup_i \frac{\mathscr{H}^{n-1}(\partial K_{i^-}^N)}{\theta_i'} \right\} \sqrt{t_q - t_p} \\
&\qquad + \left\{ \sum_{i=p}^{q} (\Phi(\partial K_{i^-}) - \Phi(\partial K_{i^+})) \right\} \sqrt{t_q - t_p}.
\end{aligned} \tag{48}$$

Now by (44),

$$\begin{aligned}
&\sum_{i=p}^{q} \Phi(\partial K_{i^-}^N) - \Phi(\partial K_{i^+}^N) \\
&\quad = \Phi(\partial K_{p^-}^N) + \left\{ \sum_{i=p+1}^{q} \Phi(\partial K_{i^-}^N) - \Phi(\partial K_{i-1^+}^N) \right\} - \Phi(\partial K_{q^+}^N) \\
&\quad \leqq \Phi(\partial K_{p^-}^N) + \int_{t_p}^{t_q} \int_{\partial K^N(r)} dA^N(x, \sigma, r)\, d\mathscr{H}^{n-1}x \\
&\qquad + \int_{t_p}^{t_q} \int_{\partial K^N(r)} dB^N(x, \sigma, r)\, d\mathscr{H}^{n-1}x.
\end{aligned}$$

By (33), we can easily deduce that $E\left\{\left(\sum_{i=p}^{q} \Phi(\partial K_{i^-}^N) - \Phi(\partial K_{i^+}^N)\right)^m\right\} \leqq C_m$.

In the next section we will show that

$$E\left\{ \sup_i \left(\frac{\mathscr{H}^{n-1}(\partial K_{i^-}^N)}{\theta_i'} \right)^m \right\} \leqq C_m \quad \text{or simply} \quad E\left\{ \sup_i \left(\theta_i'\right)^{-m} \right\} \leqq C_m \tag{49}$$

for any positive integer m. Thus by taking powers of (48), we arrive at

$$E\left| \sum_{i=p}^{q} \mathscr{L}^n(K_{i^+}^N \triangle K_{i^-}^N) \right|^{2m} \leqq C_m \left|t_q - t_p\right|^m. \tag{50}$$

8. Lower Bound for the Density Ratio under Brownian Flows

In this section, we prove (49). It is an intuitively plausible result but it takes quite a bit of work to compose a proof. The difficulty is that the required estimate is uniform in space and time. Our strategy is to replace the desired quantities by those easier to be dealt with.

We first recall the notations. K^N_{i+} is a $(\Phi, \triangle t, K^N_{i-})$-minimizer. By Proposition 3.1, it has an $(n-1)$-dimensional lower density ratio bound θ. ∂K^N_{i-} is obtained from ∂K^N_{i-1+} by the stochastic-flow deformation, $\partial K^N_{i-} = \varphi_{t_{i-1},t_i} \partial K^N_{i-1+}$. Let θ'_i (which is now a *random variable*) be the lower bound for the density ratio for ∂K^N_{i-} in the sense of (46). Our goal is to show

Theorem 8.1 (Uniform Estimate on Lower Density Ratio). *For any positive integer m,*

$$E \sup \left\{ (\theta'_i)^{-m} : i = 1, \ldots, N \right\} \leqq C_m. \tag{51}$$

The theorem is proved by first estimating $1/\theta'_i$ in terms of the spatial derivatives of the Brownian flow φ (Lemmas 8.2 and 8.3) and then establishing bounds for these derivatives (Theorem 8.4). What helps us here is that the value of θ (from minimization) is a fixed number which does not depend on what is happening before the minimization. Furthermore, in between the minimizations, the set is transported by φ for a period of $\triangle t$ so that the perturbation is roughly equal to zero. We can then concentrate on any single interval $[t_{i-1}, t_i)$.

8.1. Preliminary Computations for θ'_i

For any $p' \in \operatorname{spt} \partial [\![K^N_{i-}]\!]$ and $0 < r \leqq \triangle t$, we have

$$\begin{aligned}
&\frac{\mathscr{H}^{n-1}(\partial K^N_{i-} \cap B^n(p', r))}{r^{n-1}} \\
&\quad = \frac{1}{r^{n-1}} \int_{y \in \partial K^N_{i-}} \chi_{B^n(p',r)} \, d\mathscr{H}^{n-1} y \\
&\quad = \frac{1}{r^{n-1}} \int_{x \in \partial K^N_{i-1+}} \chi_{(\varphi^{-1}_{t_{i-1},t_i})(B^n(p',r))} \left\| \textstyle\bigwedge_{n-1} D\varphi_{t_{i-1},t_i}(x) \right\| \, d\mathscr{H}^{n-1} x \\
&\quad \geqq \frac{\inf_x \left\| \bigwedge_{n-1} D\varphi_{t_{i-1},t_i}(x) \right\|}{r^{n-1}} \int_{x \in \partial K^N_{i-1+}} \chi_{(\varphi^{-1}_{t_{i-1},t_i})(B^n(p',r))} \, d\mathscr{H}^{n-1} x \\
&\quad \geqq \frac{\inf_x \left\| \bigwedge_{n-1} D\varphi_{t_{i-1},t_i}(x) \right\|}{r^{n-1}} \int_{x \in \partial K^N_{i-1+}} \chi_{B^n(p,r_{IN})} \, d\mathscr{H}^{n-1} x
\end{aligned} \tag{52}$$

where $p = \varphi^{-1}_{t_{i-1},t_i}(p')$ and $B^n(p, r_{IN})$ is the biggest ball centered at p staying inside $\varphi^{-1}_{t_{i-1},t_i}(B^n(p', r))$. Hence,

$$\theta'_i \geqq \frac{\inf_x \left\| \bigwedge_{n-1} D\varphi_{t_{i-1},t_i}(x) \right\|}{r^{n-1}} \inf_p \int_{x \in \partial K^N_{i-1+}} \chi_{B^n(p,r_{IN})} \, d\mathscr{H}^1 x$$

or

$$\frac{1}{\theta_i'} \leqq \left(\inf_x \left\|\bigwedge_{n-1} D\varphi_{t_{i-1},t_i}(x)\right\|\right)^{-1} \sup_p \left\{ r^{n-1} \left(\int_{x\in\partial K^N_{i-1+}} \chi_{B^n(p,r_{IN})}\, d\mathscr{H}^1 \right)^{-1} \right\}. \tag{53}$$

Remarks. • For any simple unit vector σ in $\bigwedge_{n-1} R^n$, we have

$$\sigma = [\textstyle\bigwedge_{n-1} D\varphi^{-1}_{t_{i-1},t_i}][\bigwedge_{n-1} D\varphi_{t_{i-1},t_i}]\sigma;$$

$$\text{hence } \left(\inf_x \left\|\textstyle\bigwedge_{n-1} D\varphi_{t_{i-1},t_i}(x)\right\|\right)^{-1} \leqq \sup_y \left\|\textstyle\bigwedge_{n-1} D\varphi^{-1}_{t_{i-1},t_i}(y)\right\|.$$

• Let ω denote an element of the probability space Ω. Then

$$\begin{aligned} & r^{n-1} \left(\int_{x\in\partial K^N_{i-1+}} \chi_{\{B^n(p,r_{IN})\}}\, d\mathscr{H}^1 \right)^{-1} \\ & \leqq \chi_{\{r_{IN}>\triangle t\}}(\omega)\frac{1}{\theta} + \chi_{\{r_{IN}\leqq\triangle t\}}(\omega)\frac{r^{n-1}}{r^{n-1}_{IN}}\frac{1}{\theta} \\ & = \frac{1}{\theta}\left\{1 + \left|\frac{r^{n-1}}{r^{n-1}_{IN}} - 1\right|\right\}. \end{aligned}$$

(Note that we have made use of the deterministic lower density ratio bound.)

Thus we can estimate $(\theta_i')^{-1}$ through the form of

Lemma 8.2.

$$\frac{1}{\theta_i'} \leqq \frac{1}{\theta}\left\{1 + \sup_x \left\| D\varphi^{-1}_{t_{i-1},t_i}(x) - I \right\|\right\}\left\{1 + \sup_p \left|\frac{r^{n-1}}{r^{n-1}_{IN}} - 1\right|\right\}. \tag{54}$$

The following result bounds the right-hand side of (54) in terms of the spatial derivatives of φ.

Lemma 8.3 (from Multivariable Calculus)**.** *Suppose φ is a C^1 diffeomorphism from $\mathscr{O}$ to $\mathscr{O}$ with C^1 inverse $\varphi^{-1} = \psi$. Let $p \in \mathscr{O}$ and $\varphi(p) = p'$. Let $r_{IN}(p)$ denote the largest radius for the ball centered at p such that*

$$B^n(p, r_{IN}(p)) \subset \psi(B^n(p', r)).$$

Then

$$\left|\frac{r}{r_{IN}} - 1\right| \leqq \sup_{x\in\mathscr{O}} \{\|D\varphi(x) - I\|, \|D\psi(x) - I\|\} \tag{55}$$

where I is the identity matrix and $\|\cdot\|$ is the matrix sup norm.

Proof. Take any straight line C_1 from p to a point on $\partial B^n(p, r_{IN})$ touching $\partial(\psi B^n(p, r))$. Then

$$r \leqq \mathscr{H}^1(\varphi \circ C_1) = \int_{x \in C_1} \left\| \bigwedge_1 D\varphi(x) \right\| \, d\mathscr{H}^1 x,$$

$$\frac{r}{r_{IN}} \leqq \frac{1}{r_{IN}} \int_{C_1} \left\| \bigwedge_1 D\varphi(x) \right\| \, d\mathscr{H}^1 x,$$

$$\frac{r}{r_{IN}} - 1 \leqq \frac{1}{r_{IN}} \int_{C_1} \left(\left\| \bigwedge_1 D\varphi(x) \right\| - 1 \right) d\mathscr{H}^1 x \leqq \sup_x \left| \left\| \bigwedge_1 D\varphi(x) \right\| - 1 \right|. \quad (56)$$

Similarly, take any straight line C_2 from p' to any point on $\partial B^n(p', r)$. Then

$$r_{IN} \leqq \mathscr{H}^1(\psi \circ C_2) = \int_{y \in C_2} \left\| \bigwedge_1 D\psi(y) \right\| \, d\mathscr{H}^1 y,$$

$$\frac{r_{IN}}{r} \leqq \frac{1}{r} \int_{y \in C_2} \left\| \bigwedge_1 D\psi(y) \right\| \, d\mathscr{H}^1 y,$$

$$\frac{r_{IN}}{r} - 1 \leqq \frac{1}{r} \int_{y \in C_2} \left(\left\| \bigwedge_1 \psi(y) \right\| - 1 \right) \mathscr{H}^1 y \leqq \sup_y \left| \left\| \bigwedge_1 D\psi(y) \right\| - 1 \right|.$$

Thus

$$\left(\sup_y \left| \left\| \bigwedge_1 \psi(y) \right\| - 1 \right| + 1 \right)^{-1} \leqq \frac{r}{r_{IN}},$$

$$\frac{-\sup_y \left| \left\| \bigwedge_1 D\psi(y) \right\| - 1 \right|}{1 + \sup_y \left| \left\| \bigwedge_1 D\psi(y) \right\| - 1 \right|} \leqq \frac{r}{r_{IN}} - 1. \quad (57)$$

Combining (56) and (57), we get

$$\left| \frac{r}{r_{IN}} - 1 \right| \leqq \max \left\{ \sup_x \left| \left\| \bigwedge_1 D\varphi(x) \right\| - 1 \right|, \sup_y \left| \left\| \bigwedge_1 D\psi(y) \right\| - 1 \right| \right\}. \quad (58)$$

Let A be an invertible linear transformation on R^n and let v be any vector. It is easy to show that

$$\left| \frac{\|Av\|}{\|v\|} - 1 \right| \leqq \max \left\{ \|A - I\|, \|A^{-1} - I\| \right\}.$$

Setting A equal to $D\varphi$ and $D\psi$ in (58), we obtain (55). □

We will apply the above result to $\varphi = \varphi_{t_{i-1}, t_i}(x)$ and $\psi = \varphi^{-1}_{t_{i-1}, t_i}(x) = \varphi_{t_i, t_{i-1}}(x)$.

By the previous two lemmas, the following statement suffices to prove Theorem 8.1.

Theorem 8.4 (Uniform Spatial Estimates of Brownian Flows). *Let* $\{\varphi_{s,t}\}_{0\leqq s\leqq t\leqq 1}$ *be a Brownian flow generated by a* $C^2(\mathscr{O}, R^n)$*-Brownian motion* F *with its local characteristic* (a, b) *belonging to* $\mathscr{B}^{2,\delta}_{ub}(\mathscr{O})$*. Then for any positive integer* m *and* $0 \leqq s \leqq t \leqq 1$*,*

$$E\left\{\sup_{x\in\mathscr{O},\lambda\in[s,t]} \left\|D\varphi_{s,\lambda}(x) - I\right\|^{2m}\right\} \leqq C_m\, |t-s|^m\,, \tag{59}$$

$$E\left\{\sup_{x\in\mathscr{O},\lambda\in[s,t]} \left\|D\varphi_{\lambda,s}(x) - I\right\|^{2m}\right\} \leqq C_m\, |t-s|^m\,. \tag{60}$$

Note that $\varphi_{\lambda,s}(x) = \varphi^{-1}_{s,\lambda}(x)$ for $\lambda \geqq s$. Estimate (60) is thus a statement about the *backward flow* of φ.

8.2. Proof of (59)

Recall that φ satisfies the stochastic differential equation (17). Upon differentiating (17) with respect to x, we obtain

$$D\varphi_{s,t}(x) = I + \int_s^t DF(\varphi_{s,r}(x), dr)D\varphi_{s,r}(x). \tag{61}$$

Hence in coordinate form,

$$\left(\partial_\beta\varphi^\alpha_{s,t}(x) - \delta_{\alpha,\beta}\right)^{2m} = \left(\sum_\gamma \int_s^t \partial_\gamma F^\alpha(\varphi_{s,r}(x), dr)\partial_\beta\varphi^\gamma_{s,r}(x)\right)^{2m}.$$

By the BDG Inequality (21) and the assumptions on the local characteristic of F, for any positive integer m,

$$\begin{aligned}
&E\left\{\sup_{\lambda\in[s,t]} \left|\partial_\beta\varphi^\alpha_{s,\lambda}(x) - \delta_{\alpha,\beta}\right|^{2m}\right\}\\
&\leqq C_m E\left(\int_s^t \sum_{\gamma,\delta} \partial_\beta\varphi^\gamma_{s,r}(x)\partial_\beta\varphi^\delta_{s,r}(x)\partial_\gamma\partial'_\delta a^{\alpha,\alpha}(\varphi_{s,r}(x), \varphi_{s,r}(x), r)dr\right)^m\\
&\leqq C_m\, |t-s|^m\,.
\end{aligned}$$

Hence,

$$E\left\{\sup_{\lambda\in[s,t]}\int_{x\in\mathscr{O}} \left\|D\varphi_{s,\lambda}(x) - I\right\|^{2m}\, d\mathscr{L}^n x\right\} \leqq C_m\, |t-s|^m\,. \tag{62}$$

Differentiating (61) again, we obtain

$$\begin{aligned}
\partial^2_{\beta,\gamma}\varphi^\alpha_{s,t}(x) = \int_s^t &\sum_{\nu,\mu} \partial^2_{\nu,\mu} F^\alpha(\varphi_{s,r}(x), dr)\partial_\gamma\varphi^\mu_{s,r}(x)\partial_\beta\varphi^\nu_{s,r}(x)\\
&+ \sum_\nu \partial_\nu F^\alpha(\varphi_{s,r}(x), dr)\partial^2_{\beta,\gamma}\varphi^\nu_{s,r}(x).
\end{aligned}$$

We can similarly deduce that

$$E\left\{\sup_{\lambda\in[s,t]}\int_{x\in\mathscr{O}}\left\|D^2\varphi_{s,\lambda}(x)\right\|^{2m}\,d\mathscr{L}^n x\right\}\leqq C_m\,|t-s|^m\,. \tag{63}$$

By the Sobolev Embedding Theorem, for $2m > n$,

$$\begin{aligned}&\sup_{x\in\mathscr{O}}\left\|D\varphi_{s,\lambda}(x)-I\right\|^{2m}\\&\leqq C_m\left\{\int_{x\in\mathscr{O}}\left\|D\varphi_{s,\lambda}(x)-I\right\|^{2m}\,d\mathscr{L}^n x+\int_{x\in\mathscr{O}}\left\|D^2\varphi_{s,\lambda}(x)\right\|^{2m}\,d\mathscr{L}^n x\right\}.\end{aligned}$$

Hence we conclude that

$$E\left\{\sup_{x\in\mathscr{O},\lambda\in[s,t]}\left\|D\varphi_{s,\lambda}(x)-I\right\|^{2m}\right\}\leqq C_m\,|t-s|^m\,. \tag{64}$$

8.3. Proof of (60)

First we state the following identity[16]: For any $0 \leqq s \leqq t \leqq 1$, $x \in \mathscr{O}$, $\left[D\varphi_{s,t}(x)\right]^{-1}$ satisfies

$$\begin{aligned}\left[D\varphi_{s,t}(x)\right]^{-1}=I&-\int_s^t\left[D\varphi_{s,r}(x)\right]^{-1}DF(\varphi_{s,r}(x),dr)\\&+\int_s^t\left[D\varphi_{s,r}(x)\right]^{-1}\tilde{A}(\varphi_{s,r}(x),\varphi_{s,r}(x),r)dr\end{aligned} \tag{65}$$

where $\tilde{A}(x,y,t)$ is the matrix $\tilde{A}^{ij}(x,y,t)=\sum_l \partial_{x_j}\partial'_{y_l}a^{il}(x,y,t)$.

Denote $J_{s,t}(x)=\left[D\varphi_{s,t}(x)\right]^{-1}$ in (65). By techniques like those in the proof of (59), it is easy to see that

$$E\left\{\sup_{\lambda\in[s,t]}\left\|J_{s,t}(x)\right\|^{2m}\right\}\leqq C_m,\quad E\left\{\sup_{\lambda\in[s,t]}\left\|J_{s,t}(x)-I\right\|^{2m}\right\}\leqq C_m\,|t-s|^m\,. \tag{66}$$

[16] This is stated in [Kun, Exercise 4.4.3]. The proof is simply obtained by first showing that (65) has a unique solution for $J_{s,t}(x)$ which replaces $\left[D\varphi_{s,t}(x)\right]^{-1}$ in the equation. By Ito's Formula, we have $d(J_{s,t}(x)D\varphi_{s,t}(x))=0$. Hence $J_{s,t}(x)D\varphi_{s,t}(x)=J_{s,s}(x)D\varphi_{s,s}(x)=I$. We can also prove (60) by using a backward stochastic differential equation (see [Kun]). The approach we use here avoids such a concept.

Since $\varphi_{s,t}$ is a C^2-diffeomorphism, by Cramer's Rule, $DJ_{s,t}(x) = D([D\varphi_{s,t}(x)]^{-1})$ also exists. Hence upon differentiating (65), we obtain

$$\begin{aligned}
DJ_{s,t}(x) = &-\int_s^t DJ_{s,r}(x)DF(\varphi_{s,r}(x), dr) \\
&-\int_s^t J_{s,r}(x)D^2F(\varphi_{s,r}(x), dr)D\varphi_{s,r}(x) \\
&+\int_s^t DJ_{s,r}(x)\tilde{A}(\varphi_{s,r}(x), \varphi_{s,r}(x), r)dr \\
&+\int_s^t J_{s,r}(x)D\tilde{A}(\varphi_{s,r}(x), \varphi_{s,r}(x), r)D\varphi_{s,r}(x)dr \\
&+\int_s^t J_{s,r}(x)D'\tilde{A}(\varphi_{s,r}(x), \varphi_{s,r}(x), r)D\varphi_{s,r}(x)dr.
\end{aligned}$$

From this equation and (66), it follows that

$$E\left\{\sup_{\lambda\in[s,t]}\left\|DJ_{s,\lambda}(x)\right\|^{2m}\right\} \leqq C_m\left|t-s\right|^m. \tag{67}$$

Invoking the Sobolev Embedding Theorem again, for $2m > n$,

$$\begin{aligned}
&E\left\{\sup_{x\in\mathscr{O},\lambda\in[s,t]}\left\|J_{s,\lambda}(x)-I\right\|^{2m}\right\} \\
&\leqq C_m E\left\{\sup_{\lambda\in[s,t]}\left(\int_{x\in\mathscr{O}}\left\|J_{s,\lambda}(x)-I\right\|^{2m}\, d\mathscr{L}^n x+\int_{x\in\mathscr{O}}\left\|DJ_{s,\lambda}(x)\right\|^{2m}\, d\mathscr{L}^n x\right)\right\} \\
&\leqq C_m\left|t-s\right|^m.
\end{aligned} \tag{68}$$

Finally, for $\lambda \geqq s$,

$$\begin{aligned}
&\left[D\varphi_{\lambda,s}(x)\right] = \left[D\varphi_{s,\lambda}(\varphi_{s,\lambda}(x))\right]^{-1} = J_{s,\lambda}(\varphi_{\lambda,s}(x)), \\
&\sup_{x\in\mathscr{O}}\left\|D\varphi_{\lambda,s}(x)-I\right\| = \sup_{x\in\mathscr{O}}\left\|J_{s,\lambda}(\varphi_{\lambda,s}(x))-I\right\| = \sup_{x\in\mathscr{O}}\left\|J_{s,\lambda}(x)-I\right\|,
\end{aligned}$$

from which we conclude that $E\left\{\sup_{x\in\mathscr{O},\lambda\in[s,t]}\left\|D\varphi_{\lambda,s}(x)-I\right\|^{2m}\right\} \leqq C_m\left|t-s\right|^m$.

The whole Theorem 8.4 is thus proved.

8.4. Final Steps in Proving Stochastic Lower Density Ratio Bound

From Lemmas 8.2 and 8.3,

$$\begin{aligned}\frac{1}{\theta_i'} &\leqq \frac{1}{\theta}\left(1+\sup_{x\in\mathcal{O}}\left\|\bigwedge_{n-1}D(\varphi_{t_i,t_{i-1}}(x))-I\right\|\right)\left(1+\left|\frac{r^{n-1}}{r_{IN}^{n-1}}-1\right|\right)\\ &\leqq \frac{1}{\theta}\left(1+\sup_{x\in\mathcal{O}}\left\|\bigwedge_{n-1}D(\varphi_{t_i,t_{i-1}}(x))-I\right\|\right)\left(1+\sum_{k=1}^{n-1}c_k\left|\frac{r}{r_{IN}}-1\right|^k\right)\\ &\leqq \frac{1}{\theta}\left(1+\sup_{x\in\mathcal{O}}\left\|\bigwedge_{n-1}D(\varphi_{t_i,t_{i-1}}(x))-I\right\|\right)\\ &\quad\times\left\{1+\sum_{k=1}^{n-1}c_k\left(\sup_x\left\|D\varphi_{t_i,t_{i-1}}(x)-I\right\|+\sup_x\left\|D\varphi_{t_{i-1},t_i}(x)-I\right\|\right)^k\right\}.\end{aligned}$$

Upon expanding the terms in the parenthesis and making use of Theorem 8.4, we have $(\theta_i')^{-1} \leqq \theta^{-1} + X_i$ where the X_i's are random variables such that for any positive integer m, $EX_i^{2m} \leqq C_m \triangle t^m$. Simple arguments from probability theory give $E\left\{\sup_i X_i^m\right\} \leqq C_m$ for all m.

Theorem 8.1 is thus proved.

9. Measure-Valued Processes and Their Tightness

Theorem 4.1 gives us all the ingredients to prove Theorem 2.7. We will treat our evolving sets as a *measure valued process* and show tightness in the weak topology of such space. Then uniform estimate for the surface energy of the boundary implies tightness in the strong, L^1 or mass topology[17].

We first introduce some basic notions related to measure-valued processes[18].

Definition 9.1 (Recall that $\mathcal{O}$ is a compact domain in R^n). Let $\mathcal{M}(\mathcal{O})$ be the space of positive Radon measures μ on $\mathcal{O}$ such that $\mu(\mathcal{O}) \leqq M$. For any $f \in C(\mathcal{O})$, the space of continuous functions defined on $\mathcal{O}$, denote $\mu(f) = \int_{x\in\mathcal{O}} f(x)\,d\mu x$. A sequence $\left\{\mu^N\right\}_{N\geqq 1}$ is said to *converge weakly* to ν in $\mathcal{M}(\mathcal{O})$ if for any $f \in C(\mathcal{O})$, $\nu(f) = \lim_N \mu^N(f)$. $\mathcal{M}(\mathcal{O})$ can be metrized to be a *complete separable metric space* by the metric

$$d(\mu,\nu) = \sum_l \frac{1}{2^l}\frac{|\mu(f_l)-\nu(f_l)|}{1+|\mu(f_l)-\nu(f_l)|} \tag{69}$$

where $\{f_l\}_{l\geqq 1}$ is a sequence of functions dense in $C(\mathcal{O})$. We can choose f_l to be in C^2 on $\mathcal{O}$. They are fixed throughout this section.

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[17] The concept of tightness is described in the Appendix.

[18] General statements of these sorts can be found in [Daw] or [Jak]. We just mention those concepts sufficient to prove our theorem.

Let $C([0,1],\mathscr{M}(\mathscr{O}))$ be the space of $\{\mu_t \in \mathscr{M}(\mathscr{O})\}_{t\in[0,1]}$ such that for each $f \in C(\mathscr{O})$, $\mu_t(f) \in C([0,1],R)$. A sequence $\{\mu_t^N\}_{N\geqq 1}$ in $C([0,1],\mathscr{M}(\mathscr{O}))$ is said to *converge weakly* to ν_t if for any $f \in C(\mathscr{O})$, $\nu_t(f) = \lim_N \mu_t^N(f)$ uniformly in $t \in [0,1]$. This topology can also be metrized by

$$d(\mu_t,\nu_t) = \sum_l \frac{1}{2^l}\frac{\sup_{t\in[0,1]}|\mu_t(f_l)-\nu_t(f_l)|}{1+\sup_{t\in[0,1]}|\mu_t(f_l)-\nu_t(f_l)|}. \tag{70}$$

Proposition 9.2 (Compactness in terms of real-valued processes)**.** *Let $\{\mu_t^N\}_{N\geqq 1}$ be a sequence in $C([0,1],\mathscr{M}(\mathscr{O}))$. If for each f_l, $\{\mu_t^N(f_l)\}_{N\geqq 1}$ lies in a compact subset of $C([0,1],R)$, then $\{\mu_t^N\}_{N\geqq 1}$ has compact closure in $C([0,1],\mathscr{M}(\mathscr{O}))$.*

Proof. Since for each l, $\{\mu_t^N(f_l)\}_{N\geqq 1}$ is a compact subset of $C([0,1],R)$, we can find a subsequence $\mu_t^{N_l}(f_l)$ and a continuous function denoted by $L_t(f_l)$ such that $L_t(f_l) = \lim_N \mu_t^{N_l}(f_l)$ uniformly in $t \in [0,1]$. By the Cantor diagonal process, there is a subsequence of N (still denoted by N) such that $L_t(f_l) = \lim_N \mu_t^{N_l}(f_l)$ uniformly in $t \in [0,1]$ for all l. It is easy to see that $L_t : f_l \longrightarrow L_t(f_l)$ is a linear functional on the dense linear subspace of $C(\mathscr{O})$ spanned by $\{f_1, f_2, \ldots\}$. In addition, $|L_t(f_l)| \leqq \limsup_N |\mu_t^N(f_l)| \leqq M\,\|f_l\|_\infty$. Hence by the Riesz Representation Theorem, L_t is a $\mathscr{M}(\mathscr{O})$-valued process.

It is a simple matter to check the remaining assertions. Let $f \in C(\mathscr{O})$. Then

$$\begin{aligned}\limsup_{N,t}\left|L_t(f)-\mu_t^N(f)\right| &\leqq |L_t(f-f_l)| + \left|L_t(f_l)-\mu_t^N(f_l)\right| + \left|\mu_t^N(f-f_l)\right| \\ &\leqq 2M\,\|f-f_l\|_\infty + \limsup_{N,t}\left|L_t(f_l)-\mu_t^N(f_l)\right|,\end{aligned}$$

which can be made as small as possible. □

Theorem 9.3 (Tightness in terms of real-valued processes)**.** *Let $\{\mu_t^N\}$ be a sequence of stochastic processes in $C([0,1],\mathscr{M}(\mathscr{O}))$ such that for each f_l, $\{\mu_t^N(f_l)\}_{N\geqq 1}$ forms a tight sequence of processes in $C([0,1],R)$. Then $\{\mu_t^N\}_{N\geqq 1}$ is a tight sequence in $C([0,1],\mathscr{M}(\mathscr{O}))$.*

Proof. Denote Π^N be the probability measure on $C([0,1],\mathscr{M}(\mathscr{O}))$ induced by μ_t^N (i.e., the law of μ_t^N). Let ε be any positive number. By the hypothesis of the tightness of $\{\mu_t^N(f_l)\}_{N\geqq 1}$, for each l, there is a compact subset K_l^ε of $C([0,1],R)$ such that

$$\Pi^N\left\{x_t \in C([0,1],\mathscr{M}(\mathscr{O})) : x_t(f_l) \in K_l^\varepsilon\right\} \geqq 1-\varepsilon/2^l.$$

Hence

$$\Pi^N\left(\bigcap_{l=1}^\infty \left\{x_t \in C([0,1],\mathscr{M}(\mathscr{O})) : x_t(f_l) \in K_l^\varepsilon\right\}\right) \geqq 1-\varepsilon.$$

But by the previous proposition, $\bigcap_{l=1}^\infty \left\{x_t \in C([0,1],\mathscr{M}(\mathscr{O})) : x_t(f_l) \in K_l^\varepsilon\right\}$ is compact in $C([0,1],\mathscr{M}(\mathscr{O}))$. The theorem is thus proved. □

10. Proof of the Main Theorem

We are already very close to proving Theorem 2.7. However, the evolving sets $\{K^N(t)\}_{N\geqq 1}$ do not quite belong to $C([0,1],\mathscr{M}(\mathscr{O}))$. They are discontinuous at the t_i's. The following procedure gets around this obstacle. First we introduce
$C_1 = C([0,1],\mathscr{M}(\mathscr{O}))$.
$C_2 = C_2([0,1],\mathscr{M}(\mathscr{O}))$ = collection of $\mathscr{M}(\mathscr{O})$-valued processes which are piecewise continuous with finite number of jumps at $\{iN^{-1} : i, N \geqq 1\}$. C_2 is also given the metric d of (70).
$C_3 = C_3([0,1],\mathscr{M}(\mathscr{O}))$ = completion of C_2 under d. Then C_3 is also a complete separable metric space.
We can similarly define $C_1([0,1],R)$, $C_2([0,1],R)$ and $C_3([0,1],R)$. Note that C_1 is a Borel (actually a σ-compact) subset of C_3.

The concepts of tightness and the results of Proposition 9.2 and Theorem 9.3 are still valid with $C_3([0,1],\mathscr{M}(\mathscr{O}))$ and $C_3([0,1],R)$ replacing $C([0,1],\mathscr{M}(\mathscr{O}))$ and $C([0,1],R)$.

Now we reformulate Theorem 2.7 in the following way. (Note that we can consider $\mathscr{K}$ as a subset of $\mathscr{M}(\mathscr{O})$.)

Theorem 10.1 (Tightness of stochastic motion by mean curvature)**.** *Let Π^N be the law of $\{K^N(t)\}_{t\in[0,1]}$ on C_3. Then $\{\Pi^N\}_{N\geqq 1}$ is tight. Any weak limit Π_* of $\{\Pi^N\}_{N\geqq 1}$ satisfies*

1. $\Pi_*(C([0,1],\mathscr{K})) = 1$.
2. *For any positive integer m and $f \in C^2(\mathscr{O})$, there is a constant C_m such that*

$$\Pi_*\left(\sup_{t\in[0,1]}\left\{\Phi(\partial K(t))^m : K \in C([0,1],\mathscr{K})\right\}\right) \leqq C_m. \tag{71}$$

3. *For any positive integer m and for $f \in C^2(\mathscr{O})$, there is a constant $C(f,m)$ such that*

$$\int_{K\in C([0,1],\mathscr{K})}\left|\int_{x\in K(t)} f(x)\,d\mathscr{L}^n x - \int_{x\in K(s)} f(x)\,d\mathscr{L}^n x\right|^{2m} d\Pi_* K$$
$$\leqq C(f,m)\,|t-s|^m \tag{72}$$

for $0 \leqq s \leqq t \leqq 1$.

Proof. Step I – Tightness of $\left\{K_f^N(t)\right\}_{N\geqq 1}$ in $C_3([0,1],R)$. From Theorem 4.1, $K_f^N(t)$ can be decomposed as $K_f^N(t) = S_f^N(t)+R_f^N(t)$. Now $S_f^N(t) \in C_1([0,1],R)$. $M_f^N(t)$ is a piecewise constant function with discontinuities at the t_i's. Construct a new function

$$\tilde{M}_f^N(t) = \frac{t_{i+1}-t}{t_{i+1}-t_i} M_f^N(t_i^-) + \frac{t-t_i}{t_{i+1}-t_i} M_f^N(t_i^+) \qquad \text{for } t_i \leqq t < t_{i+1}. \tag{73}$$

Now $\tilde{M}_f^N(t)$ is piecewise linear function. Set $R_f^N(t) = \tilde{M}_f^N(t) - M_f^N(t)$. Then,

$$\sup_{t\in[0,1]} \left|R_f^N(t)\right| \leqq \sup_i \left|M_f^N(t_i^+) - M_f^N(t_i^-)\right|. \tag{74}$$

By (32), it is routine to check that for any positive integer m for small enough $\alpha > 0$ and for $0 \leqq s < t < 1$

$$E\left|\tilde{M}_f^N(t) - \tilde{M}_f^N(s)\right|^{2m} \leqq C(f, m)\,|t - s|^m, \tag{75}$$

$$E\left\{\sup_{t\in[0,1]} \left|R_f^N(t)\right|^m\right\} \leqq C(f, m, \alpha)\triangle t^{m\alpha}. \tag{76}$$

Let Π_f^N be the law of $K_f^N(t)$ on $C_3([0,1], R)$ and $\Pi_{f,1}^N$ be the law of $S_f^N(t) + \tilde{M}_f^N(t)$ on $C_1([0,1], R)$. By (31), (75) and Theorem A.2, there is a weak limit $\Pi_{f,*}$ of $\left\{\Pi_{f,1}^N\right\}_{N\geqq 1}$ on $C_1([0,1], R)$. Now for any bounded uniformly continuous function G on $C_3([0,1], R)$, we have

$$\begin{aligned}
\int_{x\in C_3([0,1],R)} G(x)\,d\Pi_f^N x &= \int_\Omega G(K_f^N)\,dP \\
&= \int_\Omega G(S_f^N + \tilde{M}_f^N)\,dP \\
&\quad + \int_\Omega \left(G(K_f^N) - G(S_f^N + \tilde{M}_f^N)\right) dP \\
&= \int_{x\in C_1([0,1],R)} G(x)\,d\Pi_{f,1}^N x \\
&\quad + \int_\Omega \left(G(K_f^N) - G(S_f^N + \tilde{M}_f^N)\right) dP \\
&\longrightarrow \int_{x\in C_1([0,1],R)} G(x)\,d\Pi_{f,*}x \quad \text{as } N \longrightarrow \infty.
\end{aligned} \tag{77}$$

In this convergence, we have used the facts that $\sup_t |R_f^N(t)|$ converges to zero in probability (by (76)) and that G is uniformly continuous.

Hence Π_f^N has a weak limit $\Pi_{f,*}$ with $\Pi_{f,*}(C_1([0,1], R)) = 1$. By the Prokhorov Criterion (Section A), Π_f^N is tight on $C_3([0,1], R)$. This also establishes the tightness of Π^N on $C_3([0,1], \mathscr{M}(\mathscr{O}))$.

Let Π_* be a weak limit of (a subsequence of) $\left\{\Pi^N\right\}_{N\geqq 1}$.

Step II. $\Pi_*(C([0,1], \mathscr{K})) = 1$. The conclusion of *Step I* states that for any $\varepsilon > 0$, there is a compact subset B_l^ε of $C_3([0,1], R)$ such that

$$\Pi^N\left(\left\{x_t \in C_3([0,1], \mathscr{M}(\mathscr{O})) : x_t(f_l) \in B_l^\varepsilon\right\}\right) \leqq 1 - \left(\frac{\varepsilon}{2}\right)^{l+1}.$$

In addition, Theorem 4.1 (33) implies the existence of an $M^\varepsilon > 0$ such that

$$\Pi^N\left(C_3([0,1], \mathscr{K}) \cap \left\{K_t : \sup_t \Phi(\partial K_t) \leqq M^\varepsilon\right\}\right) \geqq 1 - \frac{\varepsilon}{2}.$$

Now define A^ε by

$$A^\varepsilon = \quad C_3([0,1], \mathscr{K}) \cap \left\{ K_t : \sup_t \Phi(\partial K_t) \leqq M^\varepsilon \right\}$$
$$\cap \bigcap_{l=1}^{\infty} \left\{ x_t \in C_3([0,1], \mathscr{M}(\mathscr{O})) : x_t(f_l) \in B_l^\varepsilon \right\},$$

Then $\Pi^N(A^\varepsilon) \geqq 1 - \varepsilon$. But A^ε is a compact (and hence closed) subset of $C_3([0,1], \mathscr{M}(\mathscr{O}))$ by the compactness property of sets of finite perimeter (or integral currents) (Section 2.1). So $\Pi_*(A^\varepsilon) \geqq 1 - \varepsilon$. This immediately leads to $\Pi_*(C([0,1], \mathscr{K})) = 1$.

The remaining statements (71) and (72) of the theorem are direct consequences of weak convergence of probability measures. (All the functionals in the statements are lower semi-continuous with respect to the metric of $C_3([0,1], \mathscr{K})$.) The whole Theorem 10.1 is thus proved. □

A. Weak Convergence of Probability Measures

We describe here and in the next section two topics from probability theory used in this paper. The references for this section are [Bil] and [KS, 2.2 and 2.4].

Let S be a metric space. We use $\mathscr{P}(S)$ to denote the space of probability measures defined on S. A sequence $\{P_n\}_{n \geqq 1} \subset \mathscr{P}(S)$ is said to *converge weakly* to $P \in \mathscr{P}(S)$, denoted by $P = w \lim_n P_n$ or $P \rightharpoonup P_n$, if for any bounded and continuous function f defined on S, we have

$$\lim_n \int_S f(x)\, P_n(dx) = \int_S f(x)\, P(dx), \tag{78}$$

In this definition, we can restrict f to be only uniformly continuous.

We have the following important compactness criterion for $\mathscr{P}(S)$, developed by *Prokhorov*. A collection of probability measures $\Gamma \subset \mathscr{P}(S)$ is called *tight* if for all $\varepsilon > 0$, there is a compact set $K \subset S$ such that

$$P(K) \geqq 1 - \varepsilon \qquad \forall P \in \Gamma. \tag{79}$$

If Γ is tight, then it is *relatively compact*. The reverse is true if S is *complete*.

Now let S be a complete separable metric space. We describe a tightness criterion for probability measures defined on $C([0,1], S)$, the space of continuous functions from $[0,1]$ to S. The condition is summarized by the following two statments.

Theorem A.1 (Kolmogorov-Čentsov). *Suppose an S-valued stochastic process X satisfies the condition*

$$E\,|X_t - X_s|^\alpha \leqq C\,|t-s|^{1+\beta}, \quad \textit{for all } 0 \leqq s, t \leqq 1 \tag{80}$$

where $\alpha, \beta, C > 0$. Then X has a continuous version $\tilde{X}$ which is locally Hölder continuous with exponent γ for every $0 < \gamma < \beta/\alpha$, i.e.,

$$P\left\{\omega : \sup_{\substack{0<t-s<h(\omega) \\ s,t\in[0,1]}} \frac{\left|\tilde{X}_t(\omega) - \tilde{X}_s(\omega)\right|}{|t-s|^\gamma} \leqq \delta_\gamma\right\} = 1 \tag{81}$$

where $h(\omega)$ is an almost surely positive random variable and δ_γ is some appropriate constant depending on γ.

Theorem A.2. *Let $\left\{X_t^N\right\}_{t\in[0,1]}$ be a sequence of continuous processes satisfying*

$$E\left|X_0^N\right|^p \leqq M < \infty, \tag{82}$$

$$E\left|X_t^N - X_s^N\right|^\alpha \leqq C\,|t-s|^{1+\beta} \tag{83}$$

where p, α, β, M and C are positive numbers independent of N. Then the collection of probability measures $\left\{P^N\right\}_{N\geqq 1}$ induced on $C([0,1], S)$ by $\left\{X^N\right\}_{N\geqq 1}$ is tight.

B. Stochastic Flows

In this part, we introduce various concepts related to stochastic flows. It is a collection of results from [Kun, Chapters 3 and 4].

B.1. Definitions

Given a probability space $(\Omega, \mathscr{F}, P)$, a *stochastic flow of homeomorphisms* on R^n is a 2-parameter family of random variables taking values in the space of continuous functions from R^n to R^n, i.e., the space $C(R^n, R^n)$:

$$\varphi_{s,t}(\cdot, \omega) : R^n \longrightarrow R^n, \quad \omega \in \Omega, \ \ s, t \in [0,1] \tag{84}$$

such that for P almost every $\omega \in \Omega$, the following statements are true for all $s, t \in [0, 1]$:

1. $\varphi_{s,u}(x, \omega) = \varphi_{t,u}(\varphi_{s,t}(x, \omega), \omega), \quad x \in R^n$, i.e., $\varphi_{s,t}(\omega) = \varphi_{t,u}(\omega) \circ \varphi_{s,t}(\omega)$.
2. $\varphi_{s,s}(x, \omega) = x$, i.e. $\varphi_{s,s}(\omega)$ is the identity on R^n.
3. $\varphi_{s,t}(\cdot, \omega) : R^n \longrightarrow R^n$ is a homeomorphism.

Such flows satisfy $\varphi_{s,t}(\cdot, \omega)^{-1} = \varphi_{t,s}(\cdot, \omega)$ and $\varphi_{s,t}(\cdot, \omega) = \varphi_{0,t}(\cdot, \omega) \circ \varphi_{0,s}^{-1}(\cdot, \omega)$.

The *forward flow* of φ is the collection of maps, $\left\{\varphi_{s,t} : \ 0 \leqq s \leqq t \leqq 1\right\}$. Similarly, the *backward flow* is $\left\{\varphi_{t,s} : \ 0 \leqq s \leqq t \leqq 1\right\}$.

All stochastic flows are assumed to be continuous in the time variables with the topology of $C(R^n, R^n)$ being uniform convergence on compact subsets of R^n.

If almost surely, $\varphi_{s,t}(\cdot, \omega)$ is k-times differentiable with respect to x for all s, t, and, if the derivatives are continuous in (s, t, x), then φ is called *C^k-differentiable*. It is simply called a *diffeomorphism* if $\varphi_{s,t}$ is C^∞ in x.

A stochastic flow φ is called a *Brownian flow* if for any $0 \leqq t_0 < t_1 < \cdots < t_l \leqq 1$, the following $C(R^n, R^n)$-valued random variables are independent:

$$\left\{\varphi_{t_0,t_1}, \varphi_{t_1,t_2}, \cdots, \varphi_{t_{l-1},t_l}\right\}.$$

If in addition, the law of $\varphi_{s+h,t+h}$ is the same as $\varphi_{s,t}$ for all h, φ is then called a *temporally homogeneous Brownian flow*. From now on, we only consider Brownian flows of diffeomorphisms.

The main result we need is the relationship between Brownian flows and stochastic differential equations. Loosely speaking, given such a flow φ with certain regularity conditions in terms of the spatial variables, we can find a "Brownian motion" F in some appropriate function space such that the following Ito stochastic differential equation is satisfied:

$$\varphi_{s,t}(x) = x + \int_s^t F(\varphi_{s,r}(x), dr). \tag{85}$$

Moreover, there is a one-to-one correspondence between φ and F. F is called the *(Ito) random infinitesimal generator of* φ.

B.2. Martingales with Spatial Parameters

In order to study equation (85), [Kun] introduces the notion of martingales $\{M(x,t)\}_{t\geqq 0}$ with spatial parameter $x \in R^n$, and defines stochastic integrations with respect to M, $\int_0^t M(x, dr)$.[19] More generally, let $f_t(\lambda)$ be an R^n-valued predictable process with parameter λ, [Kun] also defines $\int_0^t M(f_r(\lambda), dr)$.

Let G be a domain in R^n. We use $C^m(G, R^e)$ to denote the space of m-times continuously differentiable functions from G to R^e and $C^{m,\delta}(G, R^e)$ the space of functions having m-th derivatives Hölder continuous with exponential δ. We also use the notations $\tilde{C}(G, R^e) = C^m(G \times G, R^e)$ and $\tilde{C}^{m,\delta}(G, R^e) = C^{m,\delta}(G \times G, R^e)$.[20]

Now let $\{F(x,t)\}_{t\geqq 0}$ be a family of R^e-valued stochastic process with spatial parameter $x \in G$. Then we can regard $\{F(\cdot,t)\}_{t\geqq 0}$ as a $C^m(G, R^e)$- or $C^{m,\delta}(G, R^e)$-valued stochastic process depending on the smoothness of F in x. Similarly, $\{G(x,y,t)\}_{t\geqq 0, x,y\in G}$ can be treated as $\tilde{C}^m(G, R^e)$ and $\tilde{C}^{m,\delta}(G, R^e)$-processes.

Using this language, a stochastic flow $\left\{\varphi_{0,t}\right\}_{t\geqq 0}$ is actually a continuous $C(R^n, R^n)$ or $C^{m,\delta}(R^n, R^n)$-valued process satisfying the properties of a flow and homeomorphism. In addition, a continuous $C^{m,\delta}(G, R^e)$-valued process $F(t)$ is called a $C^{m,\delta}$*-Brownian motion* ($m \geqq 0, 0 \leqq \delta \leqq 1$) if for any $0 \leqq t_0 < t_1 < t_2 < \cdots < t_l$,

$$\{F(t_0), F(t_{i+1}) - F(t_i) \colon i = 0, 1, \ldots, l-1\}$$

are independent random variables.

[19] Note that for *each* x, just as in the case of 1-dimensional Ito's integration, $\int_0^t M(x, dr)$ can be defined up to a probability-measure-zero set. However, now there are uncountably many x's in R^n. The total collection of the null sets for the x's might have positive probability. In order to have a definition of the integral not depending on x, [Kun] imposes a smoothness condition on M in x.

[20] [Kun] imposes a growth condition with respect to x in the definitions of these spaces. But this does not concern us as we only work on a compact domain.

B.2.1. Quadratic Variations and Regularities in Space

Now let the probability space $(\Omega, \mathscr{F}, P)$ be equipped with a filtration $\{\mathscr{F}_t\}_{t\geqq 0}$. Consider two $\mathscr{F}_t$-martingales $M(x,t)$ and $N(y,t)$ with spatial parameters $x, y \in G$. Let $A^{MN}(x,y,t)$ be the cross-variation process of $M(x,t)$ and $N(y,t)$. (When $M = N$, we omit the superscript MM.) We need the following relationships between the regularities of $M(x,t)$ in x and those of $A(x,y,t)$ in x, y.

1. Let $\{M(x,t)\}_{t\geqq 0}$ and $\{N(y,t)\}_{t\geqq 0}, x, y \in G$ be two continuous $C^{m,\delta}$-martingales ($m \geqq 0,\ 0 < \delta \leqq 1$). Then the cross-variation process $A^{MN}(x,y,t)$ has a version which is a continuous $\tilde{C}^{m,\varepsilon}$-process for any $\varepsilon < \delta$. In addition, the following identities hold[21]:

$$D_x^\alpha D_y^\beta \langle M(x,t),\ N(y,t)\rangle = \left\langle D_x^\alpha M(x,t),\ D_y^\beta N(y,t)\right\rangle,\quad |\alpha|, |\beta| \leqq m, \tag{86}$$

$$\begin{aligned} \frac{\partial}{\partial x_i}\langle M(x,t),\ N(x,y)\rangle &= \left\langle \frac{\partial}{\partial x_i} M(x,t),\ N(x,t)\right\rangle \\ &\quad + \left\langle M(x,t),\ \frac{\partial}{\partial x_i} N(x,t)\right\rangle \quad (\text{if } m \geqq 1). \end{aligned} \tag{87}$$

2. Conversely, let $M(x,t)$ be as before but with $M(x,0) \equiv 0$. If $A(x,y,t)$ has a version which is a continuous $\tilde{C}^{m,\delta}$-process for some $m \geqq 0$ and $0 < \delta \leqq 1$, then M in turn has a version which is a continuous $C^{m,\varepsilon}$-process for any $\varepsilon < \delta$. In addition, for $|\alpha| \leqq m$, $D^\alpha M(x,t)$ is a continuous martingale with cross-variation process $D_x^\alpha D_y^\alpha A(x,y,t)$.

B.2.2. Local Characteristic of M

Let $A(x,y,t)$ be the cross-variation process between $M(x,t)$ and $M(y,t)$. Then there is a continuous increasing process A_t and a predictable process $a(x,y,t)$ such that

$$A(x,y,t) = \int_0^t a(x,y,r)\, dA_r. \tag{88}$$

The pair $(a(x,y,t), A_t)$ is called the *local characteristic* of $M(x,t)$. We omit A_t if it equals t.

For the construction of stochastic integral, we first define the following classes of spatial martingales:

$$\begin{aligned} &\mathscr{B}^{m,\delta} : P \text{ a.s. } \int_0^1 \|a(r)\|_{\tilde{C}^{m,\delta}(K,R^e)}\, dA_r < \infty \quad \text{for all compact subset } K \text{ of } G, \\ &\mathscr{B}_b^{m,\delta} : P \text{ a.s. } \int_0^1 \|a(r)\|_{\tilde{C}^{m,\delta}(G,R^e)}\, dA_r < \infty, \\ &\mathscr{B}_{ub}^{m,\delta} : \|a(t)\|_{\tilde{C}^{m,\delta}(G,R^e)} < C, \quad \text{a deterministic number.} \end{aligned} \tag{89}$$

[21] We use $\langle\cdot,\ \cdot\rangle$ to denote cross-variation processes.

B.3. Stochastic Integration Using $M(x, t)$

Now we define $\int_0^t M(f_r, dr)$ where $M(x, t)$ is a spatial martingale and $f(t)$ is a predictable R^n-valued process.

Let $\{M(x, t)\}_{t \geqq 0}$ $(x \in G)$ be a continuous martingale with its local characteristic belonging to $\mathscr{B}^{m,\delta}$ $(m \geqq 0, \delta > 0)$. In addition, $f(t)$ is a G-valued predictable process such that

$$E \int_0^1 a(f_r, f_r, r)\, dA_r < \infty. \tag{90}$$

Then there is a continuous square integrable martingale denoted by

$$\int_0^t M(f_r, dr) \tag{91}$$

with quadratic variation $\displaystyle\int_0^t a(f_r, f_r, r)\, dA_r$. Furthermore, let $g(t)$ be another predictable process satisfying (90). Then

$$\left\langle \int_0^t M(f_r, dr), \int_0^t M(g_r, dr) \right\rangle = \int_0^t a(f_r, g_r, r)\, dA_r. \tag{92}$$

These formulas can be extended to stochastic integrations with respect to two martingales M and N. Let their local characteristics be $(a^M(x, y, t), A_t)$ and $(a^N(x, y, t), A_t)$ respectively. They both belong to $\mathscr{B}^{m,\delta}$ $(m \geqq 0, \delta > 0)$. In this case, we can also write

$$A^{MN}(x, y, t) = \int_0^t a^{MN}(x, y, r)\, dA_r. \tag{93}$$

Then for any two predictable processes $f(t)$ and $g(t)$ satisfying (90), we have

$$\left\langle \int_0^t M(f_r, dr), \int_0^t N(g_r, dr) \right\rangle = \int_0^t a^{MN}(f_r, g_r, r)\, dA_r. \tag{94}$$

This integration can also be defined for a *continuous semi-martingale* $F(x, t)$. F is called such a process if it can be decomposed as

$$F(x, t) = B(x, t) + M(x, t) \tag{95}$$

where $\{B(x, t)\}_{t \geqq 0}$ is a continuous process with bounded variation in t and M is a continuous local martingale. Let $(a(x, y, t), A_t)$ be the local characteristic of M. We assume that

$$B(x, t) = \int_0^t b(x, r)\, dA_r \tag{96}$$

for some predictable process $b(x, t)$. $(a(x, y, t), b(x, t), A_t)$ is also called the *local characteristic* of F.

With these definitions of stochastic integrals, we have the following generalization of *Ito's Formula*.

Let $\{F(x,t)\}_{t\geqq 0}$ *($x \in G$) be a continuous C^2-process which is also a continuous C^1-semi-martingale with local characteristic belonging to $\mathscr{B}^{1,\delta}$ ($\delta > 0$). In addition, $g(t)$ is a continuous semi-martingale taking values in G. Then $F(g_t, t)$ is a continuous semi-martingale satisfying*

$$F(g_t,t) = F(g_0,0) + \int_0^t F(g_r,dr) + \sum_{i=1}^n \int_0^t \frac{\partial F}{\partial x_i}(g_r,r)\,dg_r^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(g_r,r)d\left\langle g_r^i,\ g_r^j\right\rangle + \sum_{i=1}^n \left\langle \int_0^t \frac{\partial F}{\partial x_i}(g_r,dr),\ g_r^i\right\rangle. \tag{97}$$

B.4. Stochastic Differential Equation and Existence of Brownian Flow

We now formulate the existence result for the solution of a stochastic differential equation of the form

$$d\varphi_t = F(\varphi_t,dt) \quad \text{or} \quad \varphi_t(x) = x + \int_0^t F(\varphi_r(x),dr). \tag{98}$$

Here $\{F(x,t)\}_{t\geqq 0, x\in R^n}$ is an R^n-valued semi-martingale which can be decomposed as $F^i(x,t) = M^i(x,t) + B^i(x,t),\ i = 1,2,\cdots,n$, where M^i is a continuous local-martingale and B^i is a continuous process with bounded variation. We use the representations

$$\left\langle M^i(x,t),\ M^j(x,t)\right\rangle = A^{ij}(x,y,t) = \int_0^t a^{ij}(x,y,r)\,dA_r, \tag{99}$$

$$B^i(x,t) = \int_0^t b^i(x,r)\,dA_r. \tag{100}$$

Let $a(x,y,t) = (a^{ij}(x,y,t))_{i,j=1}^n$ and $b(x,y,t) = (b^i(x,t))_{i=1}^n$. Then $a(x,y,t)$ is a matrix-valued process with following properties:

Symmetry. For all $x, y \in R^n$ and $i, j = 1,2,\cdots,n$,

$$a^{ij}(x,y,t) = a^{ji}(y,x,t). \tag{101}$$

Non-Negative Definiteness. For any $\{x_p \in R^n\}_{p=1,\cdots,l}$ and $\{\xi_p \in R^n\}_{p=1,\dots,l}$,

$$\sum_{p,q=1}^l \left\langle a(x_p,x_q,t)\xi_p,\ \xi_q\right\rangle = \sum_{p,q,i,j} a^{ij}(x_p,x_q,t)\xi_p^i\xi_p^j \geqq 0. \tag{102}$$

The following statement can be proved by Picard Iteration.

Theorem B.1. *Let $\{F(\cdot, t)\}_{t \geqq 0}$ be a continuous $C(R^n, R^n)$-valued semi-martingale with local characteristic (a, b, A_t) belonging to $\mathscr{B}_b^{0,1}$. Then for each $t_0 \geqq 0$ and $x_0 \in R^n$,* (98) *has a unique solution φ_t in the sense that φ_t is a continuous adapted process satisfying*

$$\varphi_t = x_0 + \int_{t_0}^{t} F(\varphi_r, dr). \tag{103}$$

Now we are ready to state the existence results for stochastic flows. Let $\{\varphi_{s,t}\}_{0 \leqq s \leqq t \leqq 1}$ be a Brownian flow. We introduce the following sets of conditions.

Condition (A.1)**.** For each s, t, x and y, $\varphi_{s,t}(x)$ is square integrable. Also the following limits exist:

$$\lim_{h \to 0^+} \frac{1}{h} E\left[\varphi_{t,t+h}(x) - x\right], \tag{104}$$

$$\lim_{h \to 0^+} \frac{1}{h} E\left[(\varphi_{t,t+h}(x) - x)(\varphi_{t,t+h}(y) - y)^T\right]. \tag{105}$$

Let $b(x, t) = (b^1(x, t), \cdots, b^n(x, t))$ denote the first limit. It is called the *infinitesimal mean of φ*. Let $a(x, y, t) = (a^{ij}(x, y, t))_{i,j=1}^n$ be the $n \times n$ matrix-valued function denoting the second limit. It is called the *infinitesimal covariance of φ*. (It clearly satisfies the symmetry and non-negativity conditions (101) and (102).)

Condition (A.2)**.** There exists a positive constant K such that for any x, y,

$$\left|E\left[\varphi_{s,t}(x) - x\right]\right| \leqq K(1 + |x|)\,|t - s|, \tag{106}$$

$$\left\|E\left[(\varphi_{s,t}(x) - x)(\varphi_{s,t}(y) - y)^T\right]\right\| \leqq K(1 + |x|)(1 + |y|)\,|t - s|. \tag{107}$$

These inequalities imply that the infinitesimal mean and covariance of φ satisfy the growth conditions

$$|b(x, t)| \leqq K(1 + |x|), \tag{108}$$

$$\|a(x, y, t)\| \leqq K(1 + |x|)(1 + |y|). \tag{109}$$

Condition $(\mathrm{A.3})_{k,\delta}$**.** The infinitesimal mean b and covariance a belong to the class of $\mathscr{C}_{ub}^{k,\delta}$ and $\tilde{\mathscr{C}}_{ub}^{k,\delta}$ respectively. (The spaces $\mathscr{C}^{k,\delta}$, $\mathscr{C}_b^{k,\delta}$, $\mathscr{C}_{ub}^{k,\delta}$ and their $\tilde{\mathscr{C}}$ counterparts are defined exactly the same way as the $\mathscr{B}$ spaces (Section B.2).)

The existence of a Brownian flow is given by the following results.

Theorem B.2 (Existence of Brownian Flow)**.** *Let $a(x, y, t)$ be a continuous $n \times n$ matrix satisfying* (101) *and* (102) *and let $b(x, t)$ be a continuous n-vector-valued function. Suppose that $a \in \tilde{\mathscr{C}}_{ub}^{0,1}$ and $b \in \mathscr{C}_{ub}^{0,1}$. Then there exists a forward Brownian flow of homeomorphisms such that its infinitesimal mean and covariance exist and coincide with (a, b). The law of such a flow is unique and it satisfies Condition* (A.2).[22]

[22] The proof is given by first constructing a Brownian motion F (see the next proposition) and then solving the corresponding stochastic differential equation (98) for F. Of course, it is still quite a task to verify that the solution φ is a flow of homeomorphisms.

Proposition B.3 (Existence of Brownian motion). *Let $a(x, y, t)$ and $b(x, t)$ be as in the previous theorem. Now suppose $a \in \tilde{\mathscr{C}}^{0,\delta}$ and $b \in \mathscr{C}^{0,\delta}$ for some $\delta > 0$. Then there exists a C-valued Brownian motion F such that*

$$E\left[F(x,t)\right] = \int_0^t b(x,r)\,dr, \tag{110}$$

$$\begin{aligned}\operatorname{Cov}(F(x,t)F(y,s)) &= E\left[(F(x,t) - EF(x,t))\,(F(y,s) - EF(y,t))\right] \\ &= \int_0^{t\wedge s} a(x,y,r)\,dr. \end{aligned}\tag{111}$$

Theorem B.4 (Brownian flows and Brownian motions). *Let $\{\varphi_{s,t}\}_{0\leqq s\leqq t\leqq 1}$ be a C^k-Brownian flow satisfying Conditions* (A.1)–(A.3)$_{k,\delta}$. *Then there exists a unique $C^{k,\varepsilon}$-valued Brownian motion $F(x,t)$ $(\varepsilon < \delta)$ with $F(x,0) \equiv 0$ such that the flow φ is governed by*

$$\varphi_{s,t}(x) = x + \int_s^t F(\varphi_{s,r}(x), dr). \tag{112}$$

Furthermore, the mean and covariance of F coincide with $\int_0^t b(x,r)\,dr$ and $\int_0^t a(x,y,r)\,dr$, where b and a are the infinitesimal mean and covariance of the flow φ. (F is called the forward Ito random infinitesimal generator of *φ.)*

Theorem B.5 (Smoothness of Brownian flows). *Suppose that the local characteristic of a Brownian motion F belongs to $\mathscr{B}_b^{k,\delta}$ $(k \geqq 1, \delta > 0)$. Then the Brownian flow obtained by F has a modification which is a C^k-diffeomorphism. It is also a $C^{k,\varepsilon}$-semi-martingale for any $\varepsilon < \delta$. If in addition the local characteristic of F belongs to $\mathscr{B}_{ub}^{k,\delta}$, then for each α with $1 \leqq |\alpha| \leqq k$, $p \geqq 1$ and $N > 0$,*

$$E\left\{\sup_{|x|\leqq N}\left|D^\alpha \varphi_{s,t}(x)\right|^p\right\} < \infty. \tag{113}$$

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