Some Aspects of Linear Algebra – II

Aaron Yip

(I am currently teaching MA 351 Linear Algebra.)

Linear algebra concerns solving linear systems of equations.

Linear algebra concerns <u>solving linear systems</u> of equations.

A <u>single</u> equation:

$$ax = b \Longrightarrow x = a^{-1}b$$

Linear algebra concerns solving linear systems of equations.

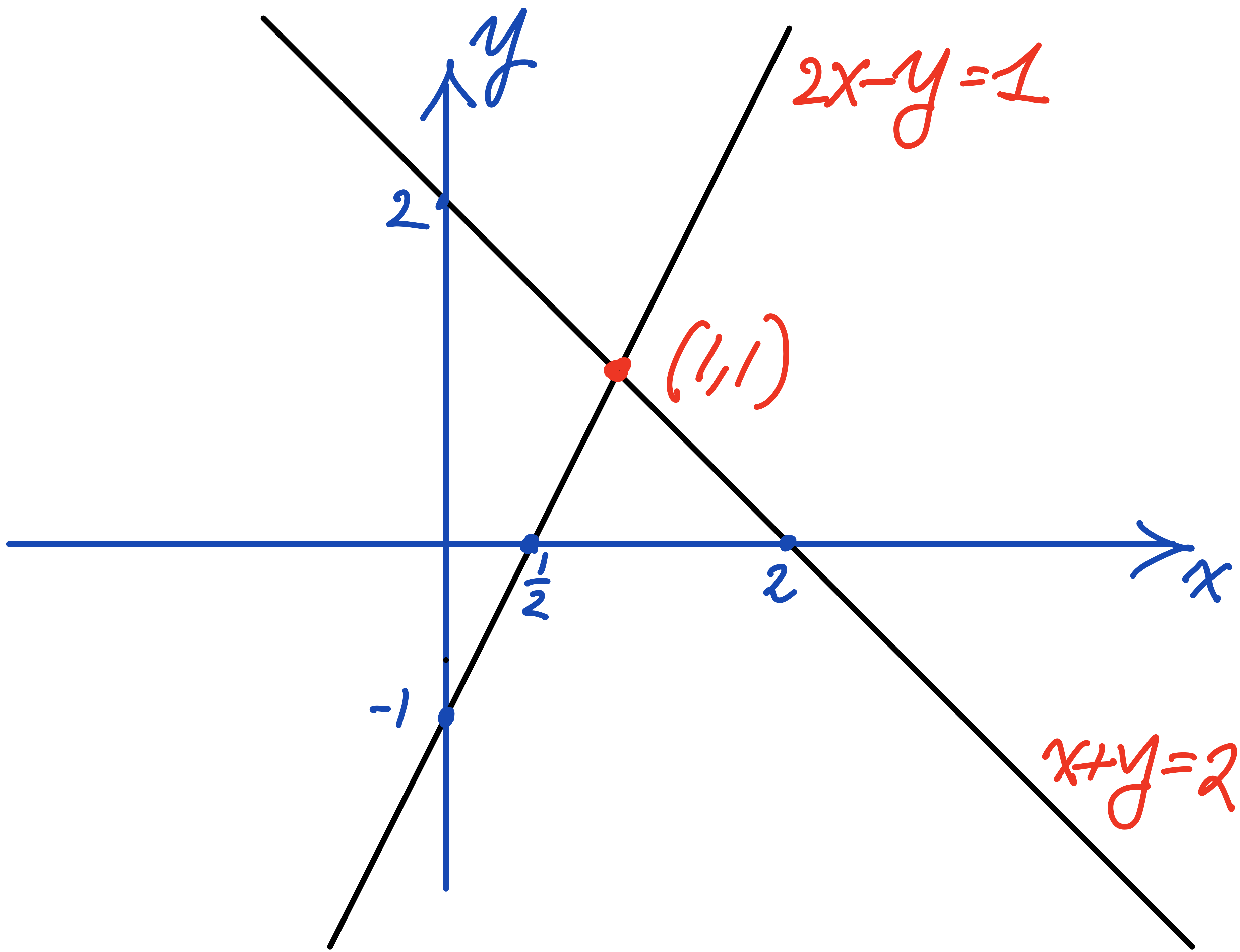
A system of equations:

$$A\vec{x}=\vec{b} \overset{?}{\Longrightarrow} \vec{x}=A^{-1}\vec{b}$$

Solve:

$$\begin{cases} 2x - y = 1 \\ x + y = 2 \end{cases}$$

$$x = 1, \quad y = 1$$



y
x
2
-1
1/2
2
2x-y=1
x+y=2
(1,1)

A somewhat different view:

$$\begin{cases} 2x - y = 1 \\ x + y = 2 \end{cases}$$

$\leftarrow$ systems of equations

$$\begin{pmatrix} 2x - y \\ x + y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\leftarrow$ vector form

Matrix form

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

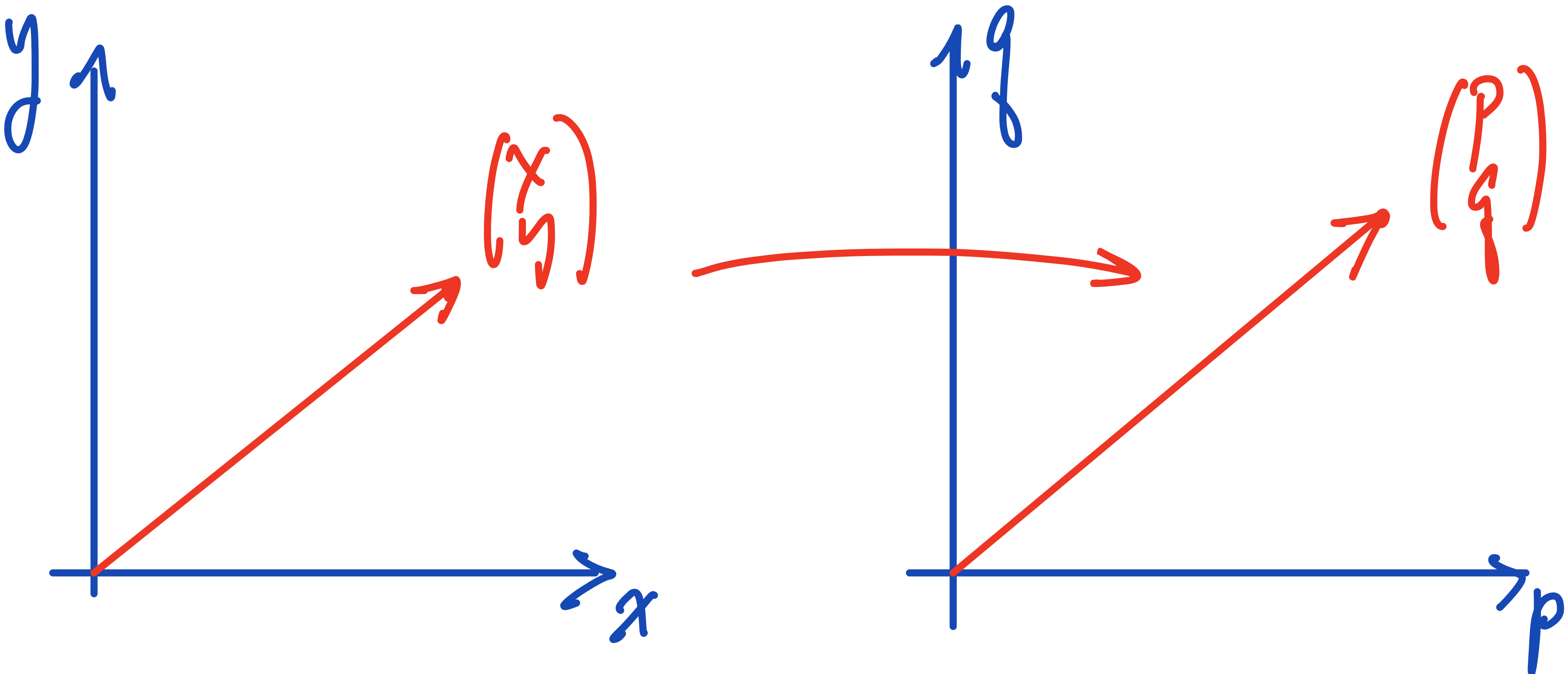
Notation matters in Linear Algebra

Use a new notion:

$$\boxed{A\vec{x} = \vec{b}}$$

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

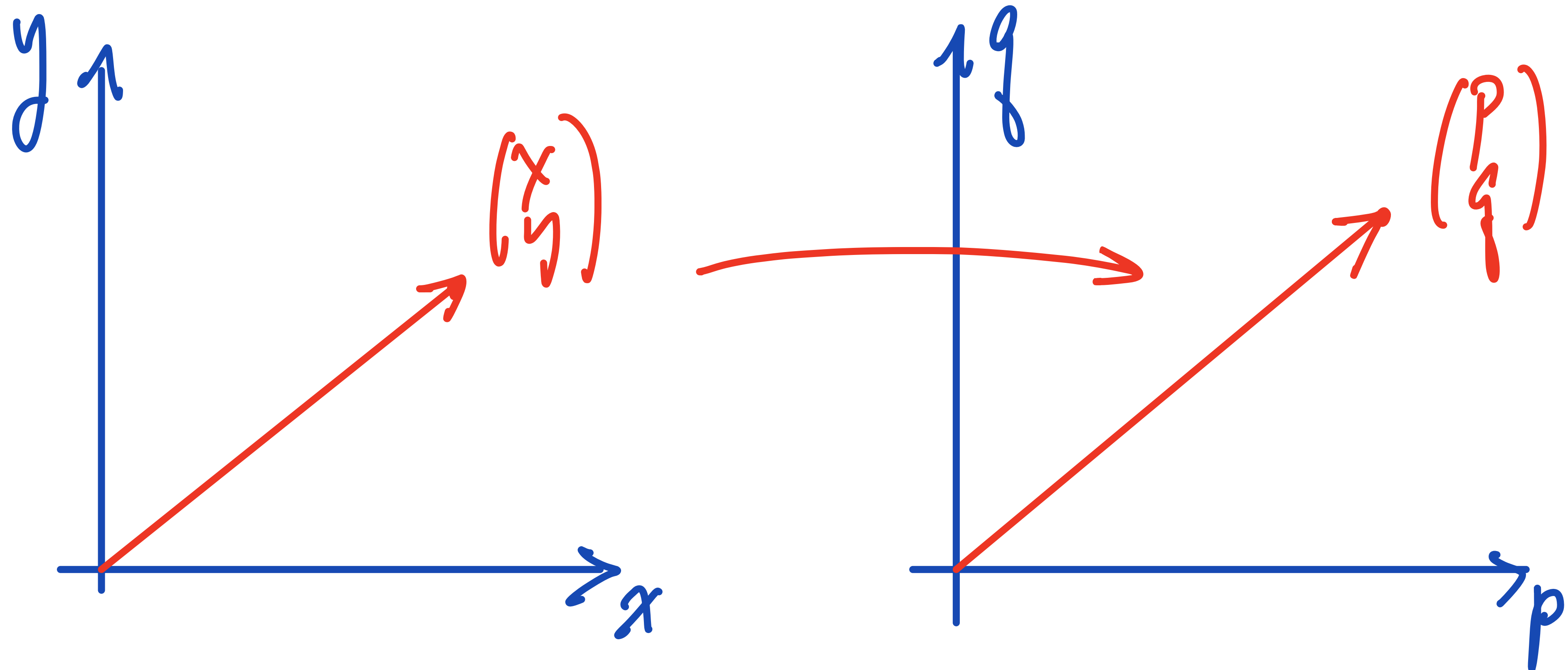
Linear Transformation



$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$p = ax + by$$
$$q = cx + dy$$

Linear Transformation



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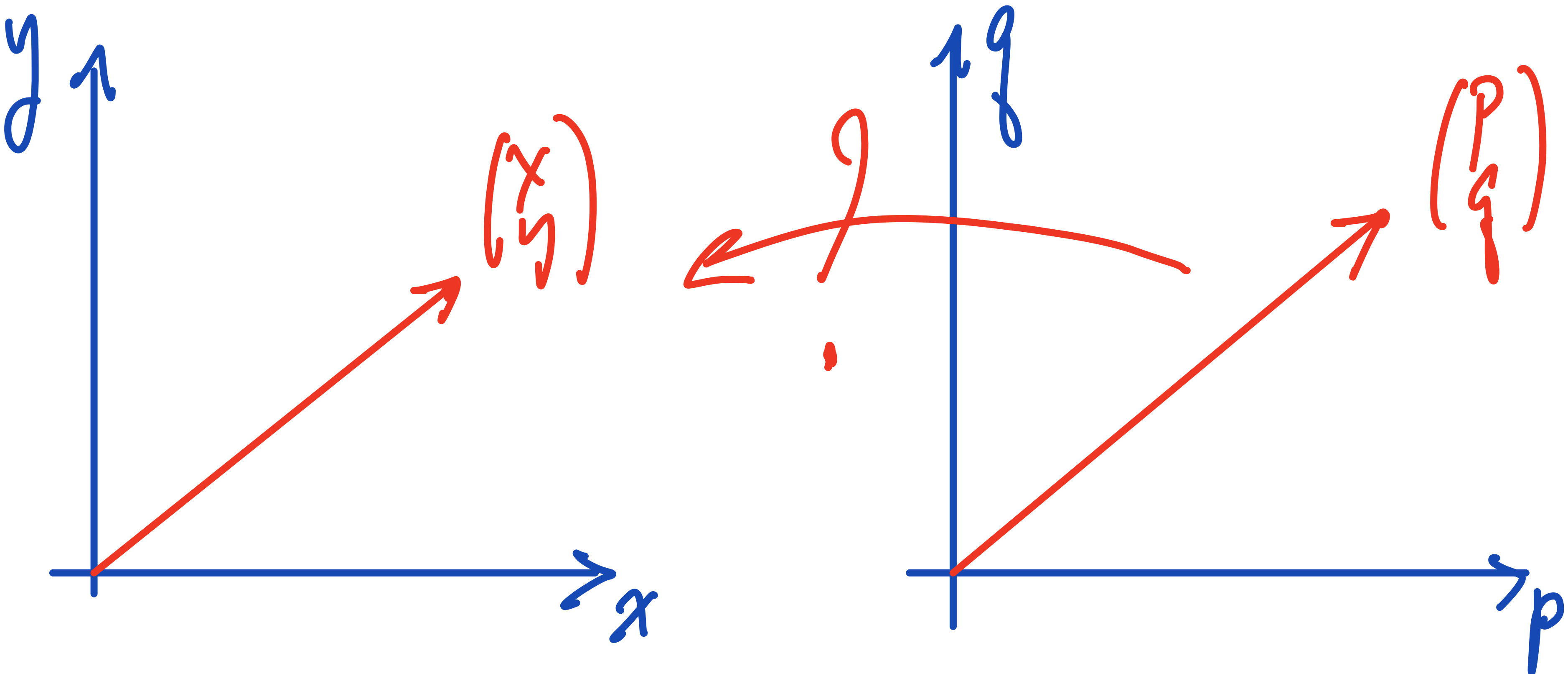
Linear Transformation



$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$p = ax + by$$
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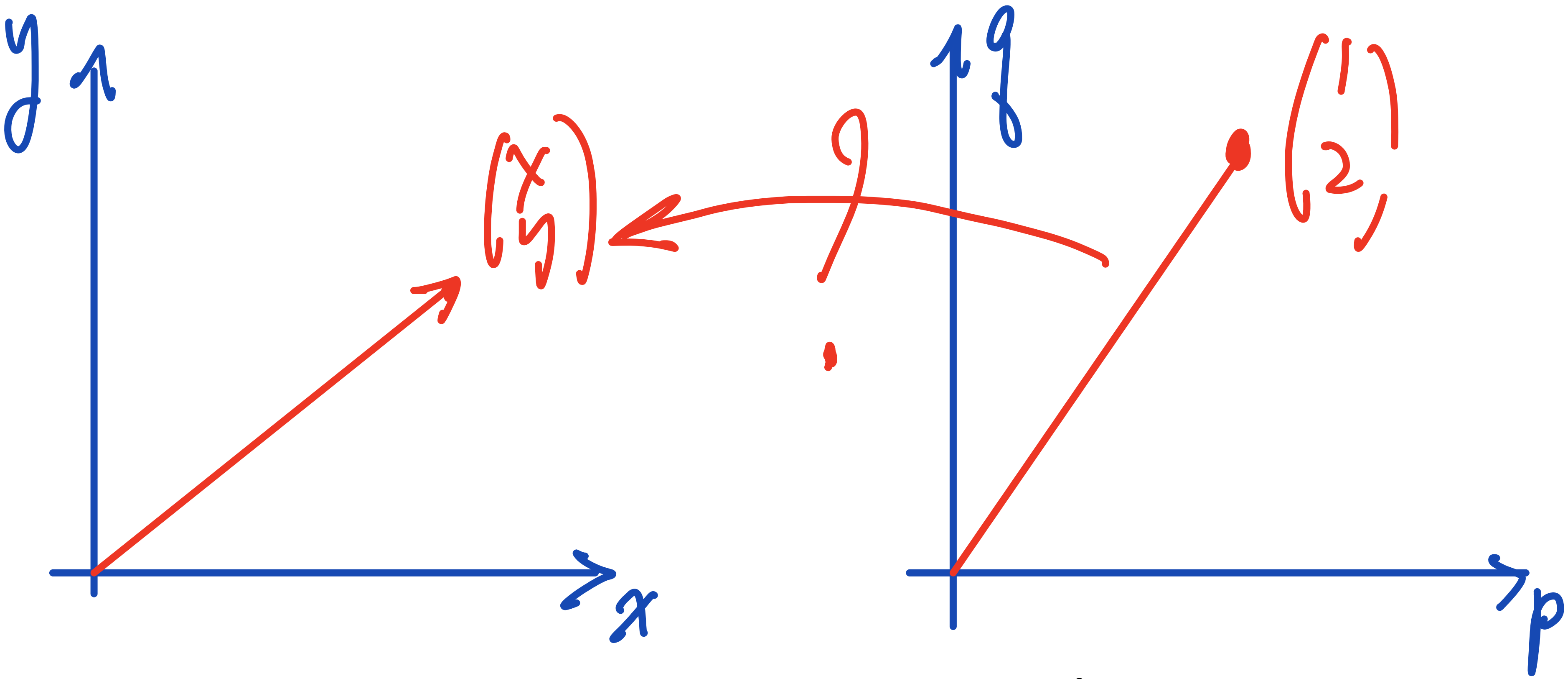
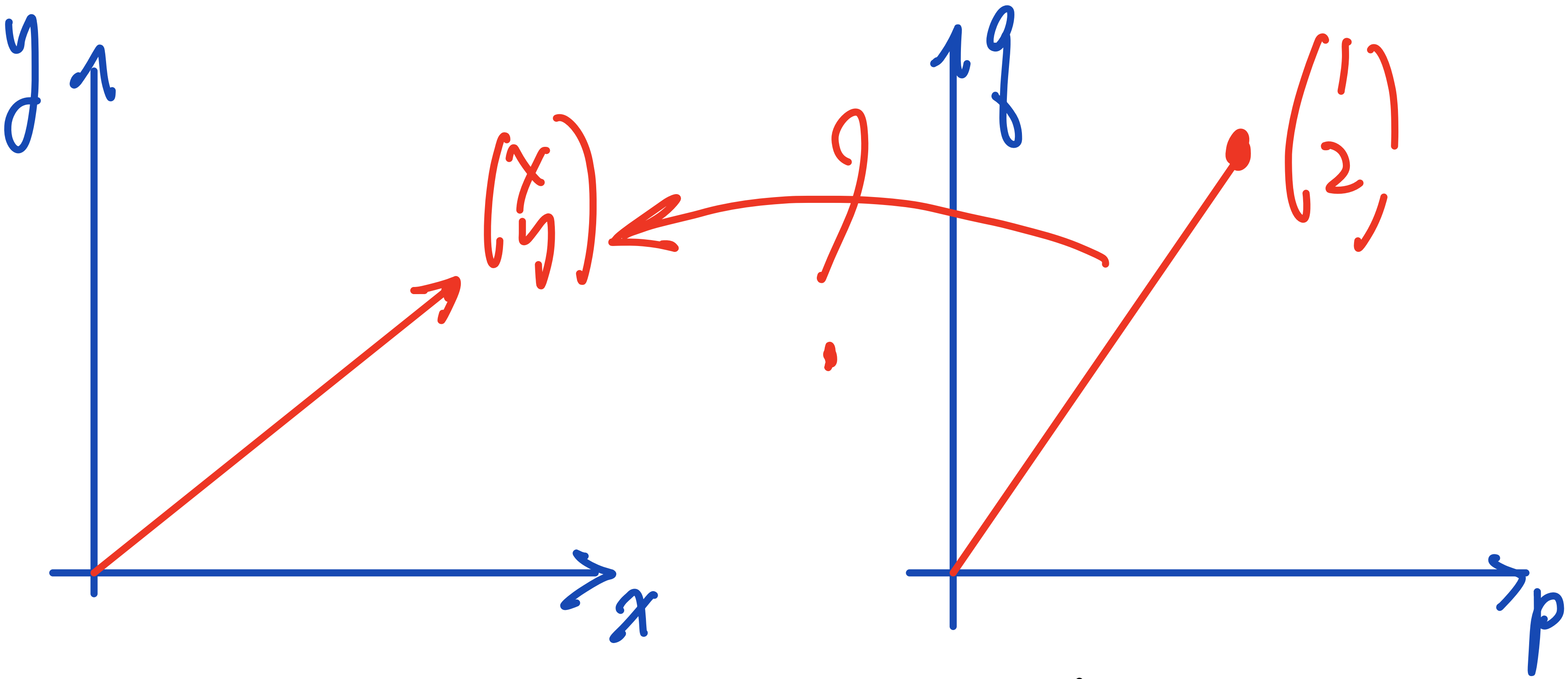
Linear Transformation



$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Linear Transformation



$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{matrix} \leftarrow 2(1) - 1(1) = 1 \\ \leftarrow 1(1) + 1(1) = 2 \end{matrix}$$

Linear Transformation

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \boxed{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}} \longrightarrow AX$$

Properties of $X \longrightarrow AX$

$$\left.\begin{array}{ll} (1) & A(X_1 + X_2) = AX_1 + AX_2 \\ (2) & A(\alpha X) = \alpha AX \end{array}\right\} \text{linearity property}$$

Linear Transformation

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \boxed{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}} \longrightarrow AX$$

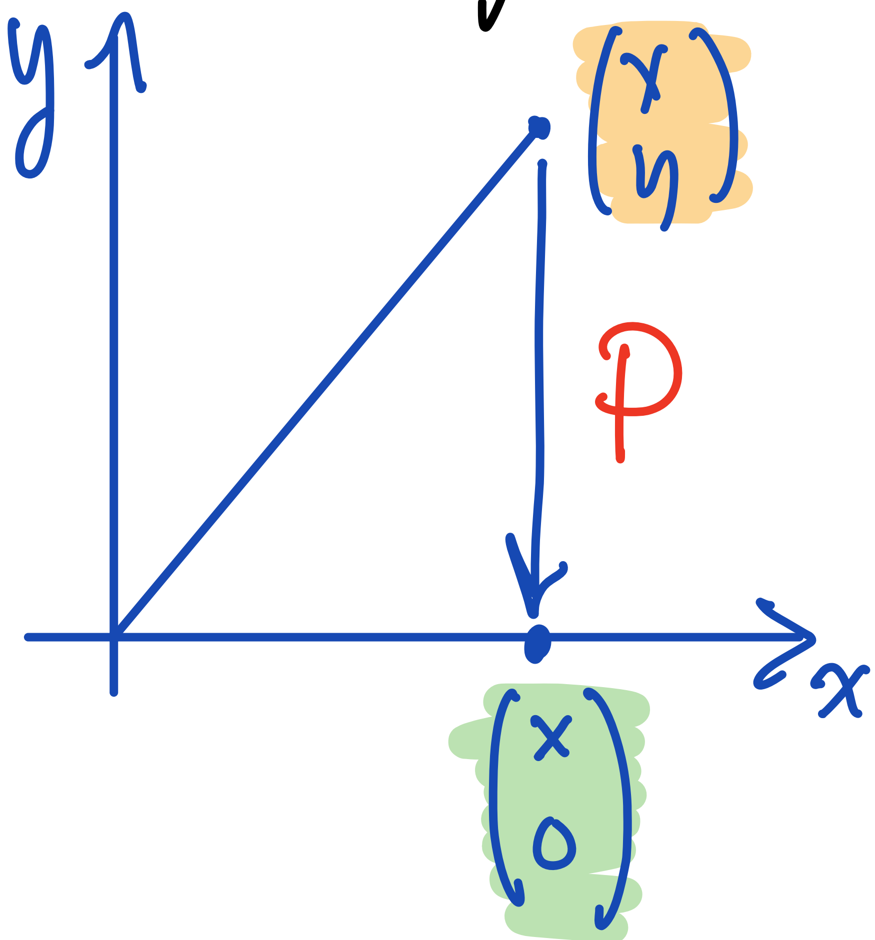
Properties of $X \longrightarrow AX$

$$\boxed{A(\alpha X_1 + \beta X_2) = \alpha A X_1 + \beta A X_2}$$

linearity property

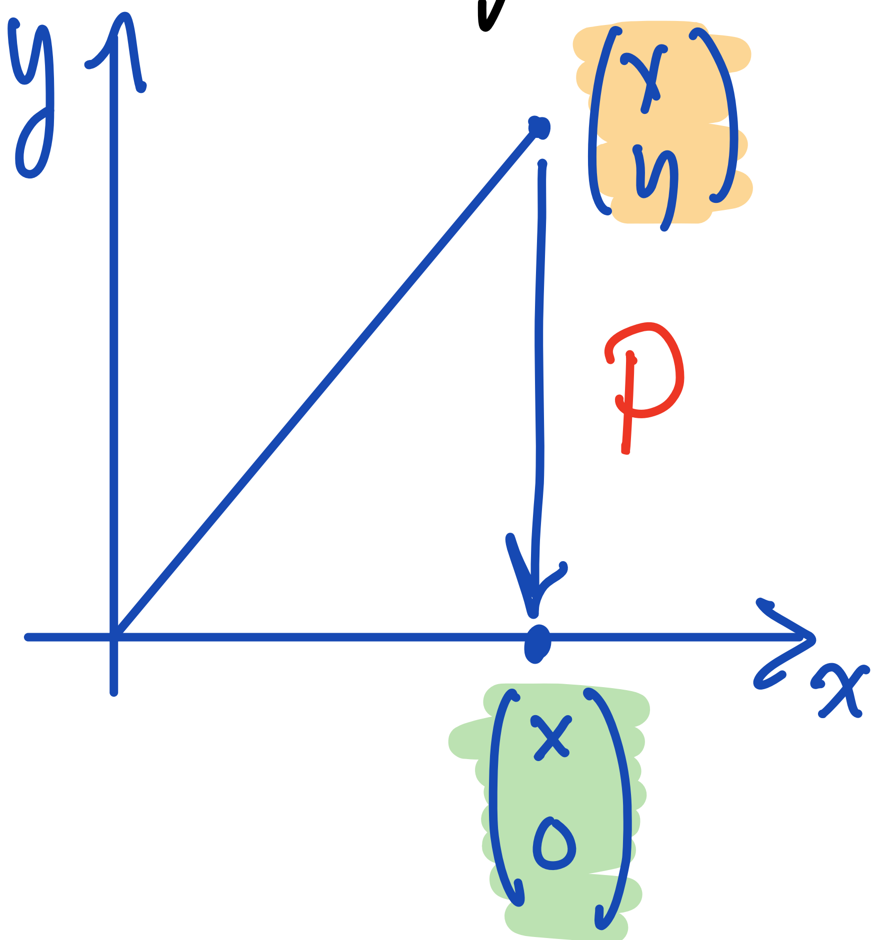
Examples of Linear Transformations

(1) Projection



Examples of Linear Transformations

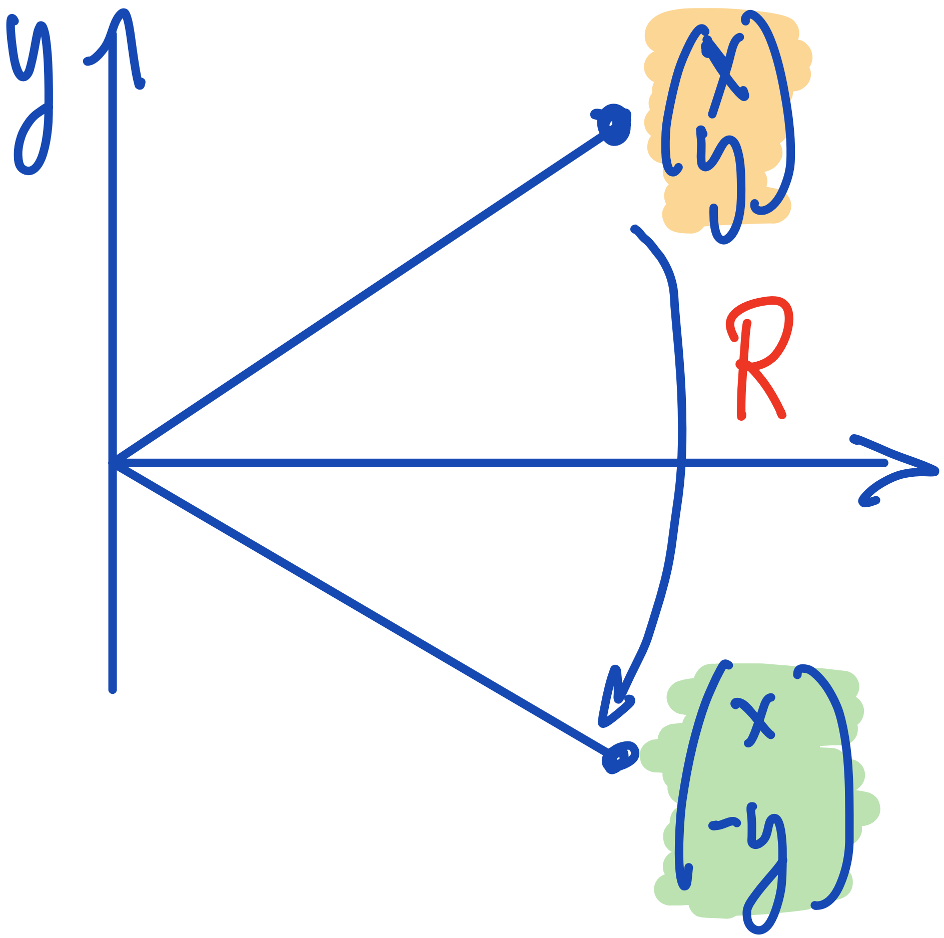
(1) Projection



$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{P} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

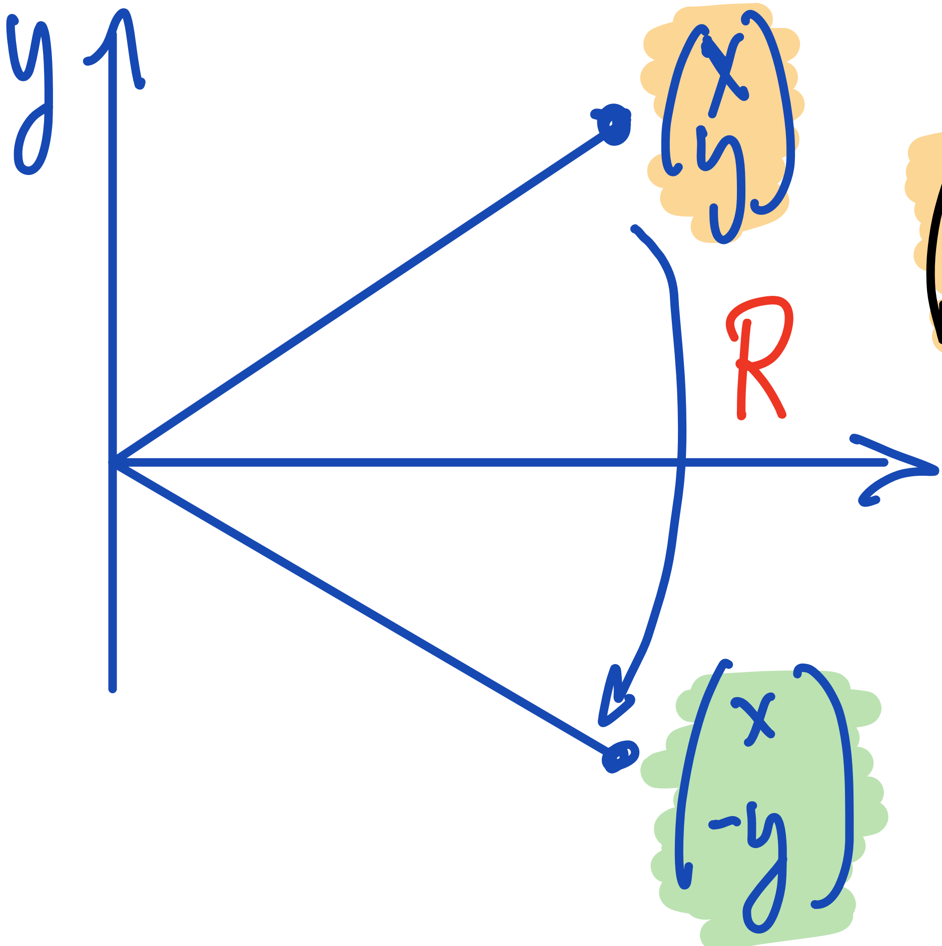
Examples of Linear Transformations

(2) Reflection



Examples of Linear Transformations

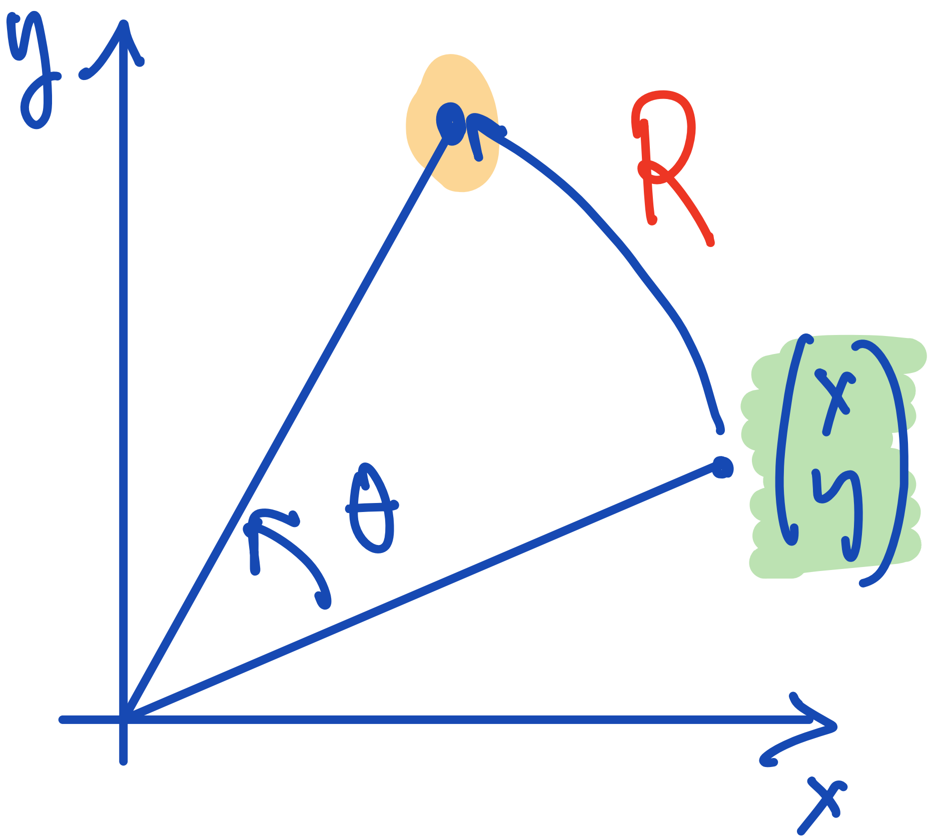
(2) Reflection



$$\begin{pmatrix} x \\ y \end{pmatrix} \to \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{R} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

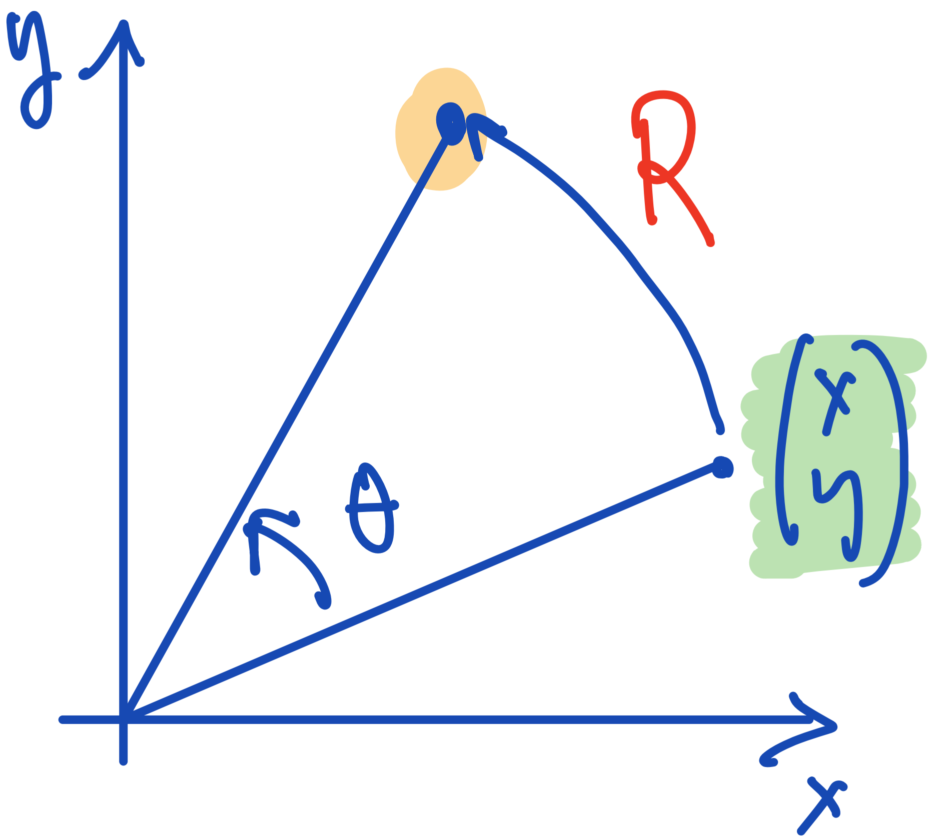
Examples of Linear Transformations

(3) Rotation



Examples of Linear Transformations

(3) Rotation



$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

Eigenvalue (λ) and Eigenvector (X)

$$AX = \lambda X \qquad (X \neq 0)$$

$$A = \begin{pmatrix} 7 & -1 \\ -6 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -1 \\ -6 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7-1 \\ -6+8 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

Eigenvalue (λ) and Eigenvector (X)

$$AX = \lambda X \quad (X \neq 0)$$

$$A = \begin{pmatrix} 7 & -1 \\ -6 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -1 \\ -6 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7-2 \\ -6+16 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

Eigenvalue (λ) and Eigenvector (X)

$$AX = \lambda X \qquad (X \neq 0)$$

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$$\begin{pmatrix} 7 & -1 \\ -6 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7-2 \\ -6+16 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Eigenvalue (λ) and Eigenvector (X)

$$AX = \lambda X \qquad (X \neq 0)$$

$$A = \begin{pmatrix} 7 & -1 \\ -6 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -1 \\ -6 & 8 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -7-3 \\ 6+24 \end{pmatrix} = \begin{pmatrix} -10 \\ 30 \end{pmatrix}$$

Eigenvalue (λ) and Eigenvector (X)

$$AX = \lambda X \quad (X \neq 0)$$

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$$\begin{pmatrix} 7 & -1 \\ -6 & 8 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -7-3 \\ 6+24 \end{pmatrix} = \begin{pmatrix} -10 \\ 30 \end{pmatrix} = 10 \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

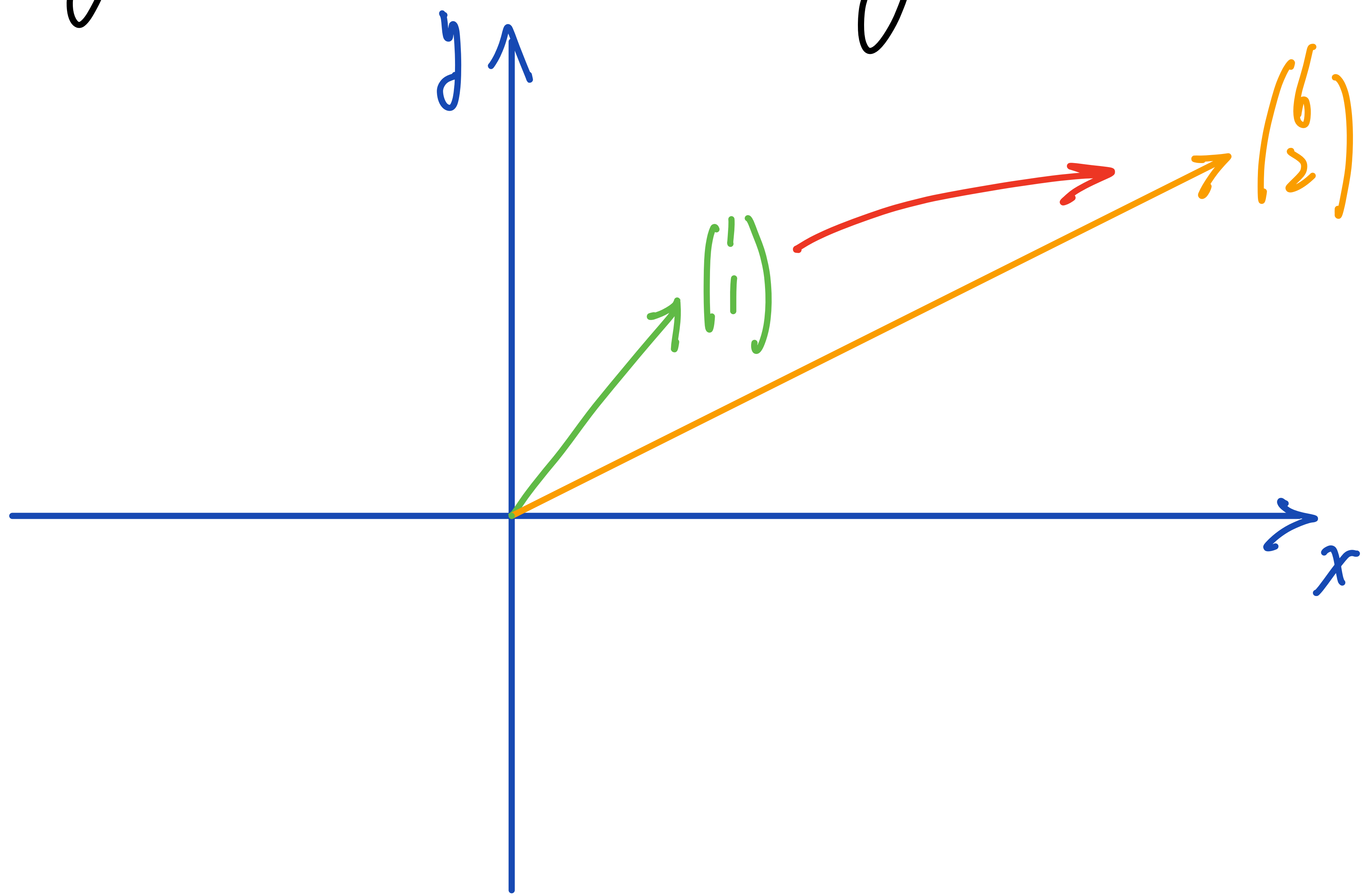
Eigenvalue (λ) and Eigenvector (X)

$$AX = \lambda X \qquad (X \neq 0)$$

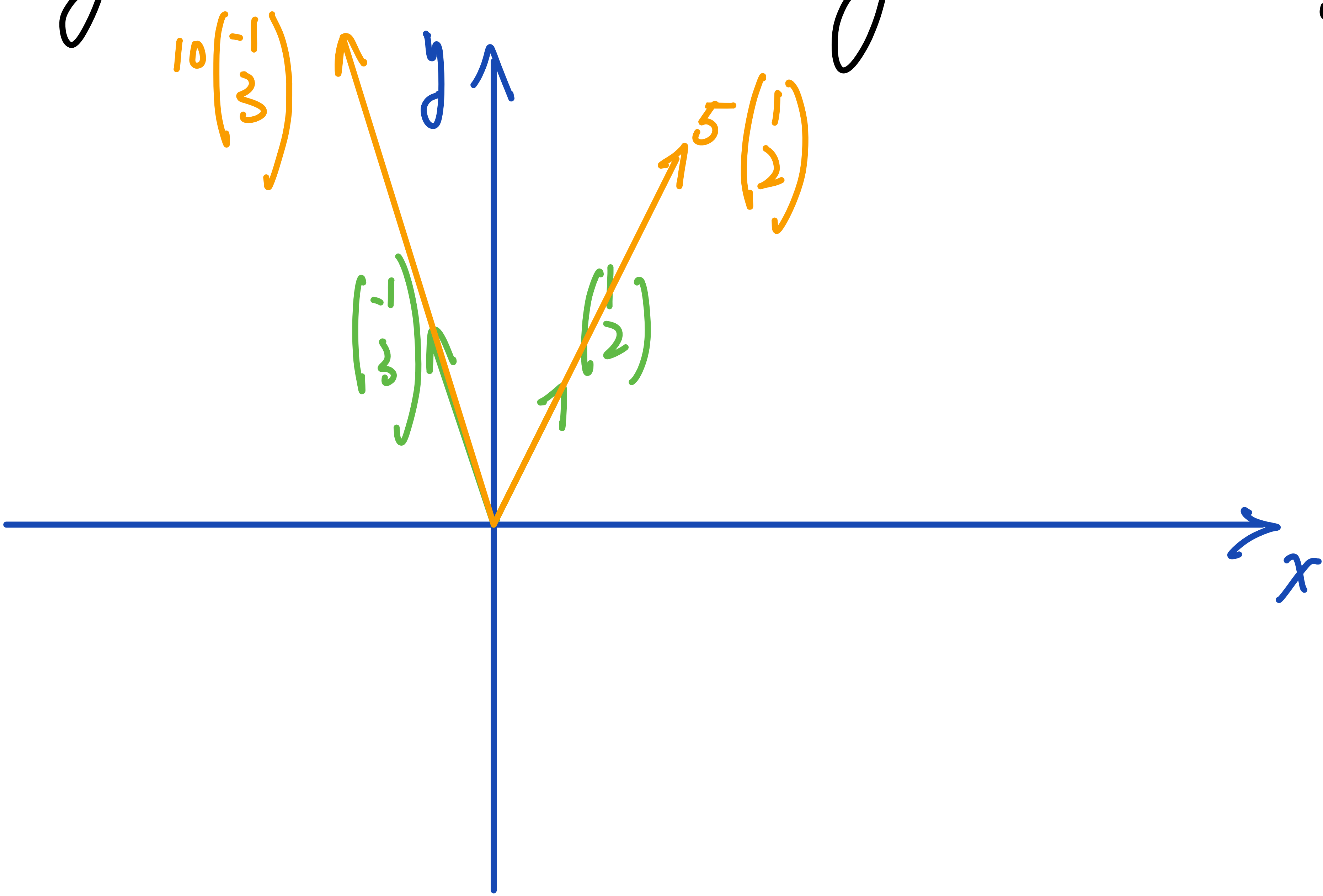
$$\begin{pmatrix} 7 & -1 \\ -6 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \begin{pmatrix} 7 & -1 \\ -6 & 8 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix} = 10\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\lambda = 5$; $\qquad X = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $\lambda = 10$

Eigenvalue (λ) and Eigenvector (X)



Eigenvalue (λ) and Eigenvector (X)



Eigenvalue (λ) and Eigenvector (X)

Iterations of A:

$$AX = \lambda X$$

$$A(AX) = A(\lambda X) = \lambda AX = \lambda(\lambda X)$$

$$A^2X = \lambda^2 X$$

$$A^3X = \lambda^3 X$$

$$A^4X = \lambda^4 X$$

Eigenvalue (λ) and Eigenvector (X)

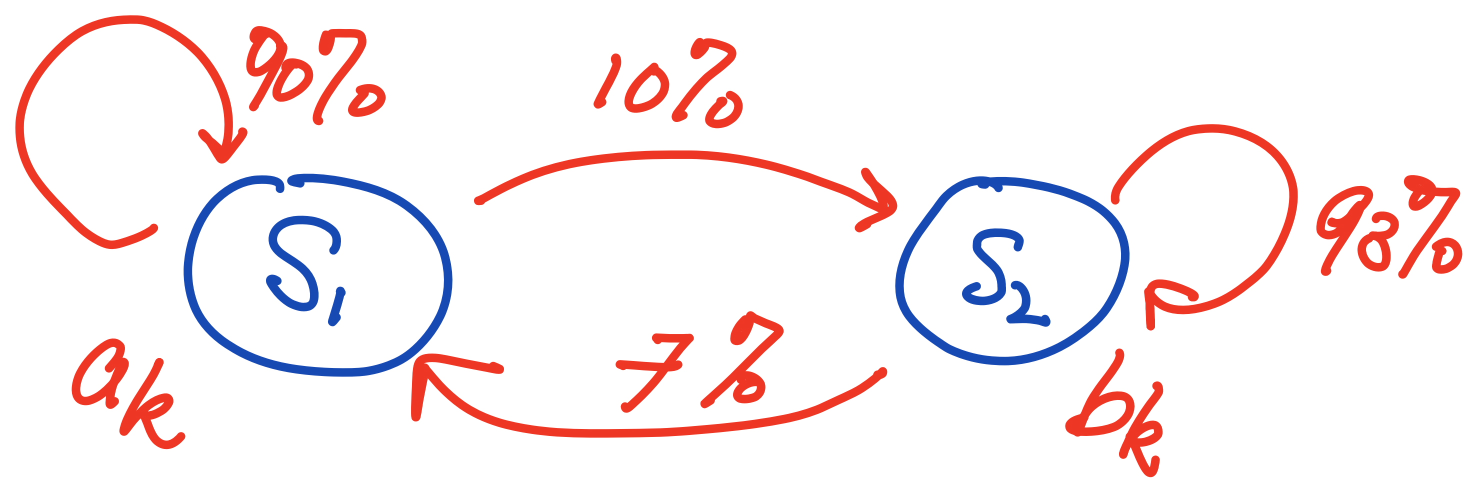
Iterations of A:

$$\underline{AX = \lambda X}$$

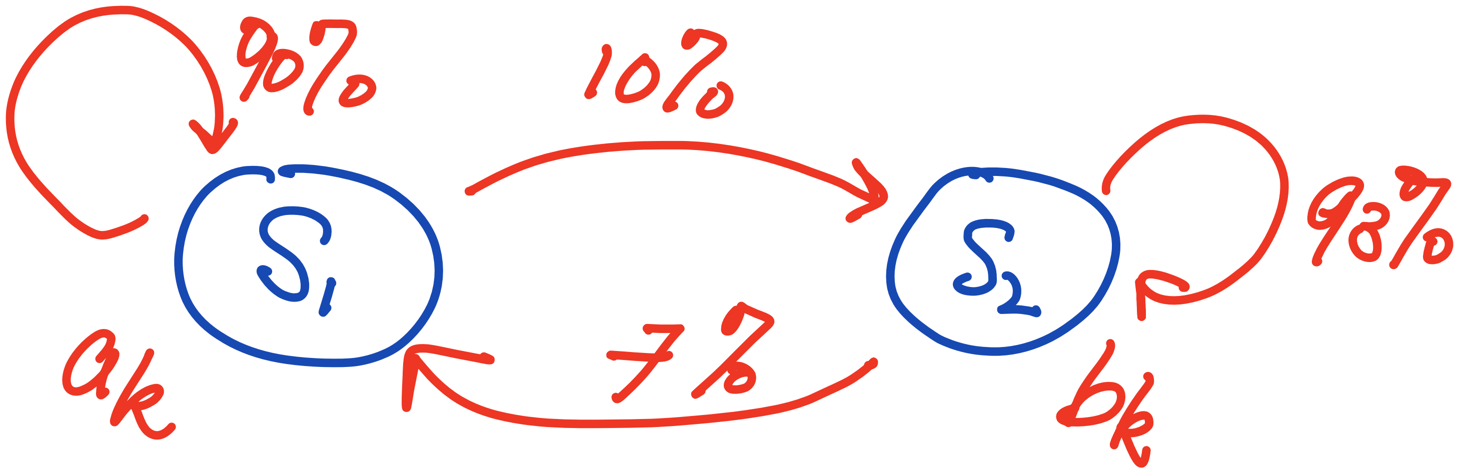
$$\Downarrow$$

$$\underline{A^n X = \lambda^n X}$$

Markov Chain



Markov Chain

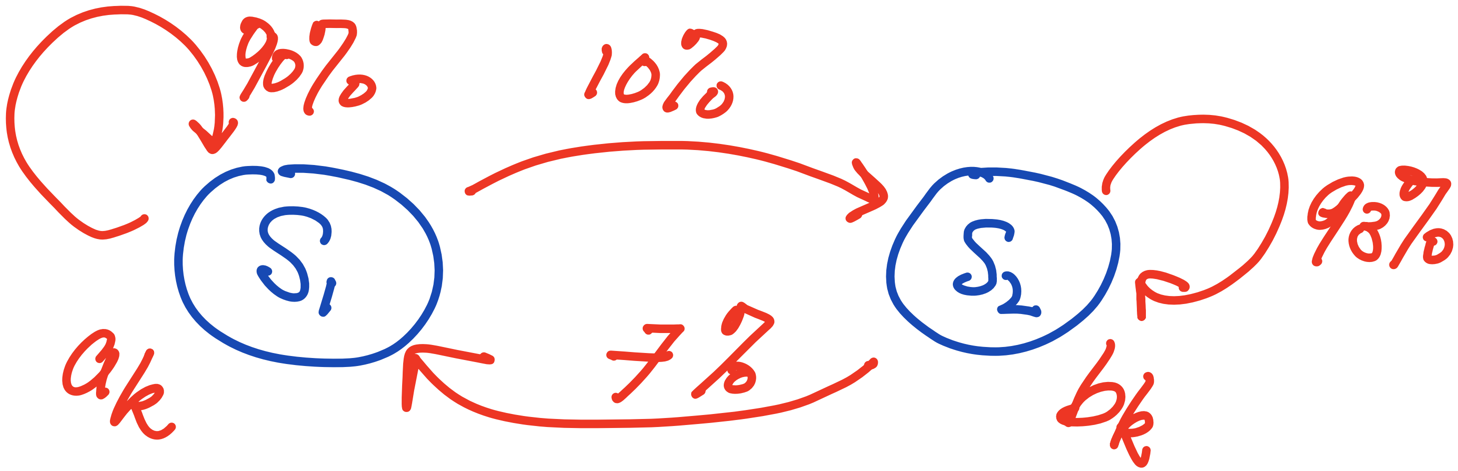


Let a_k = share/population in S_1 in time k

b_k = share/population in S_2 in time k

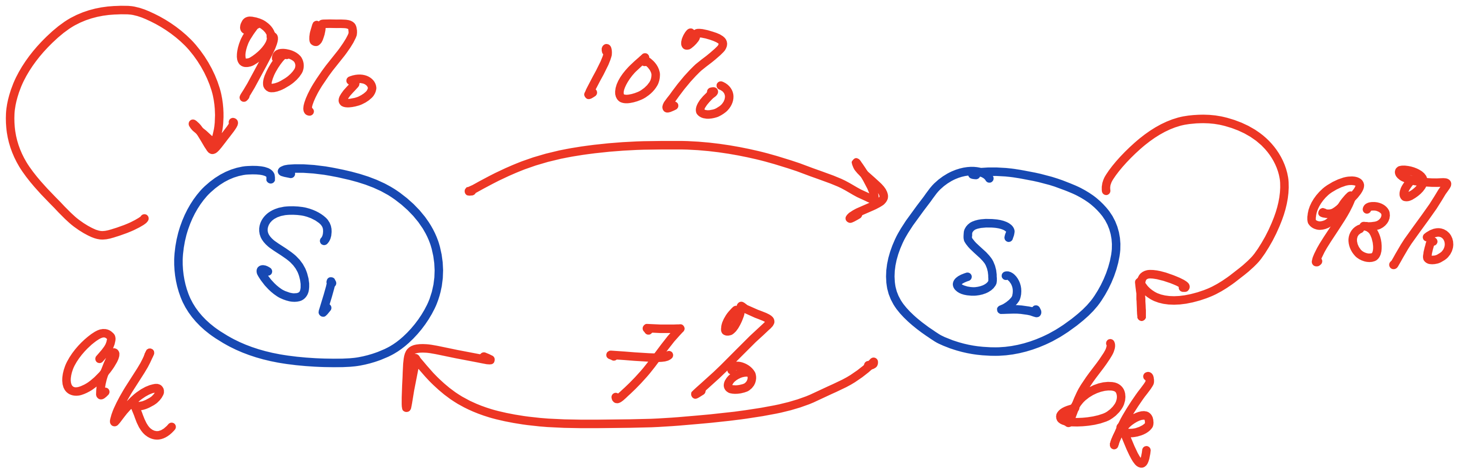
(a_0, b_0 : initial values in time 0.)

Markov Chain



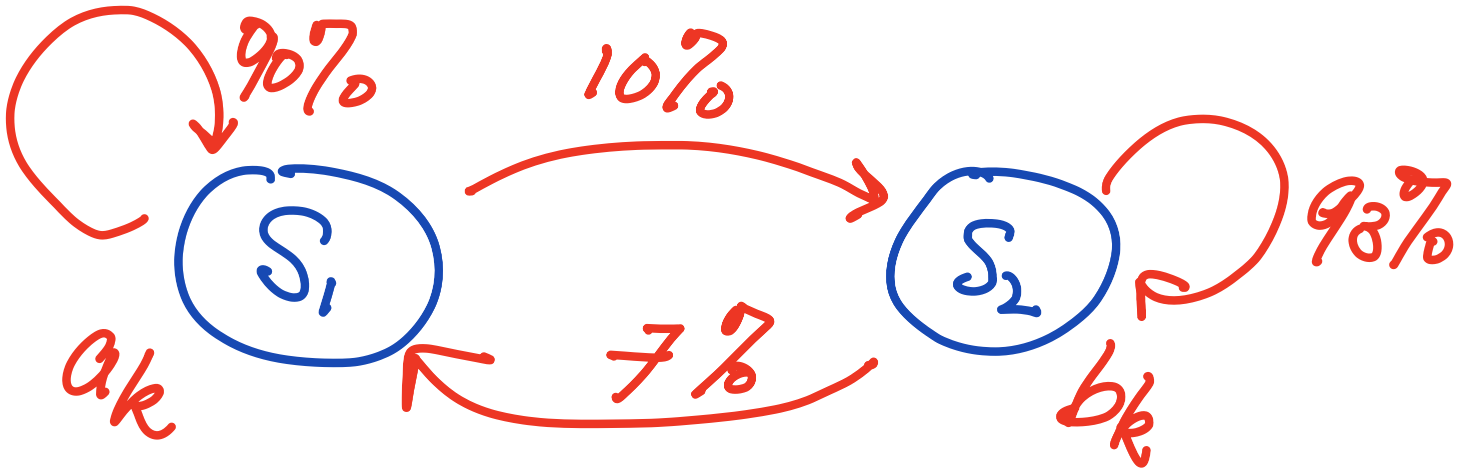
$$\begin{cases} a_1 = a_0\,0.9 + b_0\,0.07 \\ b_1 = a_0\,0.1 + b_0\,0.93 \end{cases} \iff \underbrace{\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}}_{Y_1} = \underbrace{\begin{bmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_0 \\ b_0 \end{bmatrix}}_{Y_0}$$

Markov Chain



$$\begin{cases} a_2 = a_1\,0.9 + b_1\,0.07 \\ b_2 = a_1\,0.1 + b_1\,0.93 \end{cases} \iff \underbrace{\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}}_{Y_2} = \underbrace{\begin{bmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}}_{Y_1}$$

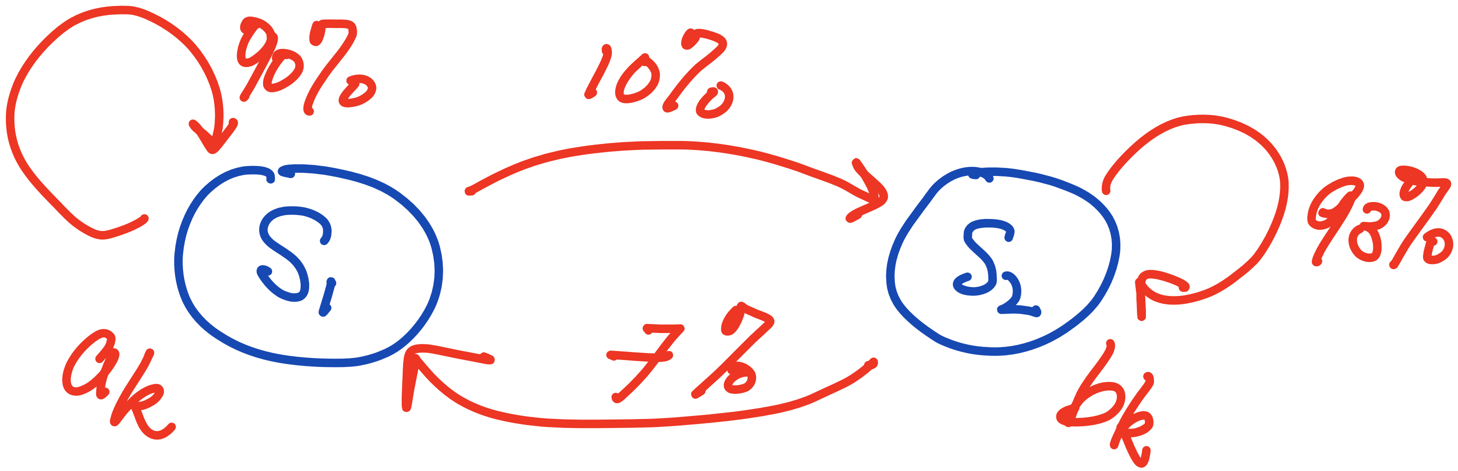
Markov Chain



$$\begin{cases} a_k = a_{k-1}0.9 + b_{k-1}0.07 \\ b_k = a_{k-1}0.1 + b_{k-1}0.93 \end{cases} \iff \underbrace{\begin{bmatrix} a_k \\ b_k \end{bmatrix}}_{Y_k} = \underbrace{\begin{bmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_{k-1} \\ b_{k-1} \end{bmatrix}}_{Y_{k-1}}$$

$$Y_k = A \, Y_{k-1}$$

Markov Chain



$$Y_1 = AY_0, \quad Y_2 = AY_1 = A(AY_0) = A^2Y_0$$

$$Y_3 = AY_2 = A(A^2Y_0) = A^3Y_0$$

$$Y_k = AY_{k-1} = \cdots = A^kY_0$$

Long Time Behavior

$A^k Y_0 \xrightarrow{k \to \infty} Y_*$ (Steady State)

$$\begin{array}{ccc} A^{(k+1)} Y_0 = A(A^k Y_0) & \longrightarrow & A Y_* \\ \downarrow & & \\ Y_* & & \end{array}$$

Long Time Behavior

$$A^k Y_0 \xrightarrow{k \to \infty} Y_*$$

Steady State

$$A^{(k+1)} Y_0 = A(A^k Y_0) \longrightarrow A Y_*$$

$$\downarrow$$

$$Y_*$$

Hence, Y_* satisfies:

$$A Y_* = Y_*$$

Long Time Behavior

$$A^k Y_0 \xrightarrow{k \to \infty} Y_*$$ **Steady State**

$$A^{k+1} Y_0 = A(A^k Y_0) \longrightarrow A Y_*$$

$$\downarrow$$

$$Y_*$$

Hence, Y_* satisfies:

$$\boxed{A Y_* = 1\, Y_*}$$

$\lambda = 1$

Long Time Behavior ($Y_* = ?$)

$$Y_* = \begin{pmatrix} a_* \\ b_* \end{pmatrix} : \quad AY_* = Y_*$$

$$\begin{pmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{pmatrix} \begin{pmatrix} a_* \\ b_* \end{pmatrix} = \begin{pmatrix} a_* \\ b_* \end{pmatrix}$$

$$\begin{cases} 0.9a_* + 0.07b_* = a_* \\ 0.1a_* + 0.93b_* = b_* \end{cases}$$

Long Time Behavior $(Y_* = ?)$

$$Y_* = \begin{pmatrix} a_* \\ b_* \end{pmatrix} : \quad AY_* = Y_*$$

$$\begin{pmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{pmatrix} \begin{pmatrix} a_* \\ b_* \end{pmatrix} = \begin{pmatrix} a_* \\ b_* \end{pmatrix}$$

$$\begin{cases} -0.1a_* + 0.07b_* = 0 \\ 0.1a_* - 0.07b_* = 0 \end{cases}$$

Long Time Behavior $(Y_* = ?)$

$Y_* = \begin{pmatrix} a_* \\ b_* \end{pmatrix}$: $AY_* = Y_*$

$$\begin{pmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{pmatrix} \begin{pmatrix} a_* \\ b_* \end{pmatrix} = \begin{pmatrix} a_* \\ b_* \end{pmatrix}$$

$$\begin{cases} -0.1 a_* + 0.07 b_* = 0 \\ \cancel{0.1 a_* - 0.07 b_* = 0} \end{cases}$$

$a_* = 0.7 b_*$

Choose a_*, b_*

$a_* + b_* = 1$

$$\begin{pmatrix} a_* \\ b_* \end{pmatrix} = \begin{pmatrix} \frac{0.7}{1.7} \\ \frac{1}{1.7} \end{pmatrix}$$

Long Time Behavior $(Y_* = ?)$

$Y_* = \begin{pmatrix} a_* \\ b_* \end{pmatrix}$: $AY_* = Y_*$

$$\begin{pmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{pmatrix} \begin{pmatrix} a_* \\ b_* \end{pmatrix} = \begin{pmatrix} a_* \\ b_* \end{pmatrix}$$

$$\begin{cases} -0.1a_* + 0.07b_* = 0 \\ \cancel{0.1a_* - 0.07b_* = 0} \end{cases}$$

$a_* = 0.7b_*$

Choose a_*, b_*

$a_* + b_* = 1$

$\begin{pmatrix} a_* \\ b_* \end{pmatrix} = \begin{pmatrix} 0.41 \\ 0.59 \end{pmatrix}$

Properties of the Matrix A

$$A = \begin{pmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{pmatrix}\begin{pmatrix} 0.41 \\ 0.59 \end{pmatrix} = 1\begin{pmatrix} 0.41 \\ 0.59 \end{pmatrix}}$$

Properties of the Matrix A

$$A = \begin{pmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{pmatrix}$$

- each entry is positive
- sum of each column equals one

Such a matrix is called a

$\boxed{\text{Markov Matrix}}$

Properties of the Matrix A

$$A = \begin{pmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{pmatrix}$$

- each entry is positive
- sum of each column equals one

Any Markov Matrix A has a Steady State Y_*: $AY_* = Y_*$

$25,000,000,000

vector

$25,000,000,000

vector

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The $25,000,000,000 Eigenvector: The Linear Algebra behind Google*

Kurt Bryan†
Tanya Leise‡

$25,000,000,000 vector

SIAM REVIEW
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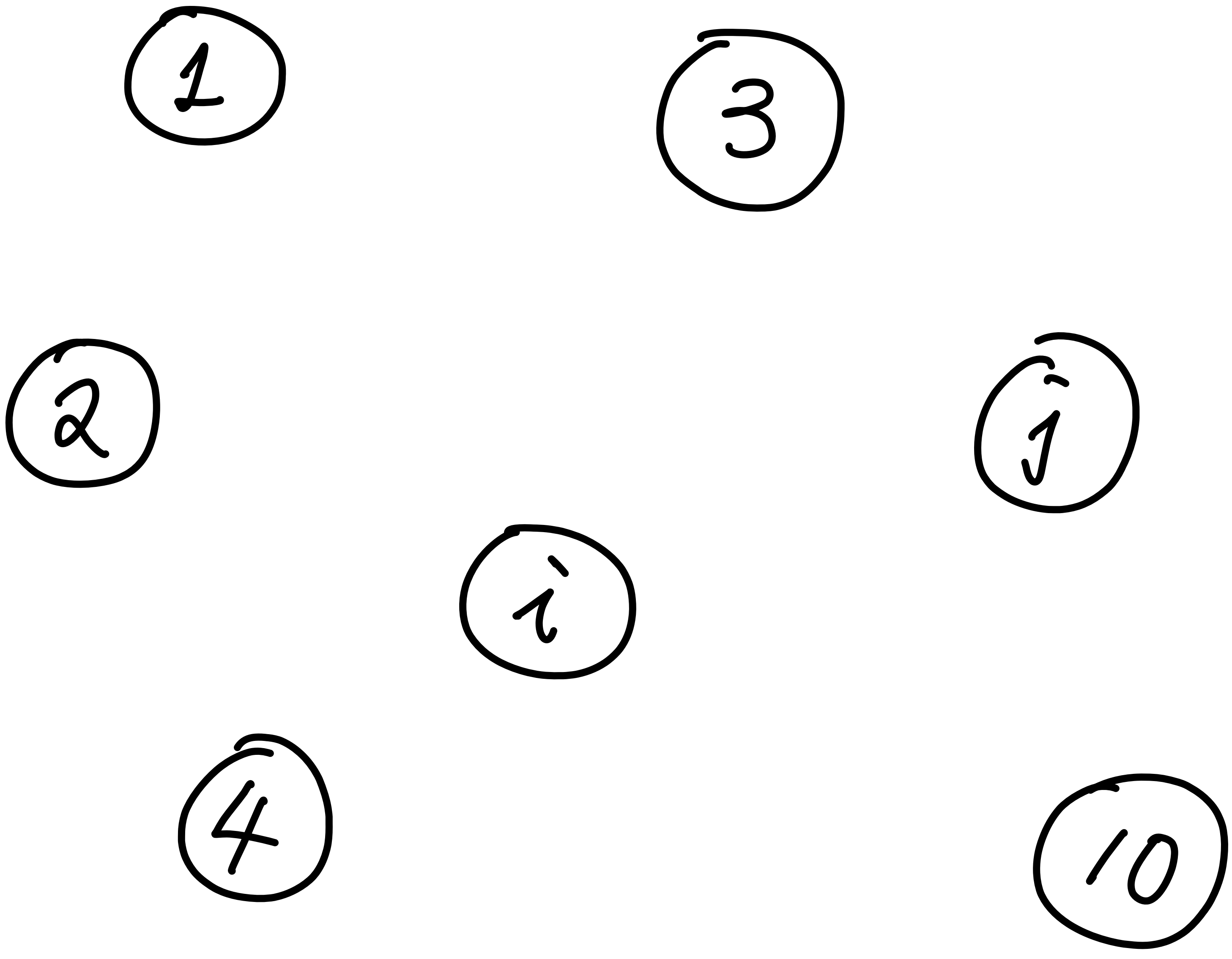
The $25,000,000,000 Eigenvector: The Linear Algebra behind Google*

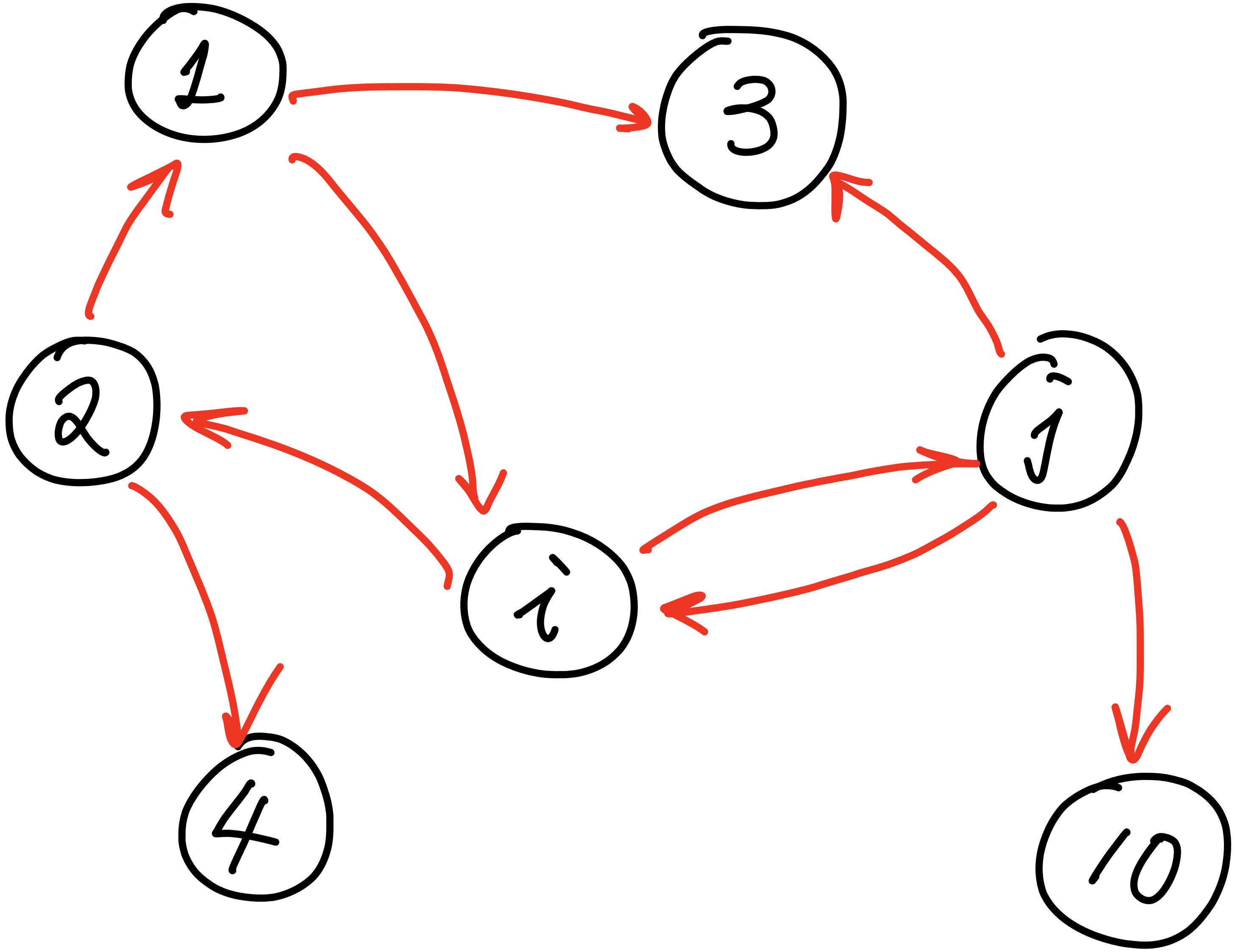
Rank webpages

Kurt Bryan†
Tanya Leise‡

Web-pages



Web-pages with links:



How to quantify / rank the importance of the webpages?

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A webpage is important if

How to quantify / rank the importance of the webpages?

A webpage is important if

(1) many other pages link to it

How to <u>quantify / rank</u> the importance of the webpages?

A webpage is <u>important</u> if

(1) <u>many other</u> pages link to it

and

(2) if an <u>important page links to it</u>.

Google's PageRank Algorithm

webpage i

i

x_i = importance of page i

$$x_i = \sum_{j}$$

other pages that link to i

Google's PageRank Algorithm

webpage i

i

x_i = importance of page i

$$x_i = \sum_j \underbrace{x_j}_{\text{importance of } j}$$

Google's PageRank Algorithm

web page i

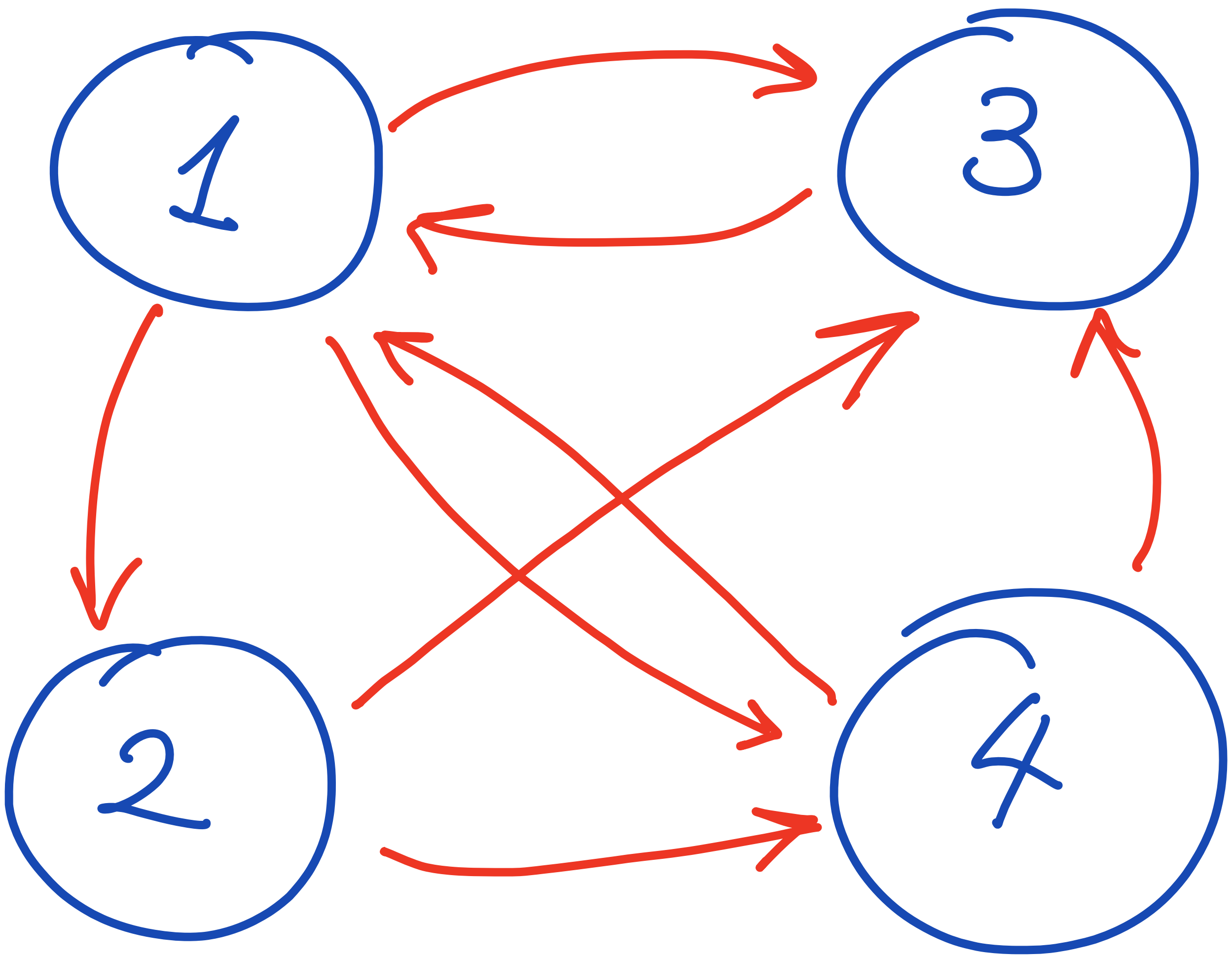
i

x_i = importance of page i

$$x_i = \sum_j \frac{x_j}{D_j}$$

no. of out-links of j

An Example



An Example

$$x_1 = \qquad x_3 + \frac{x_4}{2}$$

An Example

$$x_1 = \qquad x_3 + \frac{x_4}{2}$$

$$x_2 = \frac{x_1}{3}$$

An Example

$$x_1 = x_3 + \frac{x_4}{2}$$

$$x_2 = \frac{x_1}{3}$$

$$x_3 = \frac{x_1}{3} + \frac{x_2}{2} + \frac{x_4}{2}$$

An Example

$$x_1 = x_3 + \frac{x_4}{2}$$

$$x_2 = \frac{x_1}{3}$$

$$x_3 = \frac{x_1}{3} + \frac{x_2}{2} + \frac{x_4}{2}$$

$$x_4 = \frac{x_1}{3} + \frac{x_2}{2}$$

An Example

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\underline{X = AX, \quad \lambda = 1}$$

An Example

$$0 = -x_1 + x_3 + \frac{x_4}{2}$$

$$0 = \frac{x_1}{3} - x_2$$

$$0 = \frac{x_1}{3} + \frac{x_2}{2} - x_3 + \frac{x_4}{2}$$

$$0 = \frac{x_1}{3} + \frac{x_2}{2} - x_4$$

An Example

$$\begin{bmatrix} -1 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(A-I)\vec{x} = \vec{0}$$

An Example

$$\begin{bmatrix} -1 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Of course $\vec{x} = 0$ is a solution.

An Example

$$\begin{bmatrix} -1 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

But is there any other interesting (positive) solutions?

An Example

$$\begin{bmatrix} -1 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Yes! $x_1 = \frac{12}{31} \approx 0.387$, $x_2 = \frac{4}{31} \approx 0.129$, $x_3 = \frac{9}{31} \approx 0.29$, $x_4 = \frac{6}{31} \approx 0.194$

An Example

$$x_1 = x_3 + \frac{x_4}{2}$$

$$x_2 = \frac{x_1}{3}$$

$$x_3 = \frac{x_1}{3} + \frac{x_2}{2} + \frac{x_4}{2}$$

$$x_4 = \frac{x_1}{3} + \frac{x_2}{2}$$

An Example

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\vec{X} = A\vec{X}$$

An Example

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

More generally, equations of the form $A\vec{x} = \lambda\vec{x}$ is to find eigenvectors $\vec{x}$, eigenvalues λ

An Example

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

In the current case $\lambda = 1$,

$$A\vec{X} = 1 \cdot \vec{X}.$$

An Example

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

sum of each column gives 1

An Example

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

such a matrix is called a stochastic matrix

Some Theoretical and Practical considerations:

(1) Existence of $\vec{X}$ is given by $\boxed{\text{Perron-Frobenius Theorem}}$

Some Theoretical and Practical considerations:

(1) Existence of $\vec{X}$ is given by **Perron-Frobenius Theorem**

(2) Computation ?

Some Theoretical and Practical considerations:

(1) Existence of $\vec{x}$ is given by

$$\boxed{\text{Perron-Frobenius Theorem}}$$

(2) Computation ?

(3) Stability ?

Fibonacci Sequence

0, 1, 1, 2, 3, 5,

Fibonacci Sequence

$0, 1, 1, 2, 3, 5, 8, 13, 21 \cdots$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \cdots$

$$x_{n+2} = x_n + x_{n+1}, \qquad x_0 = 0, \; x_1 = 1$$

Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21 ...

x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 ...

$x_{n+2} = x_n + x_{n+1}$, $x_0 = 0$, $x_1 = 1$

$$x_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Fibonacci Sequence

$0, 1, 1, 2, 3, 5, 8, 13, 21 \cdots$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \cdots$

$$x_{n+2} = x_n + x_{n+1}, \qquad x_0 = 0, \; x_1 = 1$$

$$\underbrace{\begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix}}_{Y_{n+1}} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}}_{Y_n}$$

Fibonacci Sequence

$$\underbrace{\begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix}}_{Y_{n+1}} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}}_{Y_n}$$

$$Y_n = M Y_{n-1} = M^2 Y_{n-2} = \cdots = M^n Y_0$$

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} x_1 \\ x_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Fibonacci Sequence

$$\underbrace{\begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix}}_{Y_{n+1}} \underset{=}{=} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}}_{Y_n}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} = \frac{1+\sqrt{5}}{2} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} \qquad MZ_1 = \lambda_1 Z_1$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix} = \frac{1-\sqrt{5}}{2} \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix} \qquad MZ_2 = \lambda_2 Z_2$$

Fibonacci Sequence

$$\underbrace{\begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix}}_{Y_{n+1}} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}}_{Y_n}$$

$$Y_{n+1} = M \, Y_n$$

$$Y_n = c_1 \lambda_1^n Z_1 + c_2 \lambda_2^n Z_2$$

$$= c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

Fibonacci Sequence

$$\underbrace{\begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix}}_{Y_{n+1}} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}}_{Y_n}$$

$$Y_n = c_1 \lambda_1^n Z_1 + c_2 \lambda_2^n Z_2$$

$$= \underline{c_1} \left(\frac{1+\sqrt{5}}{2}\right)^n \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} + \underline{c_2} \left(\frac{1-\sqrt{5}}{2}\right)^n \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

Fibonacci Sequence

$$\underbrace{\begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix}}_{Y_{n+1}} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}}_{Y_n}$$

$$Y_{n+1} = M \, Y_n$$

$$\boxed{x_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)}$$