Some Aspects of Curvature of Curves and Surfaces

MA 108, Fall 2017

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Reading Materials and References

(A. Yip) From Engineering to Mathematics and Back to Engineering

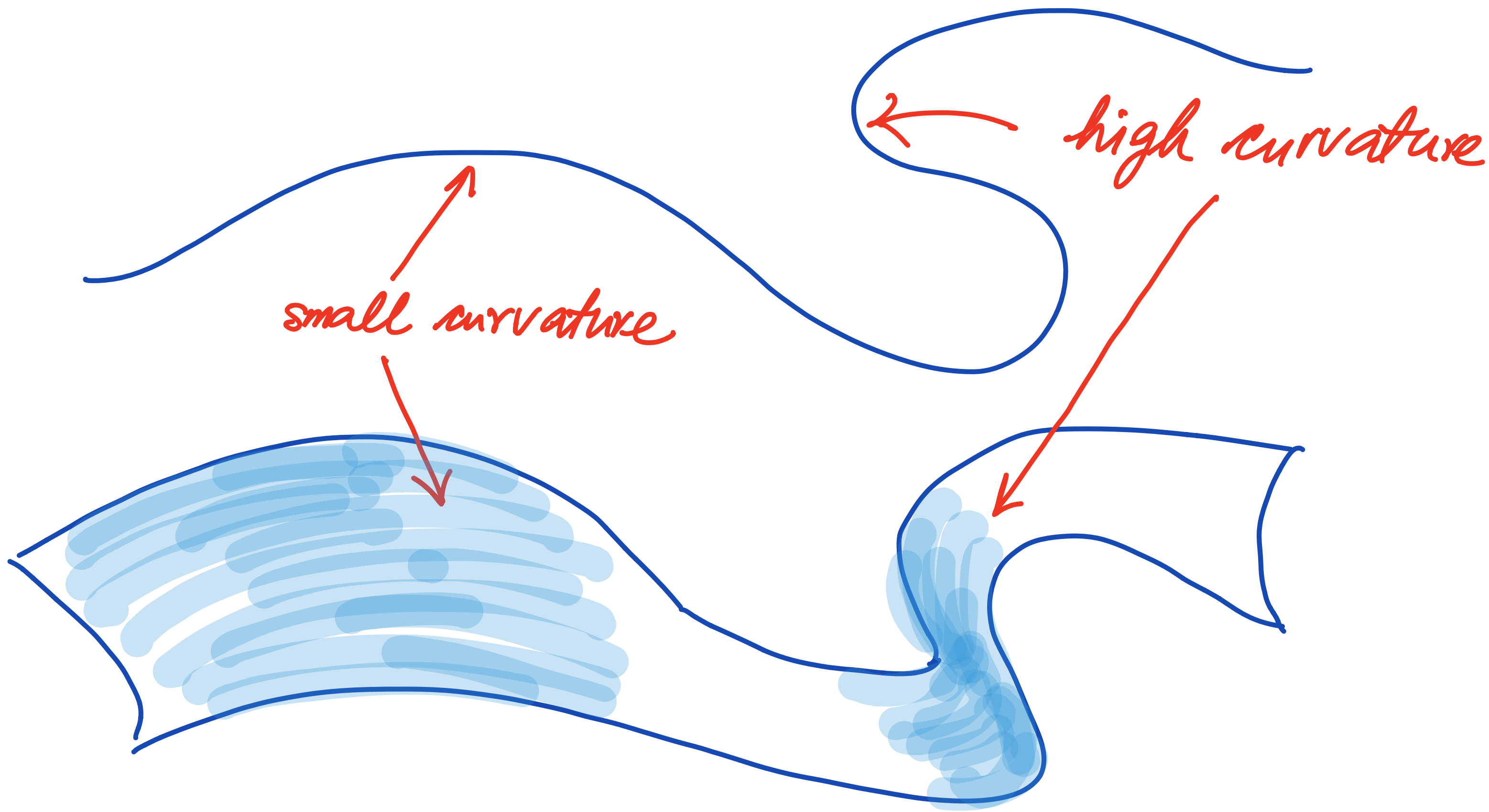
(C. Bandle) Didó's Problem and Its Impacts on Modern Mathematics

(V. Blåsjö) The Isoperimetric Problem

(M.P. Do Carmo) Differential Geometry of Curves and Surfaces

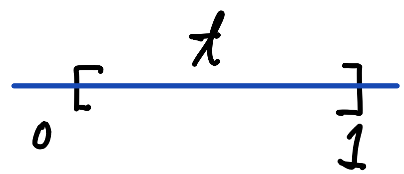
Available at http://www.math.purdue.edu/~yip/108

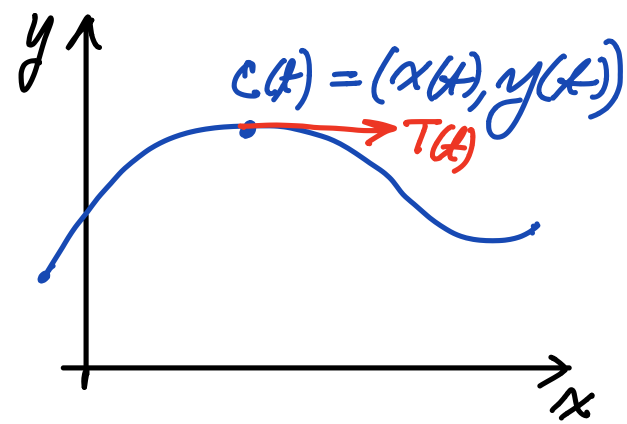
Curvature: how fast a curve turns or a surface bends



(Mathematical) Definition

Given a curve: $t \in [0,1] \longrightarrow C(t) \in \mathbb{R}^2 \ (\text{or } \mathbb{R}^3)$





Tangent Vector: $T(t) = \frac{d}{dt}C(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$

(Velocity vector) $\left(\frac{d}{dt} = \text{rate of change}\right)$

(Mathematical) Definition

rate of change $= \frac{d}{dt}$

Curvature describes how fast the tangent vector turns:

$$\frac{dT(t)}{dt} = \frac{d^2}{dt^2}C(t) = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right)$$

To be fair, the speed needs to be one:

$$\left\|\frac{dc}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \equiv 1.$$

Re-parametrize the curve $c(t)$ by $c(s)$ so that

$$\left\|\frac{dc}{ds}\right\| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} \equiv 1$$

(Mathematical) Definition

length variable

$$s \longrightarrow C(s) = (x(s), y(s))$$

velocity

$$T(s) = \frac{d}{ds}C(s) = \left(\frac{dx}{ds}, \frac{dy}{ds}\right), \quad \|T(s)\| \equiv 1$$

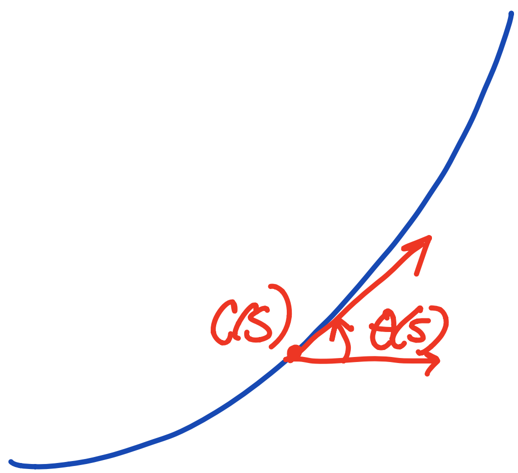
acceleration

$$\frac{dT}{ds} = \frac{d^2}{ds^2}C(s) = \left(\frac{d^2x}{ds^2}, \frac{d^2y}{ds^2}\right)$$

$$\kappa \text{ (curvature)} = \left\|\frac{dT}{ds}\right\| = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2}$$

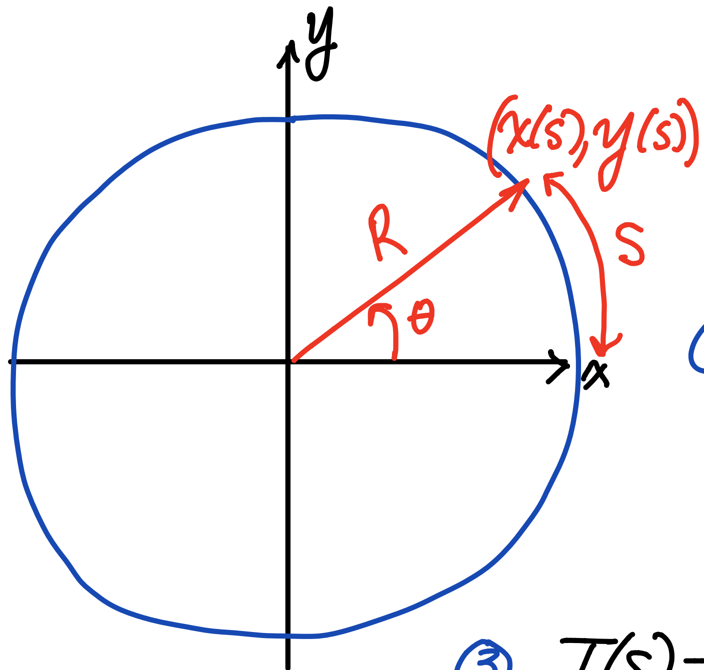
(Mathematical) Definition

Curvature also equals to the rate of change (per unit length) of the angle (θ)



$$\boxed{\kappa = \frac{d\theta(s)}{ds}}$$

A Simple Example : A Circle of Radius R



① $\theta = \frac{s}{R}$ ← arc length

② $(x(s), y(s)) = (R\cos(\theta), R\sin(\theta))$

$= \left(R\cos\left(\frac{s}{R}\right), R\sin\left(\frac{s}{R}\right)\right)$

③ $T(s) = \left(\frac{dx}{ds}, \frac{dy}{ds}\right) = \left(-\sin\left(\frac{s}{R}\right), \cos\left(\frac{s}{R}\right)\right)$

④ $\frac{dT}{ds} = \frac{1}{R}\left(-\cos\left(\frac{s}{R}\right), -\sin\left(\frac{s}{R}\right)\right)$,

⑤ $\boxed{\kappa = \left\|\frac{dT}{ds}\right\| = \frac{1}{R}}$

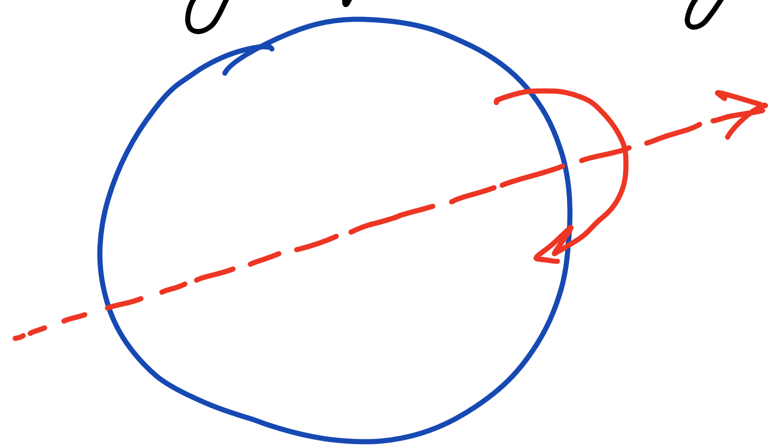
Some Special Propeities of a Circle

(1) Locus of points with constant distance to a point (origin)

(2) A curve with constant curvature,

$$\kappa = \frac{1}{R}$$

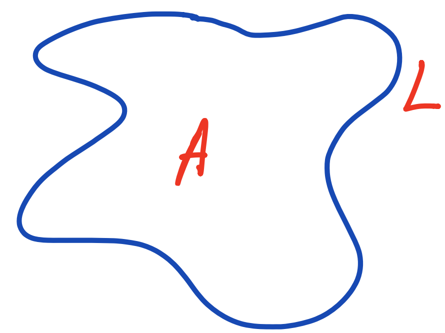
(3) Enjoys many reflection symmetry:



Extremal Property of a Circle

A circle maximizes area while minimizes length.

Three equivalent formulations:



1. Fix the area, circle has minimum length (L)
2. Fix the length, circle encloses the maximum area (A)
3. Circle minimizes the following isoperimetric ratio:

$$\frac{L^2}{A}$$

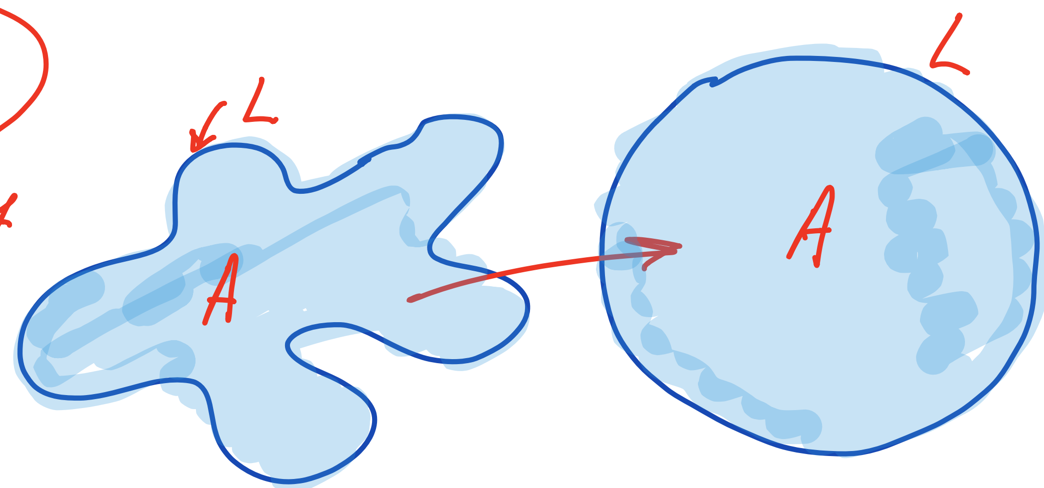
Isoperimetric Inequality

Given a closed curve,

$$\frac{L^2}{A} \geq 4\pi, \quad = \text{ holds if and only if the curve is a circle.}$$

L = length of curve

A = area enclosed by the curve



Dido's Problem and Its Impact on Modern Mathematics

Catherine Bandle

ABSTRACT. We trace the isoperimetric problem from Queen Dido to some recent applications. Emphasis is put on the developments which are significant for the applications in analysis and in partial differential equations.

Ancient Time, Origin of the Problem

Dido Purchases Land for the Foundation of Carthage. Engraving by Matthäus Merian the Elder, in *Historische Chronica*, Frankfurt a.M., 1630. Dido's people cut the hide of an ox into thin strips and try to enclose a maximal domain.

Figure 1. Queen Dido enclosed maximal area with strips of the hide of a bull.

Dido's Problem

The Roman poet Publius Vergilius Maro (70–19 B.C.) tells in his epic *Aeneid* the story of queen Dido, the daughter of the Phoenician king of the 9th century B.C. After the assassination of her husband by her brother she fled to a haven near Tunis. There she asked the local leader, Yarb, for as much land as could be enclosed by the hide of a bull. Since the deal seemed very modest, he agreed. Dido cut the hide into narrow strips, tied them together and encircled a large tract of land which became the city of Carthage. Dido faced the following mathematical problem, which is also known as the *isoperimetric problem*:

Find among all curves of given length the one which encloses maximal area.

Dido found intuitively the right answer.

The Isoperimetric Problem in Antiquity

In those days a formula for the area A of a circle of given length L was known. The Babylonians used around 1800 B.C. the formula $A = \frac{2}{25}L^2$ instead of $A = \frac{1}{4\pi}L^2$. This approximation of π by 3.125 is quite accurate. In his short treatise *On the Measurement of the Circle*, Archimedes (285–212 B.C.) circumscribed and inscribed a 96-gon around and inside the circle and determined that $3.1408 < \pi < 3.14285$.

The Greeks were interested in the isoperimetric problem also for practical reasons. It was useful to have an upper limit for the area in order to prevent the merchants from cheating when they stated the area of an island by its circumference. In those times it was commonly believed that the perimeter of a figure determines its area.

Around 150 B.C. Zenodorus proved rigorously by elementary geometrical arguments:

(i) *if there exists an n-gon having the largest area among all n-gons of given perimeter, it must be regular;*
(ii) *among all regular polygons of equal perimeter the one with more sides has a greater area; and*
(iii) *the circle encloses a greater area than any regular polygon of equal perimeter.*

Does a maximal polygon exist at all? The ancient geometers were not concerned with this question. It was settled many centuries later, for instance by Weierstrass.

No substantial mathematical progress was made for almost 1,900 years on the isoperimetric problem. During all this time it was taken for granted that the circle has the largest area among all plane domains of given perimeter. Similarly, the astronomers believed that an analogous property is also valid for domains in space namely, that the ball has the largest volume of all domains of given surface area. A detailed description of the isoperimetric problem in ancient time is given in [5].

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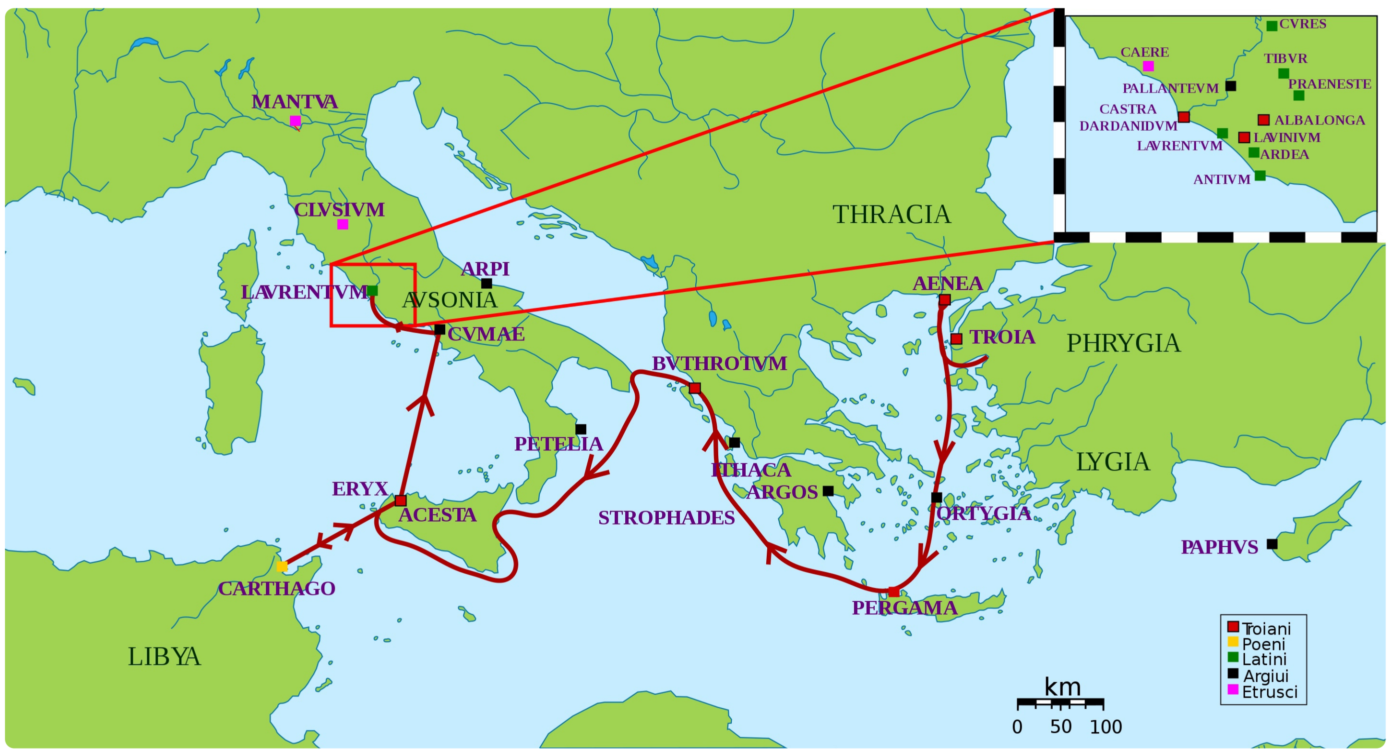
City of Carthage (Cartago)

Journey of Aeneas from Troy to Italy



Meeting Between Aeneas and Dido

The Isoperimetric Problem

Viktor Blåsjö

1. ANTIQUITY. To tell the story of the isoperimetric problem one must begin by quoting Virgil:

> At last they landed, where from far your eyes
> May view the turrets of new Carthage rise;
> There bought a space of ground, which Byrsa call'd,
> From the bull's hide they first inclos'd, and wall'd.
> (*Aeneid*, Dryden's translation)

This refers to the legend of Dido. Virgil's version has it that Dido, daughter of the king of Tyre, fled her home after her brother had killed her husband. Then she ended up on the north coast of Africa, where she bargained to buy as much land as she could enclose with an oxhide. So she cut the hide into thin strips, and then she faced, and presumably solved, the problem of enclosing the largest possible area within a given perimeter—the isoperimetric problem. But earthly factors mar the purity of the problem, for surely the clever Dido would have chosen an area by the coast so as to exploit the shore as part of the perimeter. This is essential for the mathematics as well as for the progress of the story. Virgil tells us that Aeneas, on his quest to found Rome, is shipwrecked and blown ashore at Carthage. Dido falls in love with him, but he does not return her love. He sails away and Dido kills herself. Kline concludes [**23**, p. 135]:

> And so an ungrateful and unreceptive man with a rigid mind caused the loss of a potential mathematician. This was the first blow to mathematics which the Romans dealt.

As for the mathematics of the isoperimetric problem, the Greeks pretty much solved it, by their standards, when Zenodorus proved that a circle has greater area than any polygon with the same perimeter. His work was lost. We know of it mainly through Pappus and Theon of Alexandria. Pappus's introduction to the subject (in his *Collection*, Book V) is considered a literary masterpiece. To quote Heath [**18**, p. 389]:

> It is characteristic of the great Greek mathematicians that, whenever they were free from the restraint of the technical language of mathematics, as when for instance they had occasion to write a preface, they were able to write in language of the highest literary quality, comparable with that of the philosophers, historians, and poets.

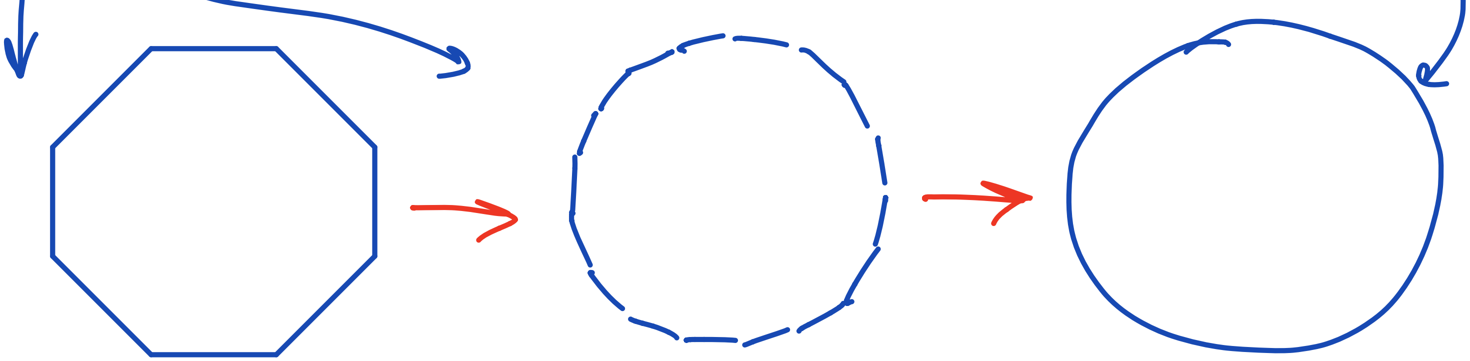
We quote Pappus's introduction from [**40**, pp. 588–593]:

> Though God has given to men, most excellent Megethion, the best and most perfect understanding of wisdom and mathematics, He has allotted a partial share to some of the unreasoning creatures as well. To men, as being endowed with reason, He granted that they should do everything in the light of reason and demonstration, but to the other unreasoning creatures He

Isoperimetric Inequality for Polygons

Around 150 B.C. Zenodorus proved rigorously by elementary geometrical arguments:

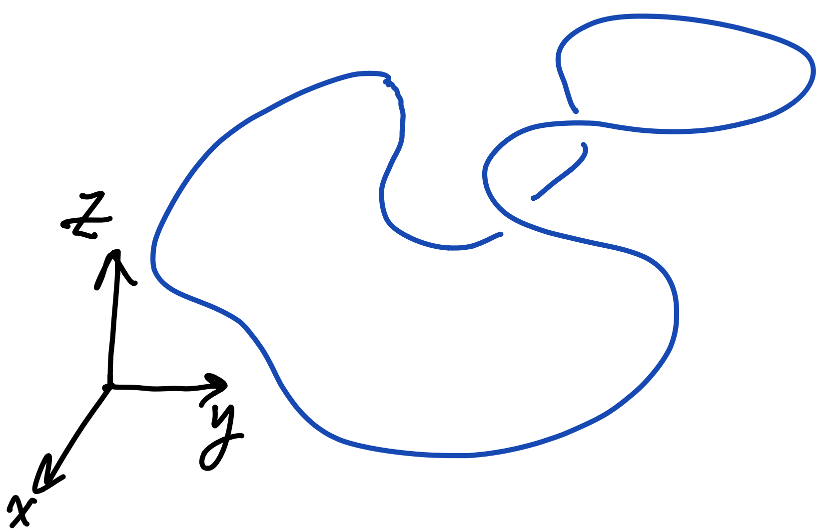
(i) *if there exists an n-gon having the largest area among all n-gons of given perimeter, it must be regular;*
(ii) *among all regular polygons of equal perimeter the one with more sides has a greater area; and*
(iii) *the circle encloses a greater area than any regular polygon of equal perimeter.*

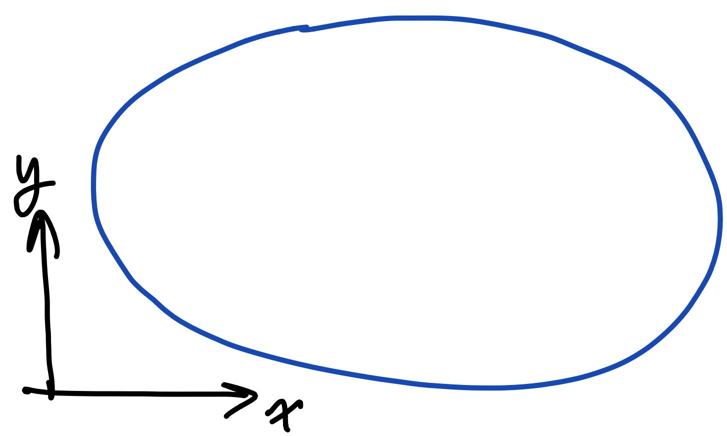


Fenchel Theorem

$\int_C |\kappa|\, ds$

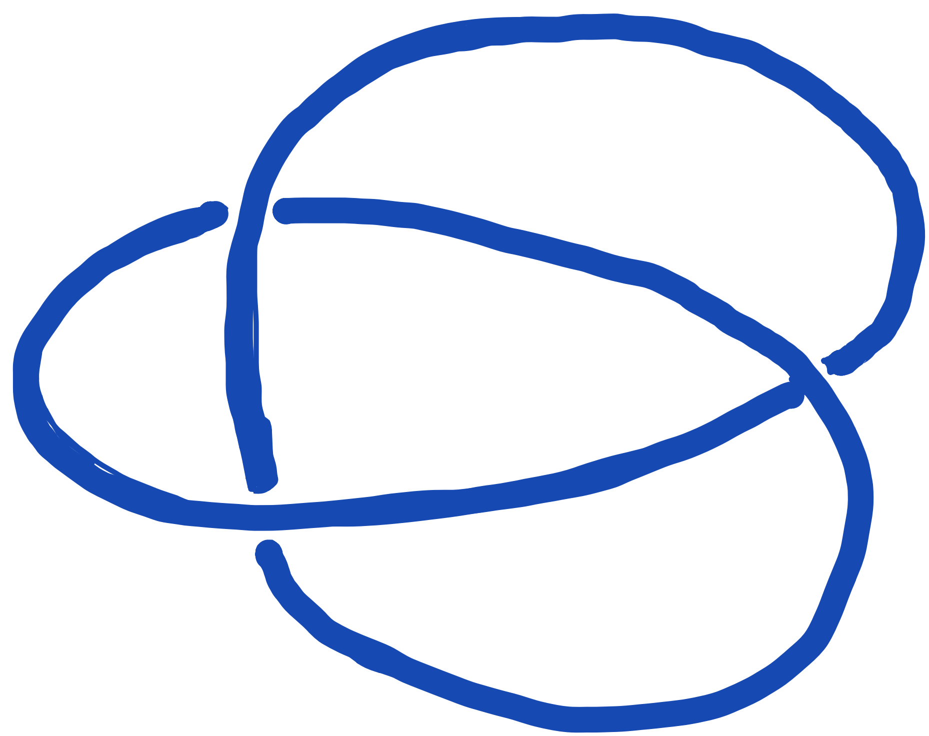
The total curvature of a simple closed curve is $\geq 2\pi$, and equality holds if and only if the curve is a plane convex curve





<u>Fary-Milnor Theorem</u>

The <u>total curvature</u> of a <u>knotted</u> simple closed curve is $\geq 4\pi$

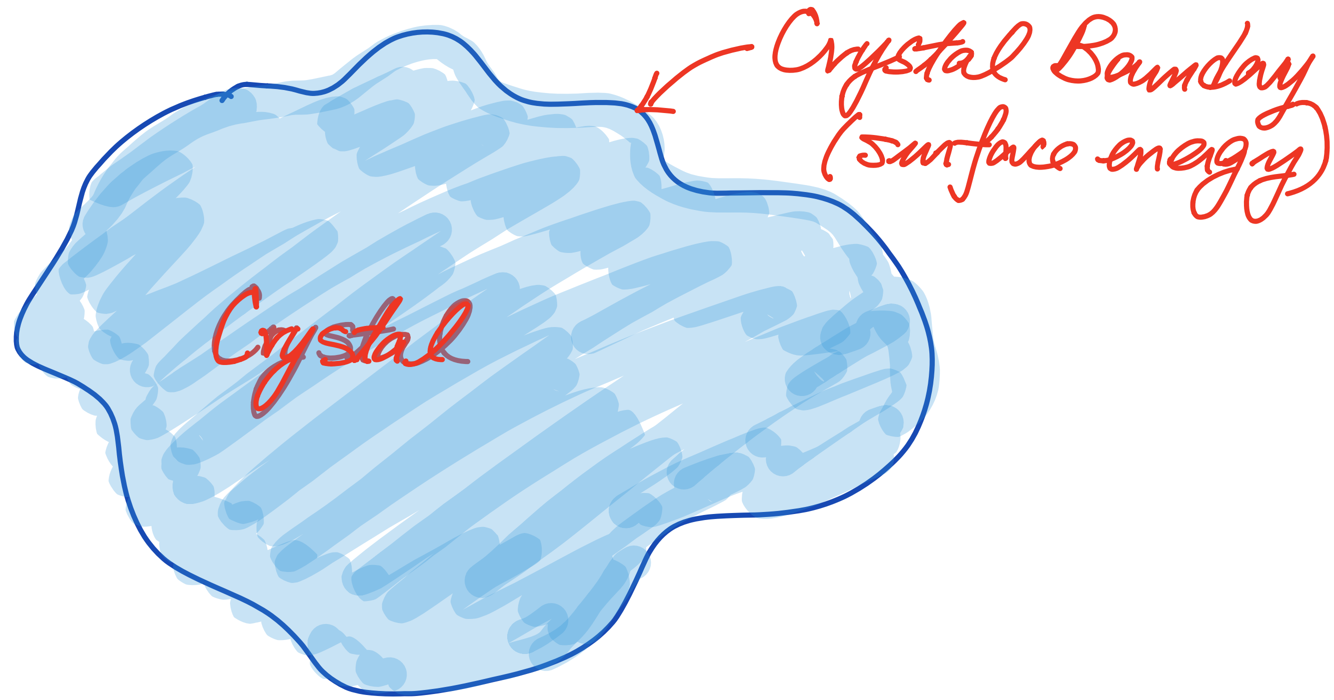


Crystal Shapes and Isoperimetric Ineq





Wulff Shape of Equilibrium Crystal

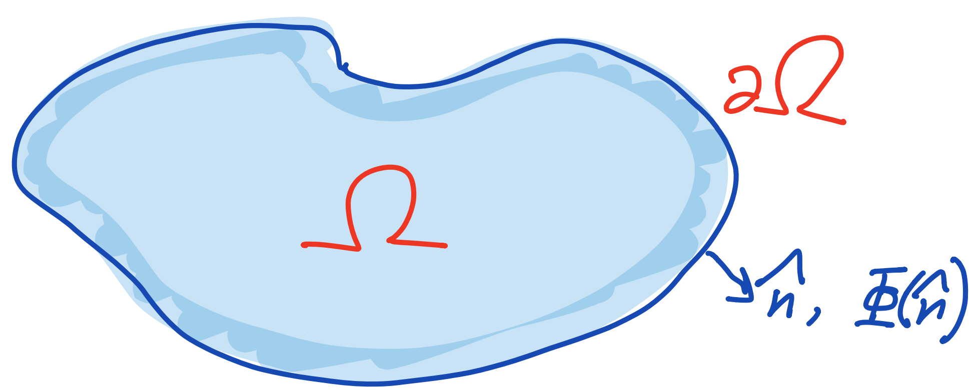


At equilibrium, the crystal attains a shape that minimizes surface energy with given volume

Wulff Shape of Equilibrium Crystal

minimize $\displaystyle\int_{\partial\Omega} \underbrace{\Phi(\hat{n})}_{\text{surface energy integrand}} \, da$, given $\underbrace{Vol(\Omega)}_{\text{volume of crystal}}$

$\partial\Omega$: boundary of crystal

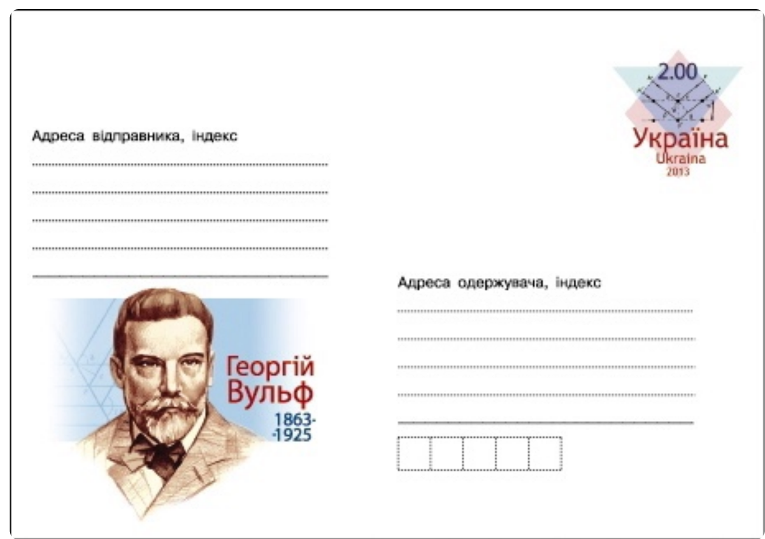


Wulff Shape of Equilibrium Crystal

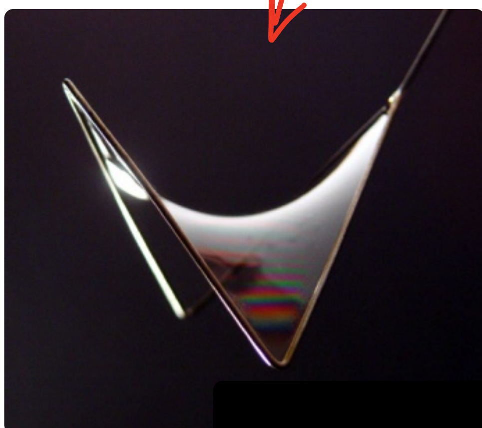
The shape that minimizes the surface energy (with given volume) is called the

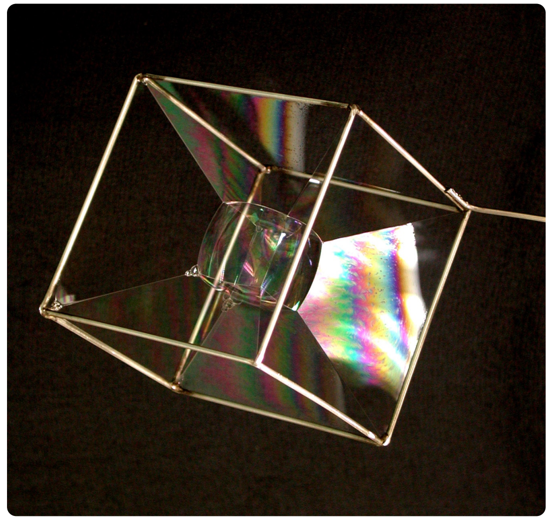
$\boxed{\text{Wulff Shape}}$

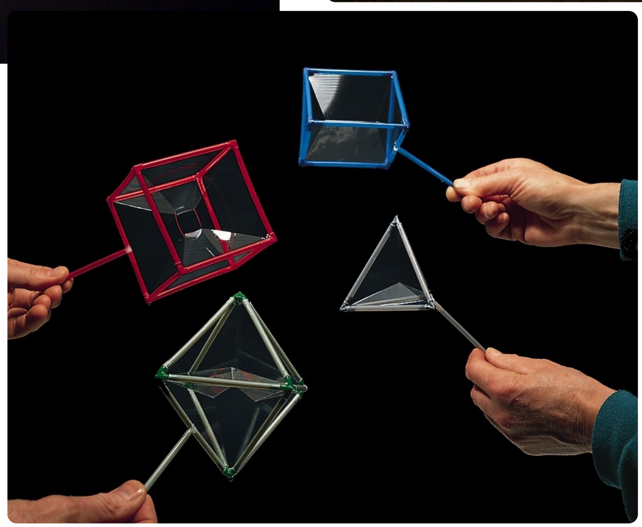
Crystallographer
George (Yuri Viktorovich) Wulff (1863–1925)



Soap Films and Soap Bubbles







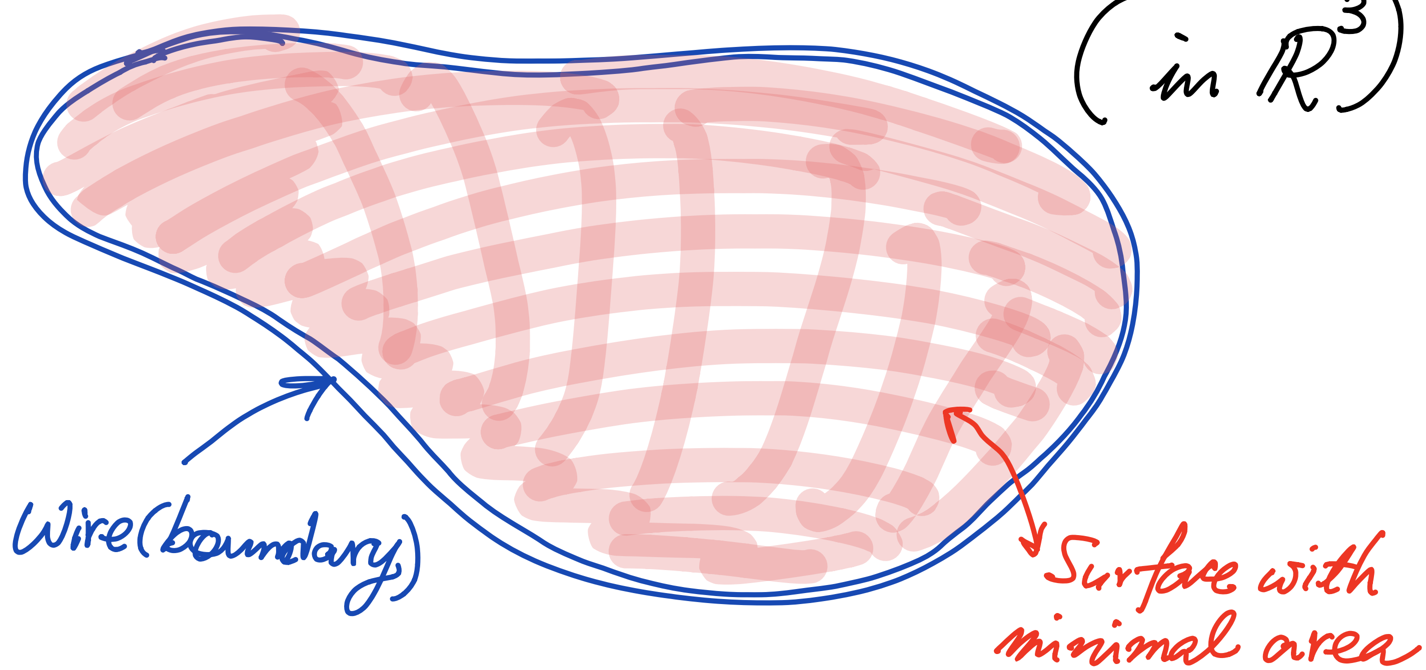
Soap Films and Soap Bubbles

The problem of finding the minimum area with given boundary is experimented by Joseph (Antoine Ferdinand) Plateau (1801–1883), a Belgium physicist.

Soap Films and Soap Bubbles

Plateau Problem: find the surface that minimizes area with given boundary.

(in $\mathbb{R}^3$)



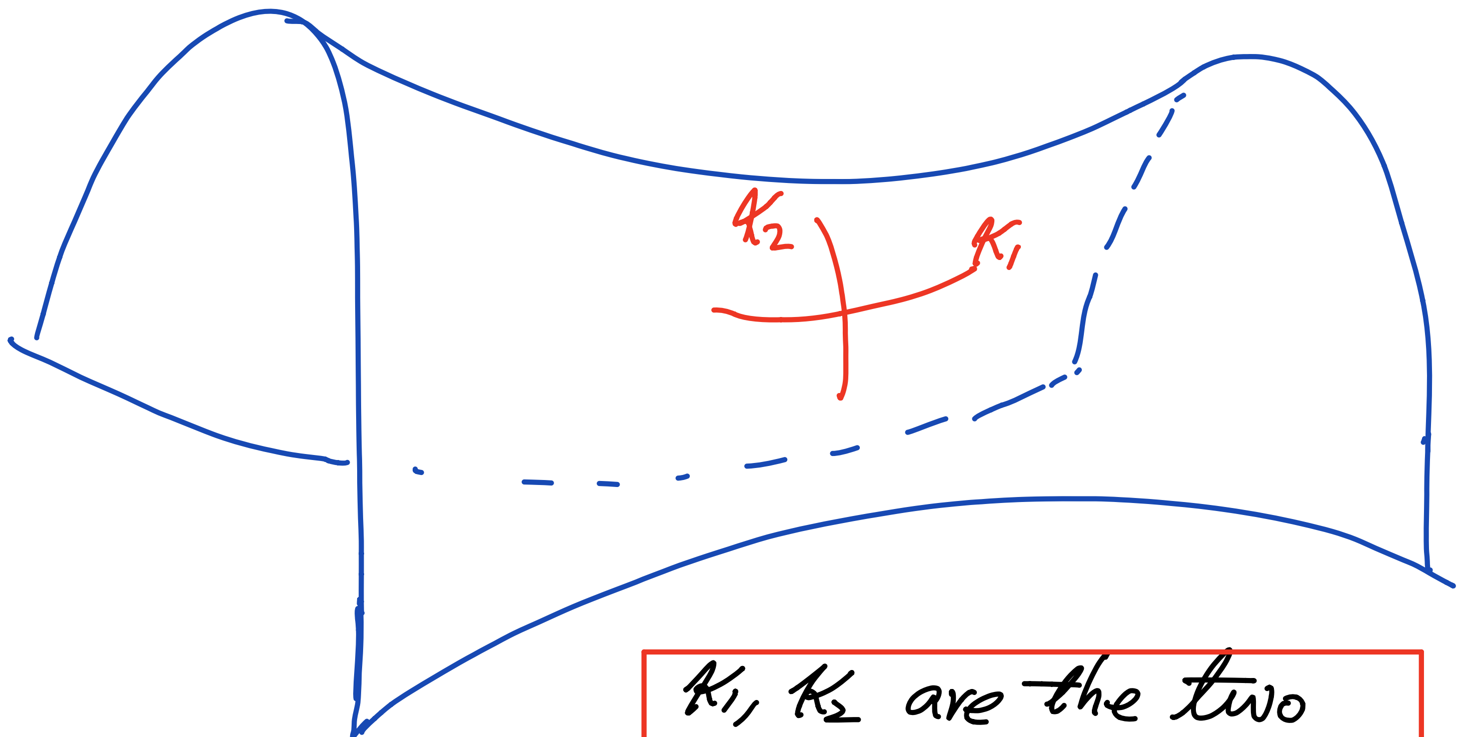
Soap Films and Soap Bubbles

Plateau Problem: find the surface that minimizes area with given boundary.

The solution is given by minimal surface with satisfies

$$\boxed{\text{mean curvature } \left(\frac{\kappa_1+\kappa_2}{2}\right) = 0}$$

Mean Curvature of a Surface



κ_1, κ_2 are the two principal curvatures