Probability Theory:

From Brain-Teasers to Differential Equations

MA 108, Fall 2023

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Probability/Statistics:

quantitative description of random phenomena

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- tossing a coin/dice
- selecting a card from a deck of cards
- raining today?
- election?

Probability/Statistics: MA/STAT 416, 519

quantitative description of random phenomena

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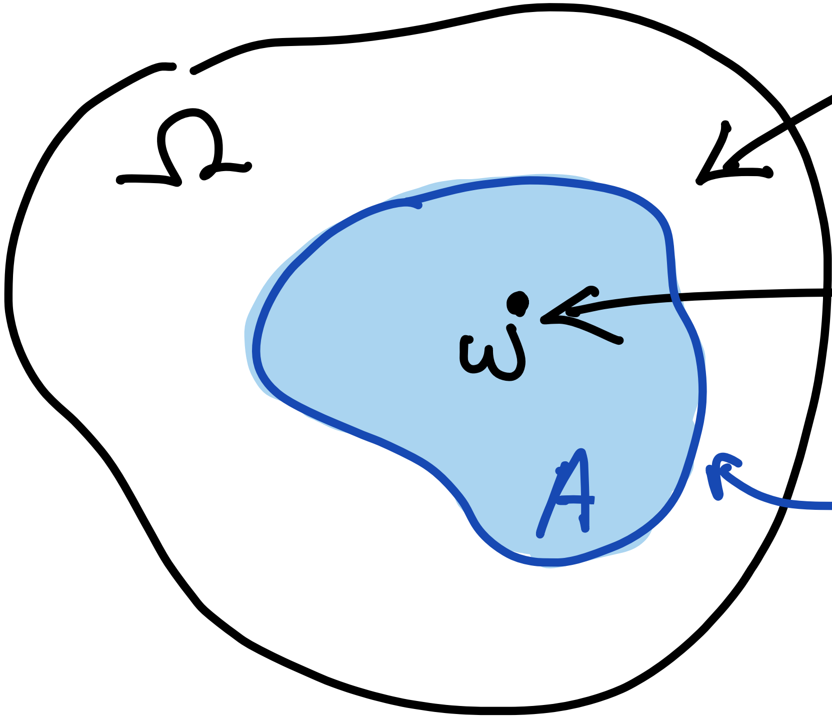
Probability Space and Probability



Space of outcomes ("universe")

a particular outcome

an event (a collection of outcomes)

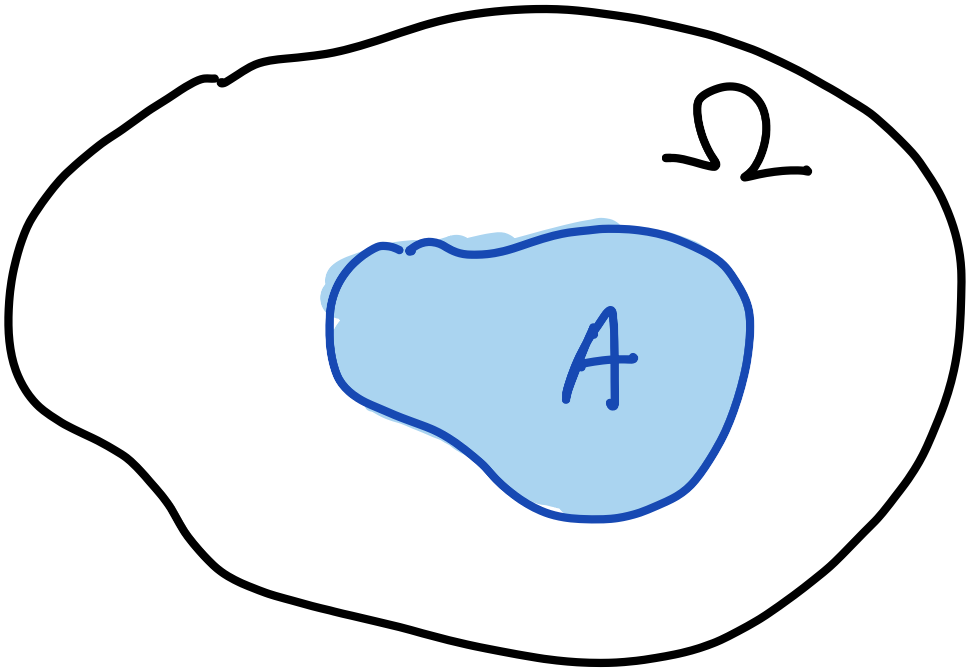
($A \subseteq \Omega$
a subset of Ω)

Probability Space and Probability

$$A \subseteq \Omega \xrightarrow{\quad P \quad} P(A) \in [0, 1]$$

an event

probability of A



$$P(\Omega) = 1$$

$$0 \leq P(A) \leq 1$$

"size" of A

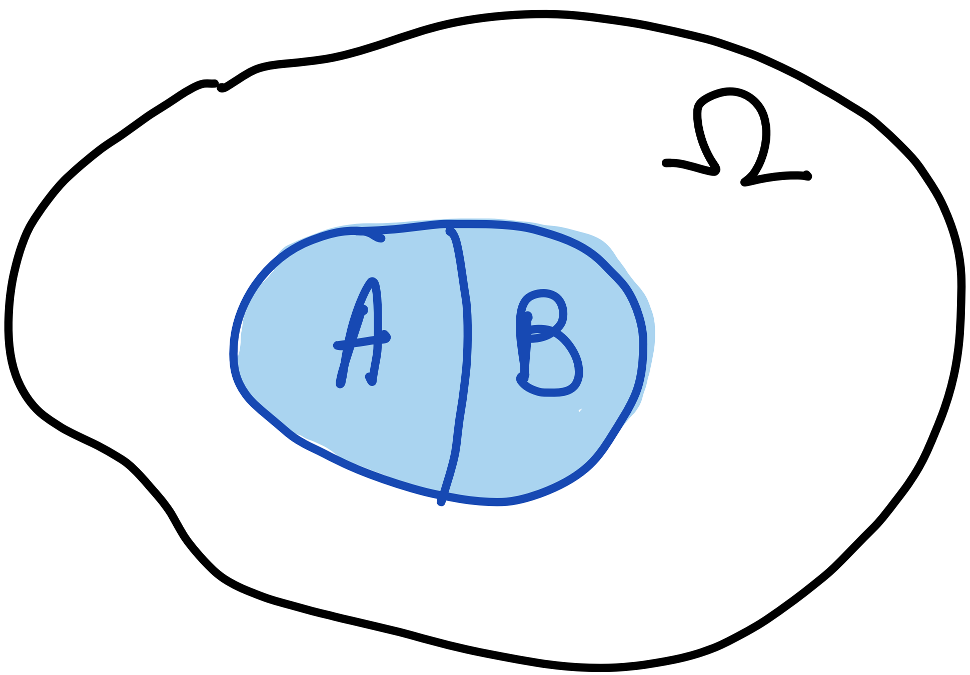
"area" of A

Probability Space and Probability

$$A \subseteq \Omega \xrightarrow{P} P(A) \in [0, 1]$$

an event

probability of A



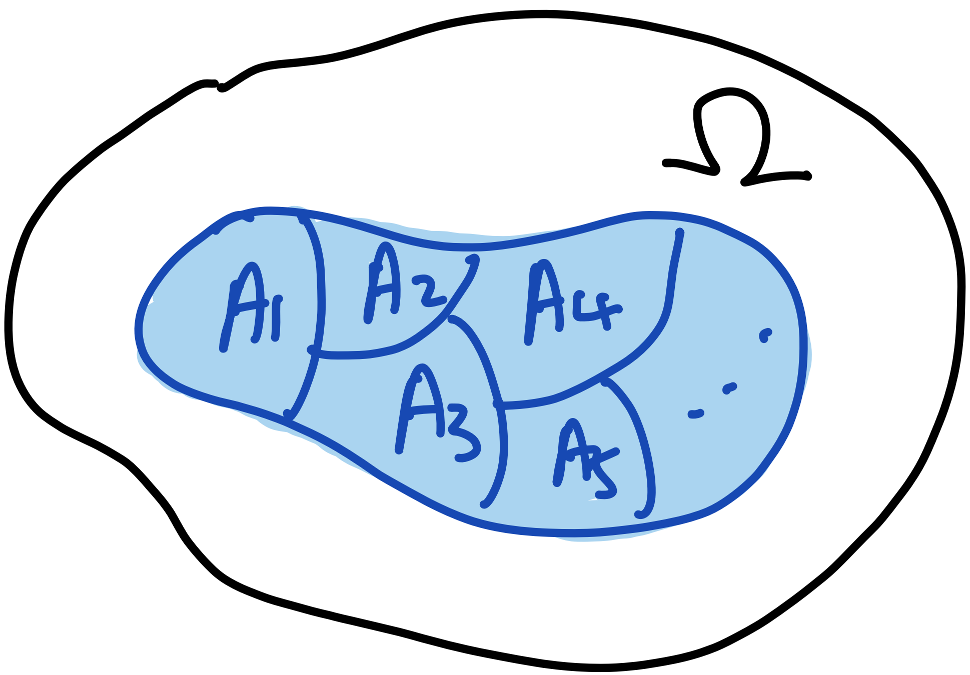
$$P(A \cup B) = P(A) + P(B)$$

Probability Space and Probability

$$A \subseteq \Omega \xrightarrow{\quad P \quad} P(A) \in [0, 1]$$

an event

probability of A



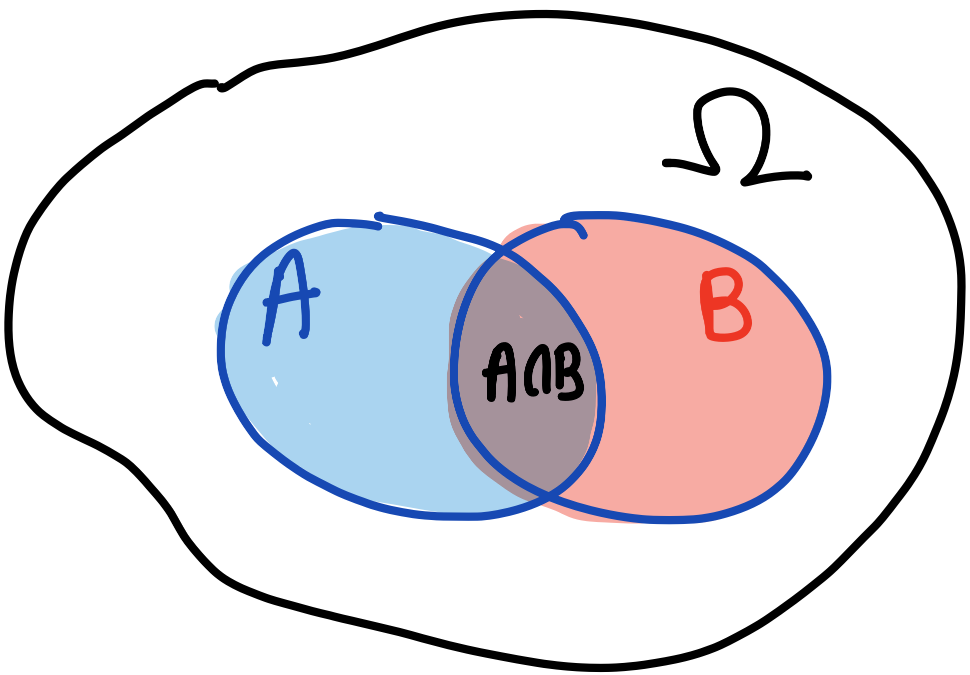
$$P(A_1 \cup A_2 \cup A_3 \cdots) = P(A_1) + P(A_2) + \cdots$$

Probability Space and Probability

$$A \subseteq \Omega \xrightarrow{\ P\ } P(A) \in [0, 1]$$

an event

probability of A



$$P(A) + P(B) = P(A \cup B) + P(A \cap B)$$

Probability Space and Probability

Tossing a coin once

$$\Omega = \{T, H\} \quad (= \{0, 1\})$$

$$\omega_1 = T, \quad \omega_2 = H$$

$$A_1 = \{T\}, \quad A_2 = \{H\}$$

$$P(A_1) = \frac{1}{2}, \quad P(A_2) = \frac{1}{2}$$

Probability Space and Probability

Tossing a coin twice

$\Omega = \{TT, TH, HT, HH\} = \{00, 01, 10, 11\}$

$\omega = TT,\ TH,\ HT,\ HH$

$\mathcal{A} = \{TT\},\ \{TH, HT\},\ \ldots\ldots$

$P(\{TT\}) = \frac{1}{4},\quad P(\{TH, HT\}) = \frac{2}{4} = \frac{1}{2}$

Independence of events

$A, B \subseteq \Omega$ are called independent (of each other)

if $$\boxed{P(A \cap B) = P(A)P(B)}$$

eg Throw a coin twice,

$$\underbrace{P(1^{st} = T \text{ and } 2^{nd} = H)}_{\frac{1}{4}} = \underbrace{P(1^{st} = T)}_{\frac{1}{2}}\underbrace{P(2^{nd} = H)}_{\frac{1}{2}}$$

Conditional Probability

$$\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)}}$$

Conditional Probability

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Probability of A given B

(extra piece of information)

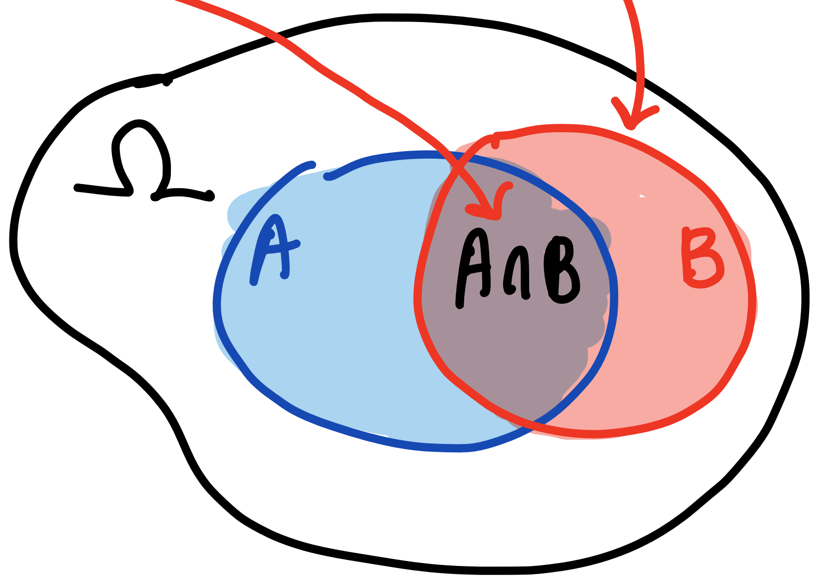
Conditional Probability

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

relative area of A inside B

Probability of A given B

(extra piece of information)



Conditional Probability

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\boxed{P(A \cap B) = P(B)\,P(A/B)}$$

Conditional Probability

Tossing a coin twice

$$P(\text{one } H) = P(TH, HT) = \frac{2}{4} = \frac{1}{2}$$

$$P(\text{one } H \,/\, \text{at least one } H)$$

Conditional Probability

Tossing a coin twice

$$P(\text{one } H) = P(TH, HT) = \frac{2}{4} = \frac{1}{2}$$

$$P(\underbrace{\text{one } H}_{TH,\ HT} \,/\, \text{at least } \underbrace{\text{one } H}_{TH,\ HT,\ HH}) = \frac{2}{3}$$

Conditional Probability

Tossing a coin twice

$$P(\text{one } H) = P(TH, HT) = \frac{2}{4} = \frac{1}{2}$$

$$P(\text{one } H \mid \text{at least one } H)$$

$$= \frac{P(\text{one } H \cap \text{at least one } H)}{P(\text{at least one } H)}$$

$\frac{2}{4}$

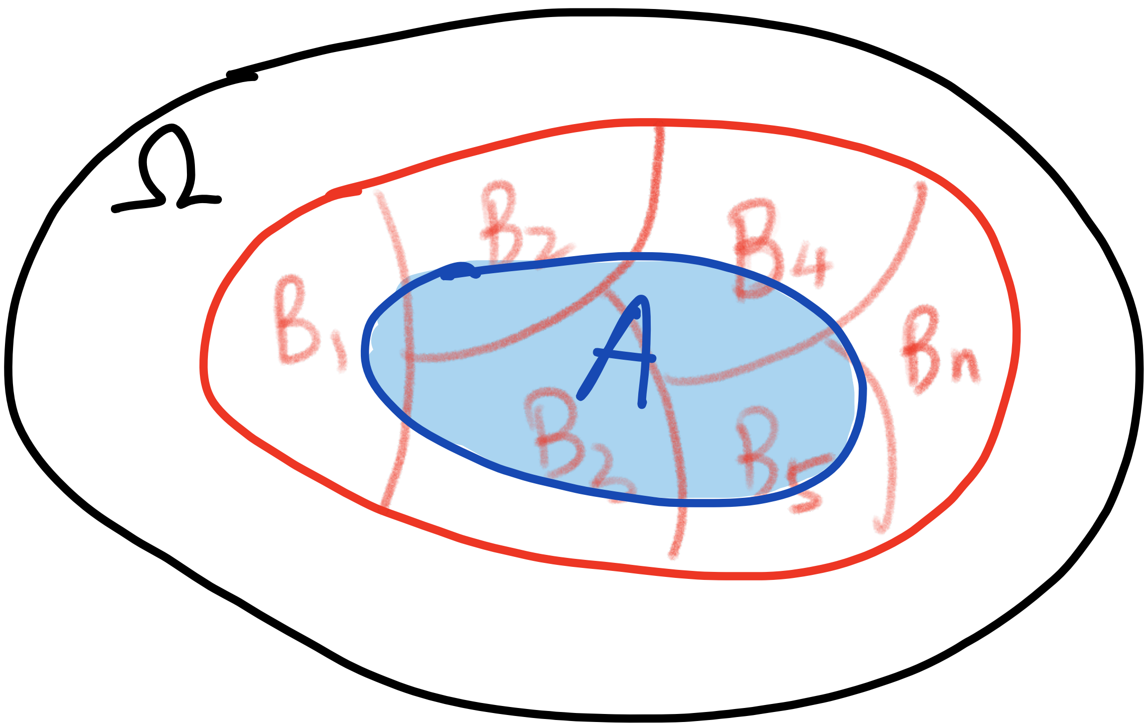
$\frac{3}{4}$

$$= \frac{2}{3}$$

Bayes Formula (1) (Forward)

If $A \subseteq B_1 \cup B_2 \cup \cdots B_n$,

(B_i – disjoint)



Bayes Formula (1) (Forward)

If $A \subseteq B_1 \cup B_2 \cup \cdots B_n$,

(B_i – disjoint)

$$\boxed{\begin{aligned} P(A) &= P(A/B_1)P(B_1) + P(A/B_2)P(B_2) \\ &\quad + \cdots + P(A/B_n)P(B_n) \end{aligned}}$$

Bayes Formula (2) (Backward)

$$\boxed{P(B/A)} = \frac{P(B\cap A)}{P(A)}$$

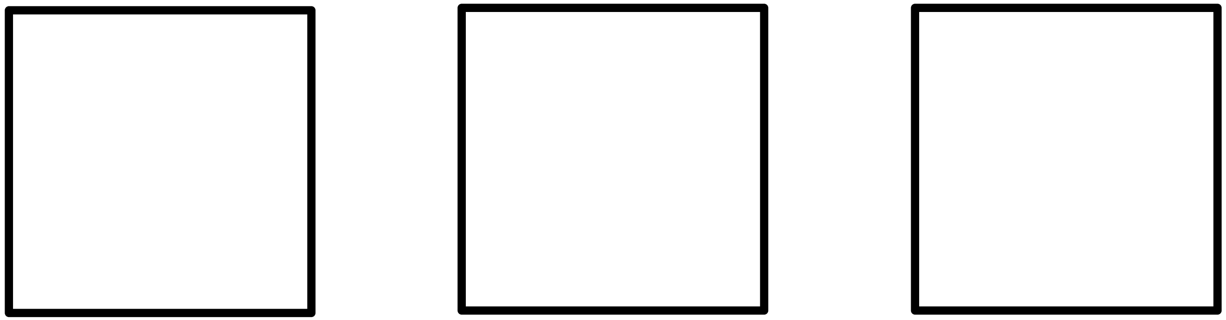
$$= \frac{P(A/B)\,P(B)}{P(A)}$$

$$= \boxed{\frac{P(A/B)\,P(B)}{P(A/B)\,P(B) + P(A/B^c)\,P(B^c)}}$$

$(B^c = \Omega \setminus B)$

Monty Hall (Car and Goat)

Three closed doors



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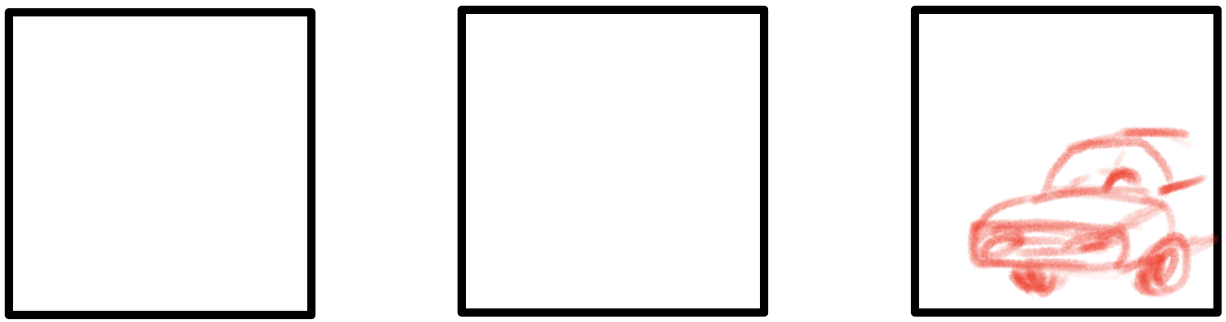

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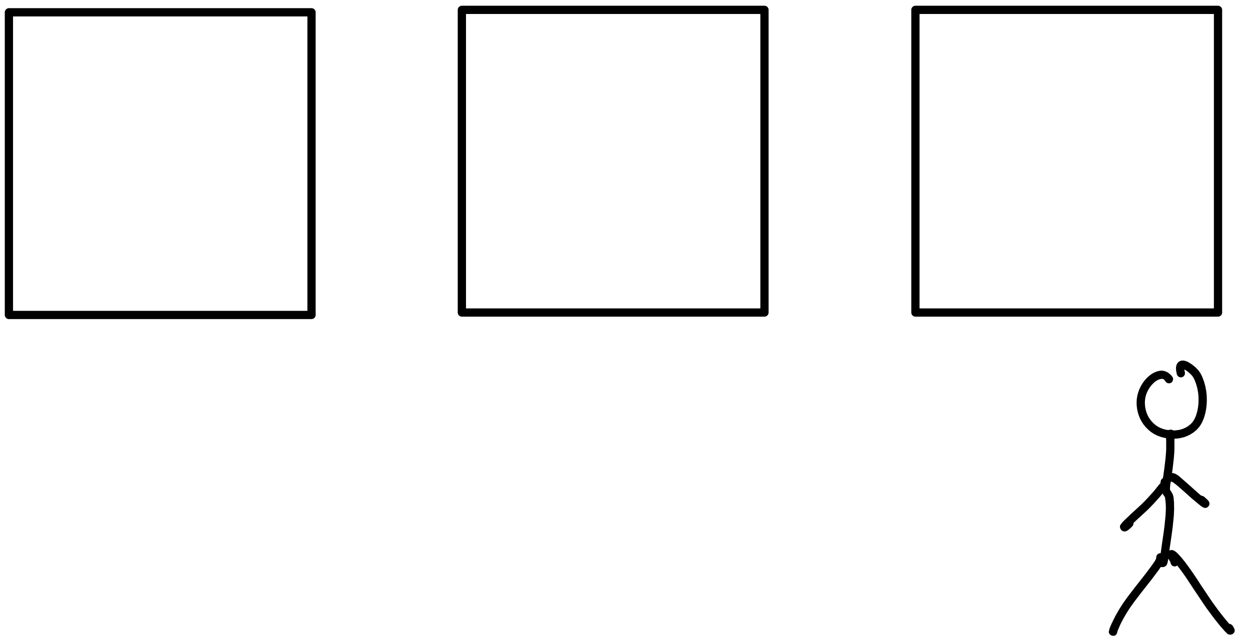
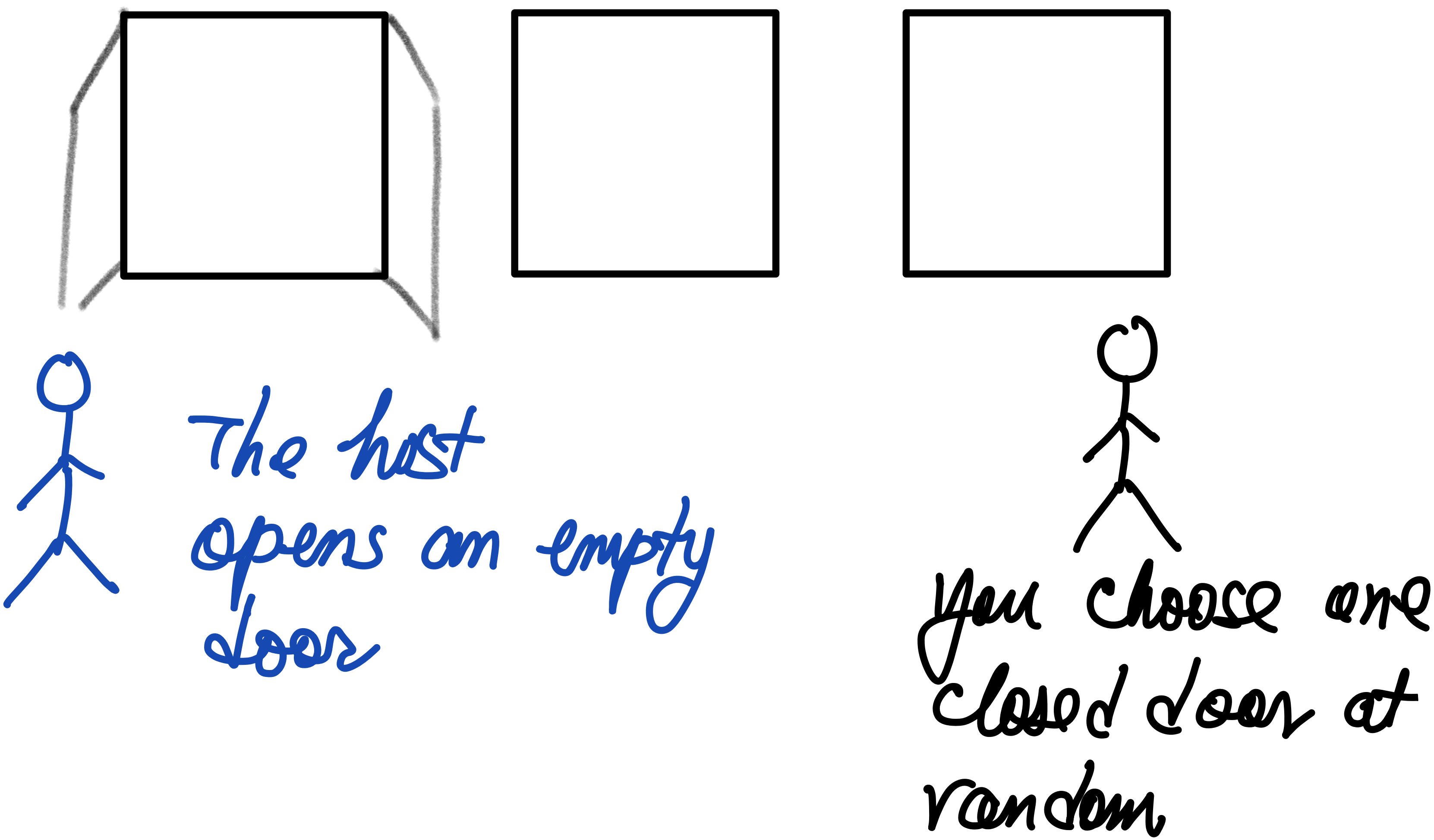


You choose one closed door at random

Monty Hall (Car and Goat)

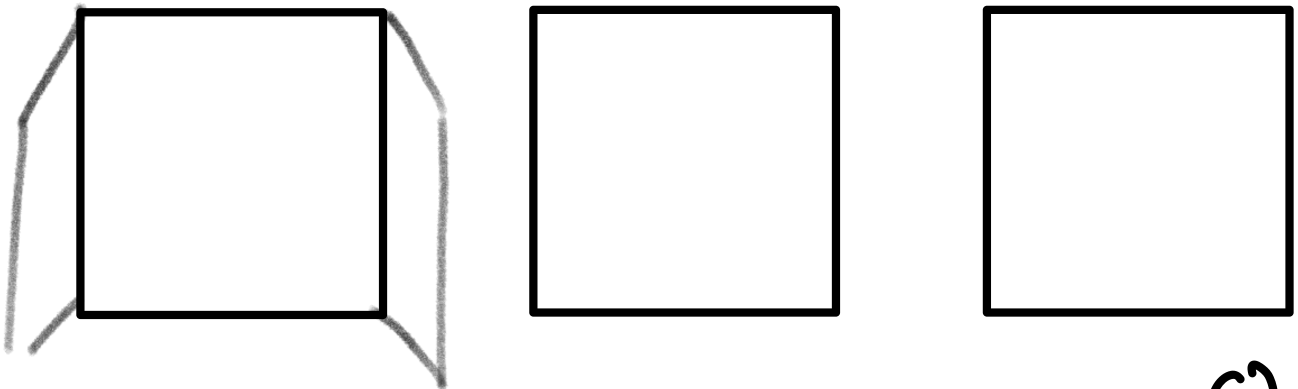
Three closed doors

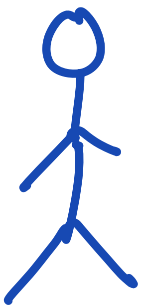


Monty Hall (Car and Goat)

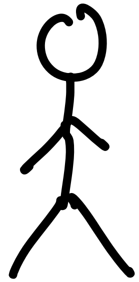
Three closed doors

Do you want to switch?



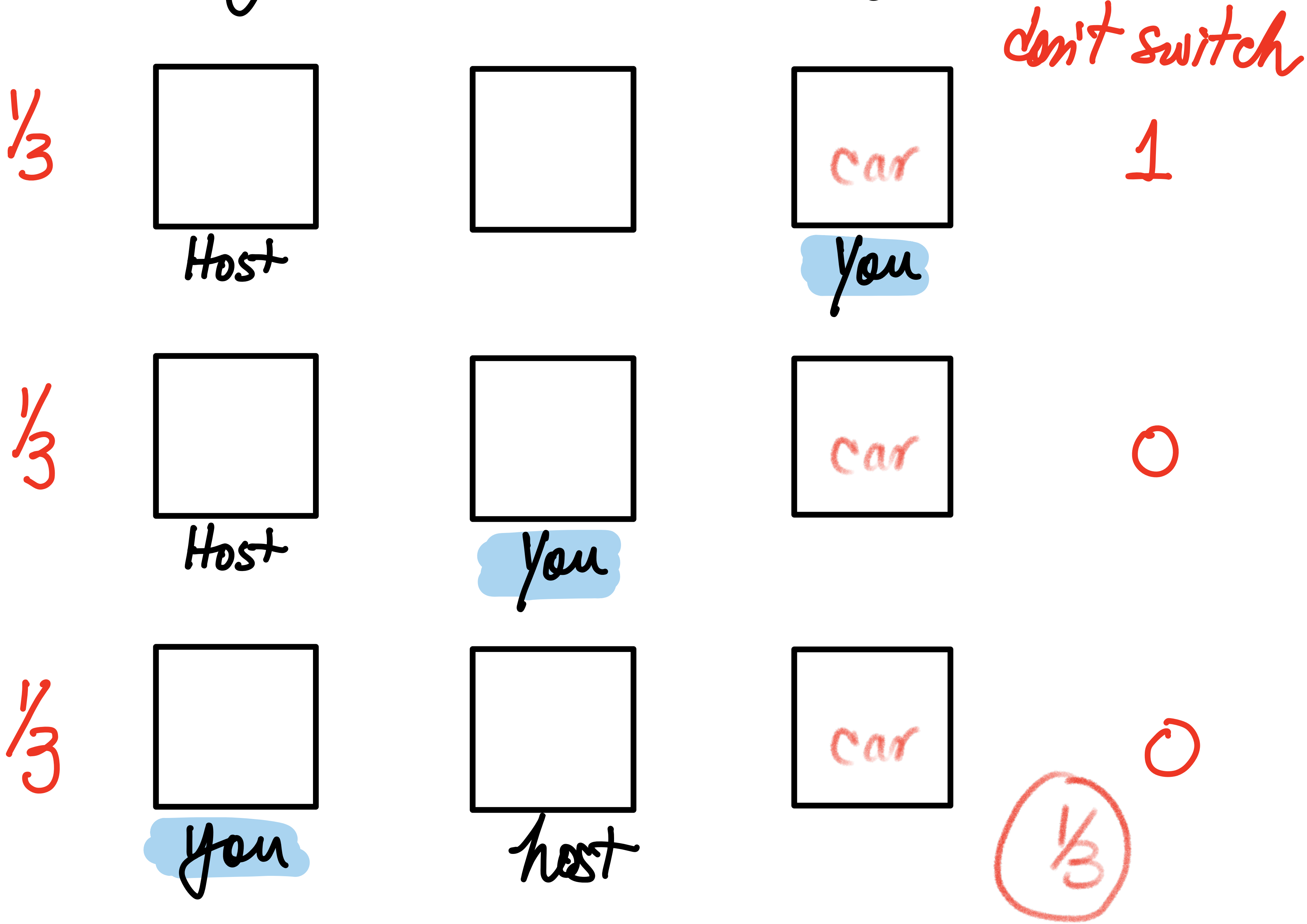


The host opens an empty door

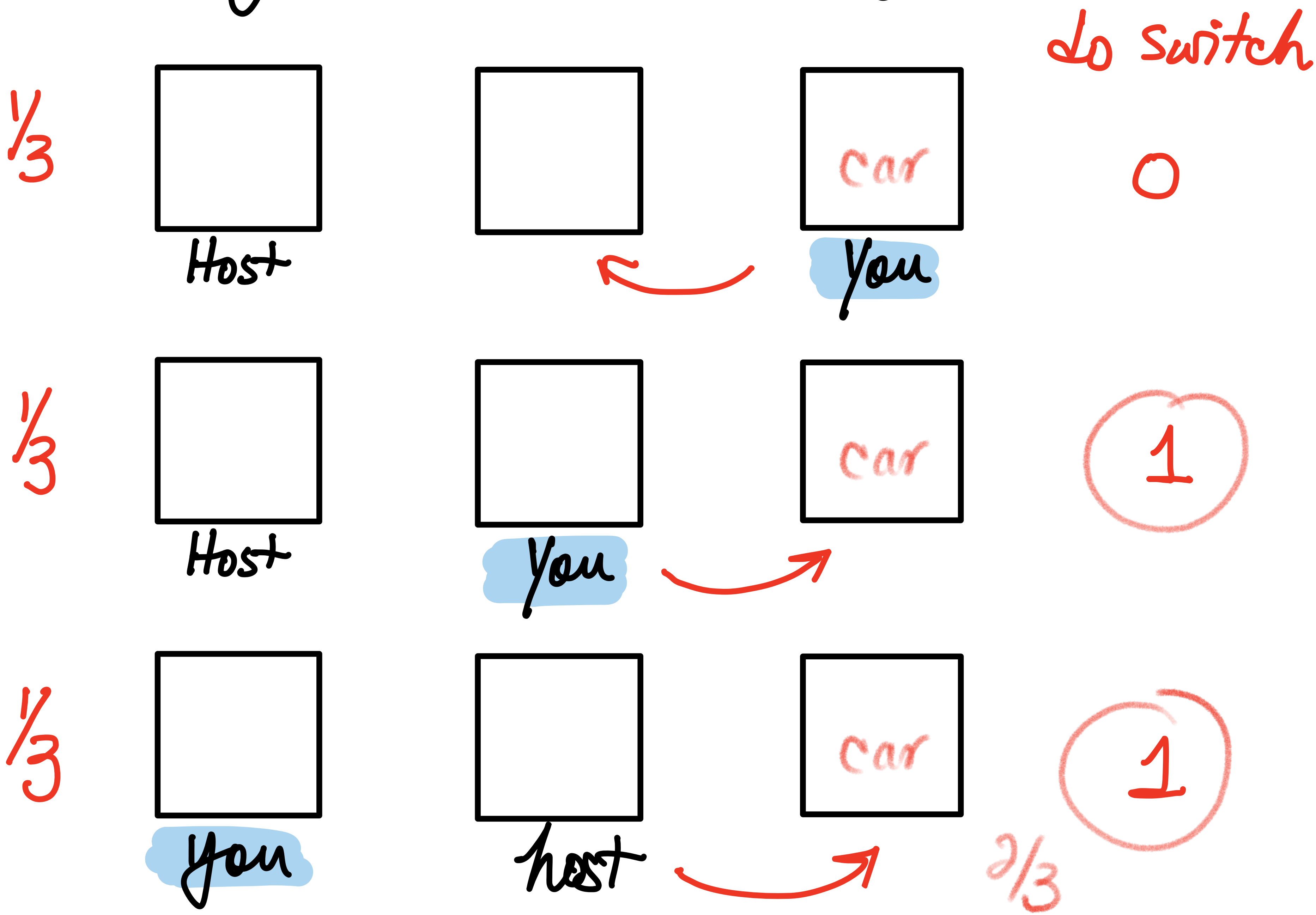


You choose one closed door at random

Monty Hall (Car and Goat)



Monty Hall (Car and Goat)



King and His Sibling

Suppose the King has one other sibling.

$$\boxed{P(\text{The sibling is a sister}) = ?}$$

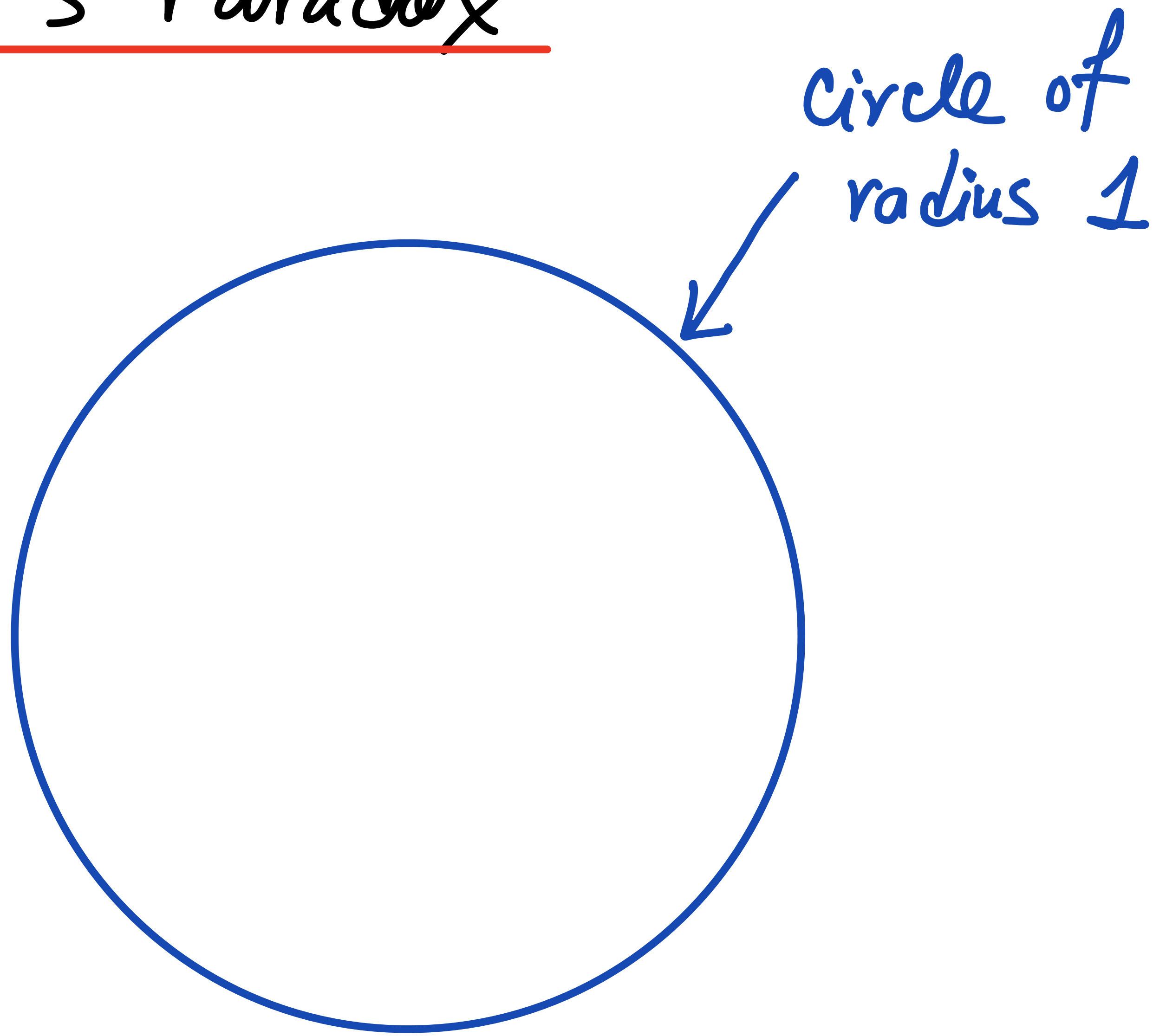
Boy vs Girl

Suppose Mr. Smith is seen with a boy (presumably his son) in a supermarket.

Suppose you know Mr. Smith has one other child.

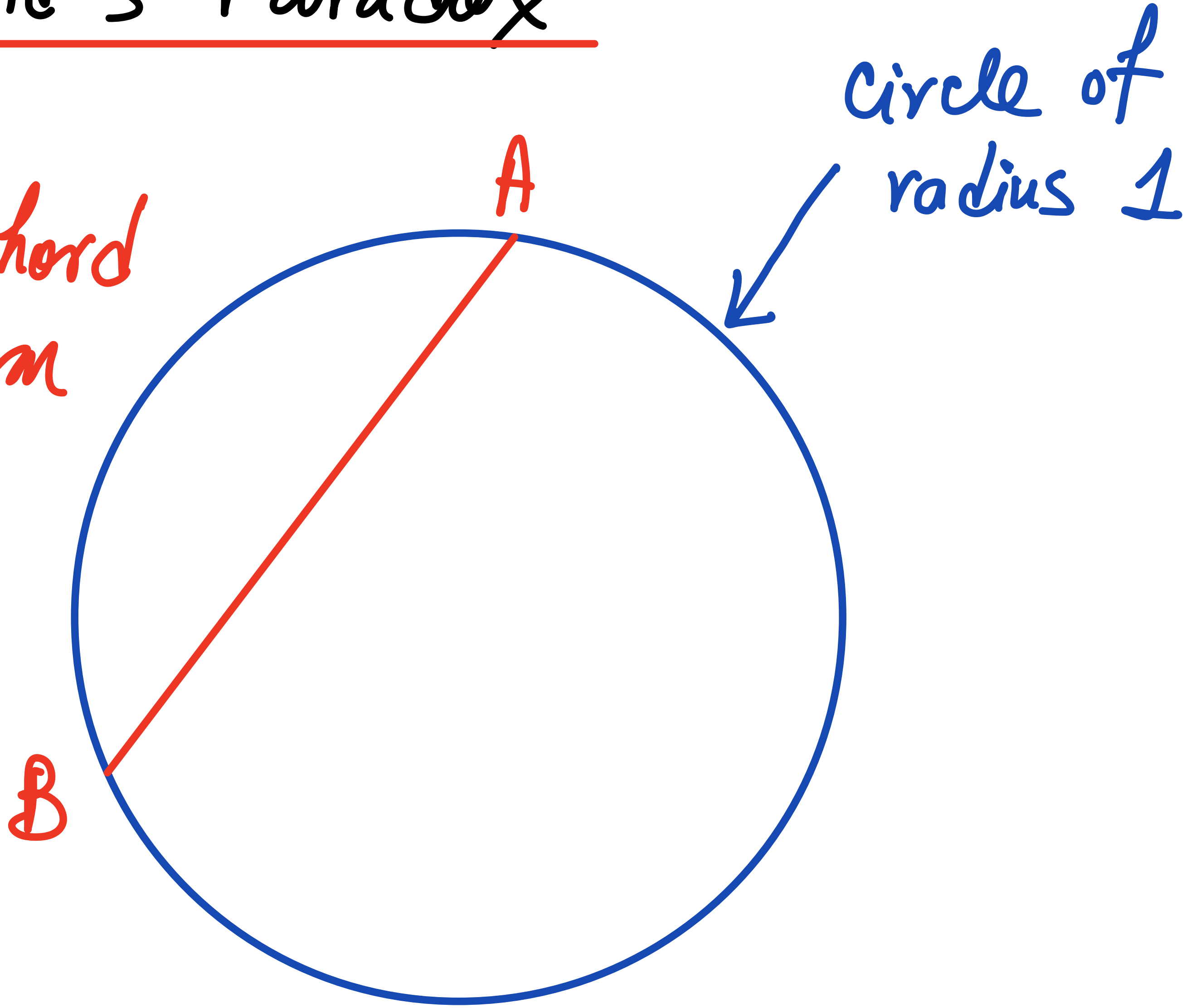
$$\boxed{P(\text{The other child is a daughter}) = ?}$$

Bertrand's Paradox



Bertrand's Paradox

choose a chord at random



Bertrand's Paradox

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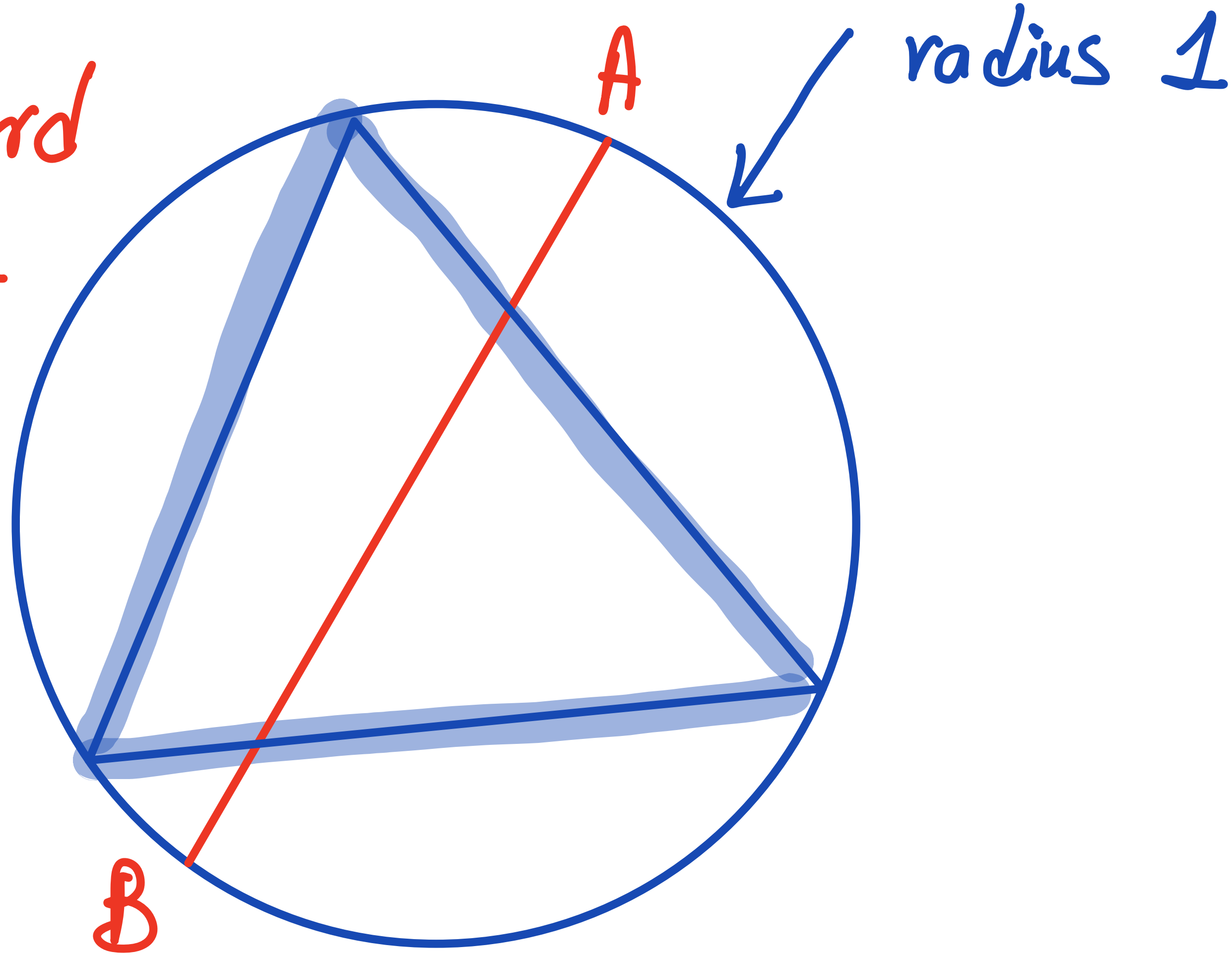
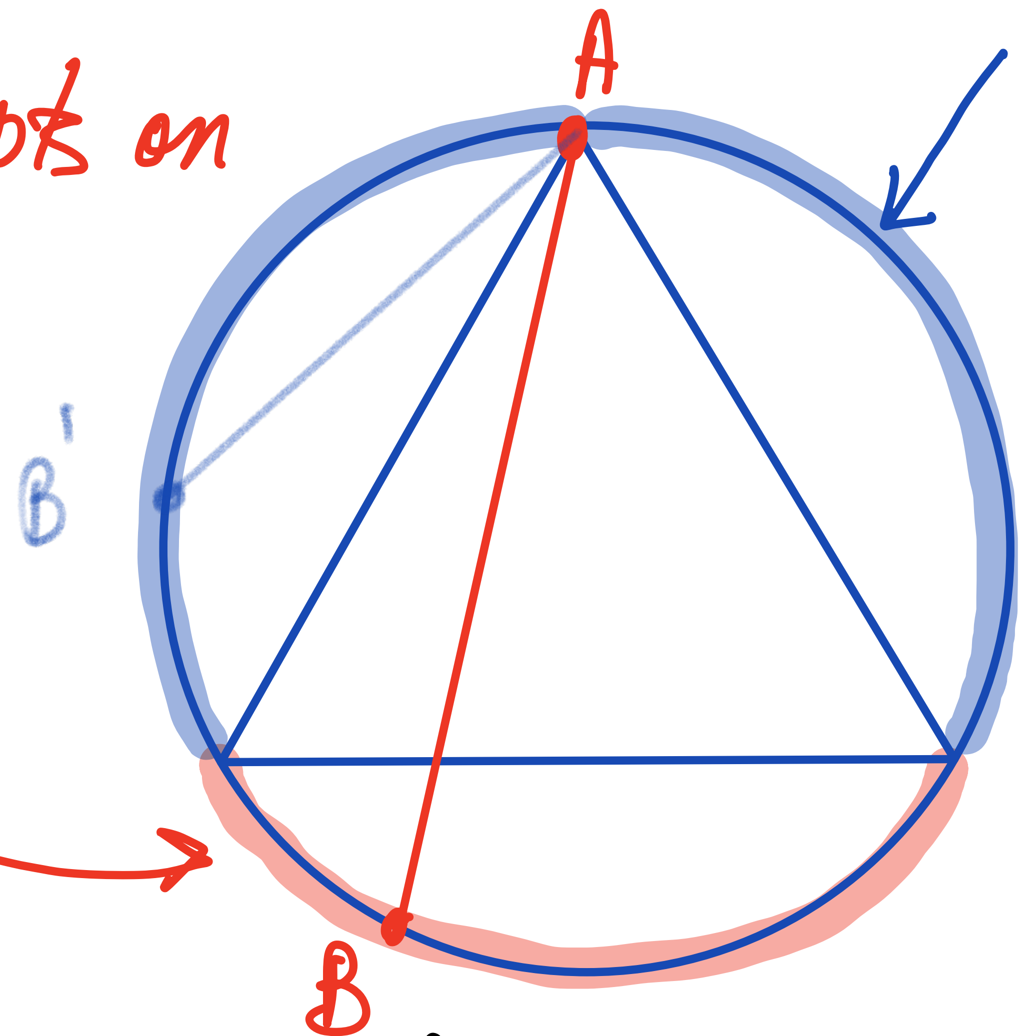


$P(|AB| > \text{side of an equilateral } \triangle) = ?$

Bertrand's Paradox

Choose 2 pts on circle at random

circle of radius 1



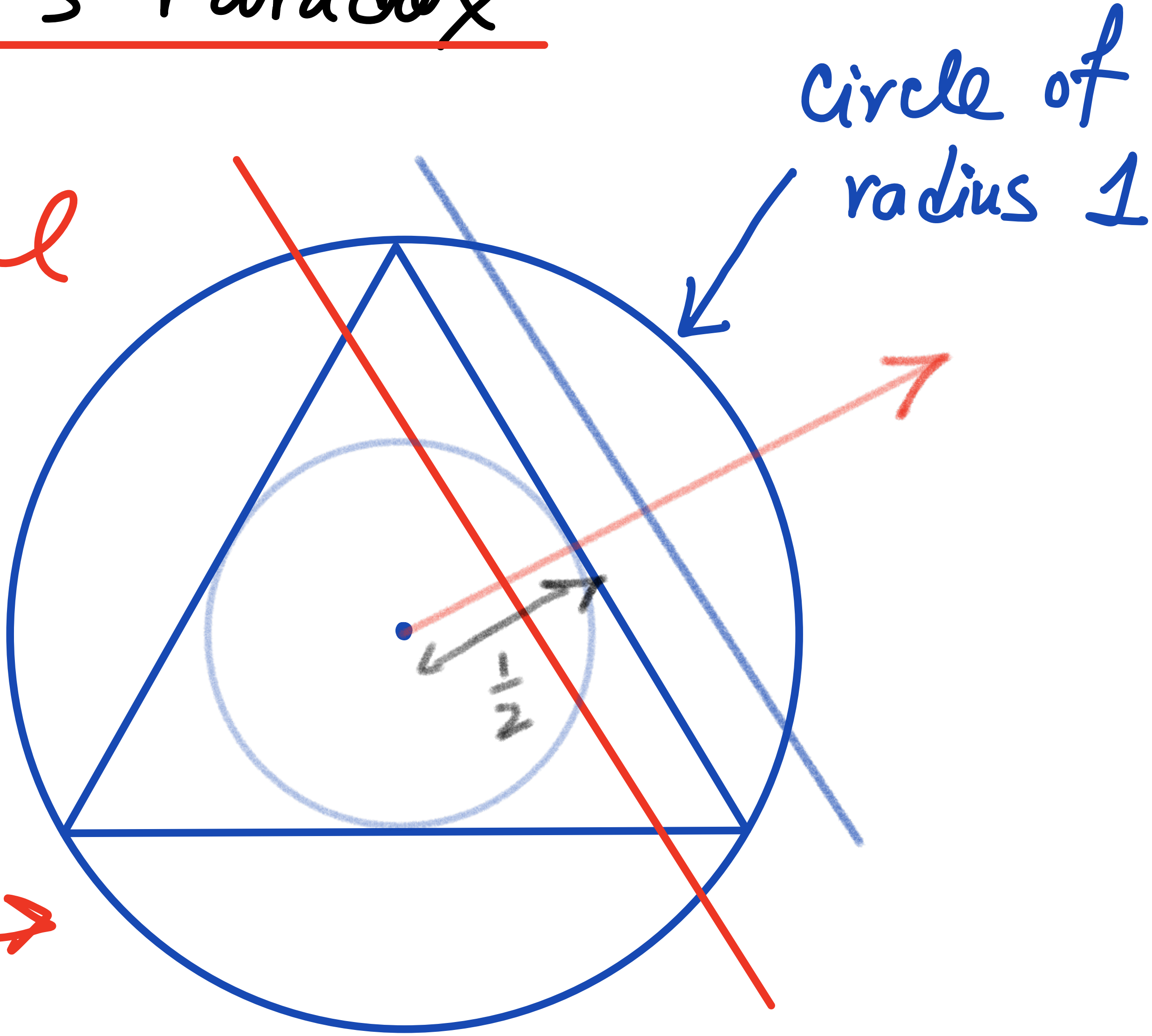
$\frac{1}{3}$

$P(|AB| > \text{side of an equilateral } \triangle) = ?$

Bertrand's Paradox

Choose a radial direction at random

$\frac{1}{2}$



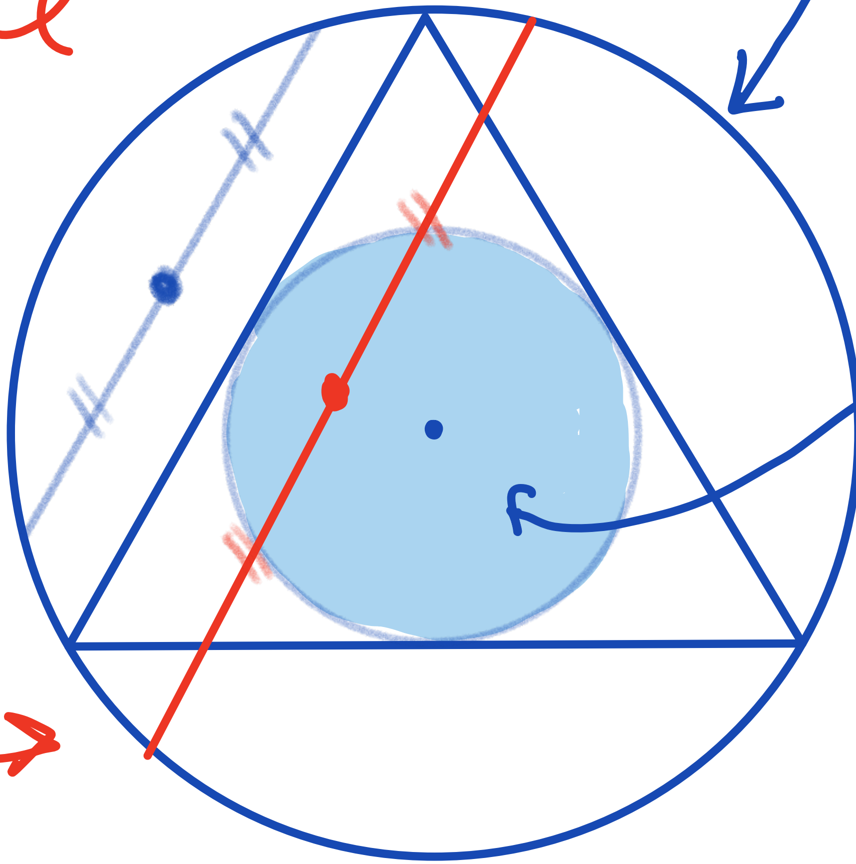
$P(|AB| >$ side of an equilateral $\triangle) = ?$

Bertrand's Paradox

Choose a radial direction at random

$\frac{1}{4}$

circle of radius 1



Area $= \frac{1}{4}$ circle

$P(|AB| > \text{side of an equilateral } \triangle) = ?$

Statistics in the Court Room

(Anderson–Seppäläinen–Valkó)

Reasoning with uncertainty in the real world ♦

Humans are not naturally good at reasoning with uncertainty. This is one lesson from the Nobel Prize winning investigations into human decision-making by Daniel Kahneman and Amos Tversky [Kah11]. Below we describe two cases of faulty reasoning with serious real-world consequences.

Example 2.45. The Sally Clark case is a famous wrongful conviction in England. Sally Clark's two boys died as infants without obvious causes. In 1998 she was charged with murder. At the trial an expert witness made the following calculation. Population statistics indicated that there is about a 1 in 8500 chance of an unexplained infant death in a family like the Clarks. Hence the chance of *two* deaths is $(1/8500)^2$, roughly 1 in 72 million. This number was presented as an indication of how unlikely Clark's innocence was. The jury convicted her.

Statistics in the Court Room

(Anderson–Seppäläinen–Valkó)

Much went wrong in the process and the reader is referred to [SC13] for the history. Two errors relevant to our topic happened: (i) the probabilities of the two deaths were multiplied as if these were independent random events, and (ii) the resulting number was misinterpreted as the probability of Clark's innocence. Let us consider these two points in turn.

(i) The assumption of independence that led to the 1 in 72 million probability can readily fail due to unknown genetic or environmental factors. Here is a simple illustration of how that can happen.

Suppose a disease appears in 0.1% of the population. Suppose further that this disease comes from a genetic mutation passed from father to son with probability 0.5 and that a carrier of the mutation develops the disease with probability 0.5. What is the probability that both sons of a particular family have the disease? If the disease strikes completely at random, the answer is 0.001^2, or 1 in a million. However, the illness of the first son implies that the father carries the mutation. Hence the conditional probability that the second son also falls ill is $0.5 \cdot 0.5 = 0.25$. Thus the correct answer is $0.001 \cdot 0.25$, which is 1 in 4000, a much larger probability than 1 in a million.

Statistics in the Court Room

(Anderson-Seppäläinen-Valkó)

Much went wrong in the process and the reader is referred to [SC13] for the history. Two errors relevant to our topic happened: (i) the probabilities of the two deaths were multiplied as if these were independent random events, and (ii) the resulting number was misinterpreted as the probability of Clark's innocence. Let us consider these two points in turn.

(ii) The second error is the interpretation of the 1 in 72 million figure. Even if it were the correct number, it is not the probability of Sally Clark's innocence. This mistake is known as *prosecutor's fallacy*. The mathematically correct reasoning proceeds with Bayes' rule: by comparing the relative likelihood of two competing explanations for the deaths, namely Sally Clark's guilt and a medical explanation. This involves the prior probability of Clark's guilt, which in turn depends crucially on the strength of the other evidence against her.

To put a different view on this point, consider a lottery. The probability of getting 6 numbers out of 40 exactly right is $\binom{40}{6}^{-1}$, about 1 in 3.8 million, extremely unlikely. Yet there are plenty of lottery winners, and we do not automatically suspect cheating just because of the low odds. In a large enough population even a low probability event is likely to happen to *somebody*. ▲

Wikipedia
The Free Encyclopedia

Sally Clark

Sally Clark	
Born	Sally Lockyer August 1964[1] Devizes, England[1]
Died	15 March 2007 (aged 42) Hatfield Peverel, England[2]
Nationality	English
Citizenship	United Kingdom
Occupation	Solicitor
Known for	Wrongly convicted of killing her sons

Sally Clark (August 1964 – 15 March 2007)[1] was an English solicitor who, in November 1999, became the victim of a miscarriage of justice when she was found guilty of the murder of her two infant sons. Clark's first son died in December 1996 within a few weeks of his birth, and her second son died in similar circumstances in January 1998. A month later, Clark was arrested and tried for both deaths. The defense argued that the children had died of sudden infant death syndrome (SIDS). The prosecution case relied on flawed statistical evidence presented by paediatrician Professor Sir Roy Meadow, who testified that the chance of two children from an affluent family suffering SIDS was 1 in 73 million. He had arrived at this figure by squaring his estimate of a chance of 1 in 8500 of an individual SIDS death in similar circumstances. The Royal Statistical Society later issued a statement arguing that there was no statistical basis for Meadow's claim, and expressed concern at the "misuse of statistics in the courts".[3]

Clark was convicted in November 1999. The convictions were upheld on appeal in October 2000, but overturned in a second appeal in January 2003, after it emerged that Alan Williams, the prosecution forensic pathologist who examined both babies, had failed to disclose microbiological reports that suggested the second of her sons had died of natural causes.[4] Clark was released from prison having served more than three years of her sentence. Journalist Geoffrey Wansell called Clark's experience "one of the great miscarriages of justice in modern British legal history".[5] As a result of her case, the Attorney General Lord Goldsmith ordered a review of hundreds of other cases, and two other women had their convictions overturned. Clark's experience caused her to develop severe psychiatric problems and she died in her home in March 2007 from alcohol poisoning.[2]

Application and Interpretation of Bayes Formula

$P(\text{evidence} / \text{guilty})$

vs

$P(\text{guilty} / \text{evidence})$

Application and Interpretation of Bayes Formula

$$P(\text{evidence} / \text{guilty}) = 1$$

vs

$$P(\text{guilty} / \text{evidence})$$

Application and Interpretation of Bayes Formula

$$P(\text{evidence} / \text{guilty}) = 1$$

vs

$$P(\text{guilty} / \text{evidence}) = P(G/E)$$

Application and Interpretation of Bayes Formula

$$P(\text{guilty} / \text{evidence})$$

$$= P(G/E) = \frac{P(G \cap E)}{P(E)}$$

$$= \frac{P(E/G)\,P(G)}{P(E/G)\,P(G) + P(E/G^c)\,P(G^c)}$$

Application and Interpretation of Bayes Formula

$$P(\text{guilty} \mid \text{evidence})$$

$$= P(G \mid E) = \frac{P(G \cap E)}{P(E)}$$

$$= \frac{\overset{1}{\cancel{P(E \mid G)}}\, P(G)}{\underset{1}{\cancel{P(E \mid G)}}\, P(G) + \underset{\approx 0,\ > 0}{\cancel{P(E \mid G^c)}}\, P(G^c)}$$

Application and Interpretation of Bayes Formula

$$P(\text{guilty} \,/\, \text{evidence})$$

$$= P(G/E) = \frac{P(G \cap E)}{P(E)}$$

$$= \frac{P(G)}{P(G) + P(E/G^c)\,P(G^c)}$$

Application and Interpretation of Bayes Formula

$$P(\text{guilty} \mid \text{evidence})$$

$$= P(G \mid E) = \frac{P(G \cap E)}{P(E)}$$

$$= \frac{P(G)}{P(G) + \underbrace{P(E \mid G^c)}_{>0,\ \approx 0} P(G^c)}$$

Application and Interpretation of Bayes Formula

$P(\text{guilty} / \text{evidence})$

$$= \frac{P(G)}{P(G) + \underbrace{P(E/G^c)}_{\text{eg. } 10^{-6} \approx 0, \text{ but } > 0} P(G^c)}$$

prior distribution (assumption)

if $P(G) = 0.5$, then $P(G/E) \simeq 1$

Application and Interpretation of Bayes Formula

$$P(\text{guilty} / \text{evidence}) = \frac{P(G)}{P(G) + \underbrace{P(E/G^c)}_{\text{eg. } 10^{-6} \approx 0, \text{ but } > 0} P(G^c)}$$

prior distribution (assumption)

if $P(G) = 10^{-9}$, then $P(G/E) \simeq 10^{-3}$