Nicolas Bacaër

A Short History of Mathematical Population Dynamics

From the beginning.....

Fibonacci sequence:

n	1	2	3	4	5	6	7	8	...
P_n	1	1	2	3	5	8	13	21	...

From the beginning.....

Fibonacci sequence: (1202)

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$$P_{n+2} = P_{n+1} + P_n$$

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Fibonacci sequence: (1202)

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$$P_{n+2} = P_{n+1} + P_n$$

P_n = # of pairs of rabbits

From the beginning.....

Fibonacci sequence:

n	1	2	3	4	5	6	7	8	...
P_n	1	1	2	3	5	8	13	21	...

$$\boxed{P_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n}$$

Chapter 2
Halley's life table (1693)

$$\boxed{P_{n+1} = P_n - D_n}$$

Table 2.1 Halley's life table showing the population P_k aged k.

Age	Number	Age	Number	Age	Number
1	1000	29	539	57	272
2	855	30	531	58	262
3	798	31	523	59	252
4	760	32	515	60	242
5	732	33	507	61	232
6	710	34	499	62	222
7	692	35	490	63	212
8	680	36	481	64	202
9	670	37	472	65	192
10	661	38	463	66	182
11	653	39	454	67	172
12	646	40	445	68	162
13	640	41	436	69	152
14	634	42	427	70	142
15	628	43	417	71	131
16	622	44	407	72	120

Chapter 2
Halley's life table (1693)

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$D_2 = 145$

$D_3 = 57$

⋮

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Chapter 2
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$$\boxed{P_{n+1} = P_n - D_n}$$

age 20.

Halley also used his life table to compute the price of annuities upon lives. During the sixteenth and seventeenth centuries, several cities and states had sold such annuities to their citizens to raise money. The buyers received each year until their death a fixed amount of money, which was equal to a certain percentage of the sum initially paid, often twice the interest rate of the time, but independently of the age of the buyer. Of course the institution was risking bankruptcy if too many people with a very long life expectancy bought these annuities. The problem could not be correctly addressed without a reliable life table.

Chapter 2
Halley's life table (1693)

After this break focusing on demography Halley returned to his main research subjects. Between 1698 and 1700 he sailed around the Atlantic Ocean to draw a map of the Earth's magnetic field. In 1704 he became professor at Oxford University. The following year he published a book on comets and predicted that the comet of 1682, which Kepler had observed in 1607, would come back in 1758: it became known as "Halley's comet". He also published a translation of the book by Apollonius of Perga on conics. In 1720 he replaced Flamsteed as Astronomer Royal. He tried to solve the problem of determining longitude at sea precisely from observation of the Moon, a problem of great practical importance for navigation. He died in Greenwich in 1742 at age 86.

Chapter 2
Halley's life table (1693)

Helena. In 1684 he visited Newton in Cambridge to discuss the link between Kepler's laws of planetary motion and the force of attraction exerted by the Sun. He encouraged Newton to write the famous *Mathematical Principles of Natural Philosophy*, a book which he finally published at his own expense. He was then working as clerk of the Royal Society. In 1689 he designed a bell for underwater diving, which he tested himself.

Chapter 3

Euler and the geometric growth of populations (1748–1761)

$$P_{n+1} = (1+r)P_n$$

$$\frac{P_{n+1} - P_n}{P_n} = r$$

rate of growth per capita

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Euler and the geometric growth of populations (1748–1761)

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rate of growth per capita

$$\boxed{P_n = (1+r)^n P_0}$$

geometric growth

Chapter 3

Euler and the geometric growth of populations (1748–1761)

This is called geometric or exponential growth. The first example asks:

Ex 1 If the population in a certain region increases annually by one thirtieth and at one time there were 100,000 inhabitants, we would like to know the population after 100 years.

Ex2 Since after the Flood all men descended from a population of six, if we suppose that the population after two hundred years was 1,000,000, we would like to find the annual rate of growth.

Ex3 If the human population increases annually by 1/100, we would like to know how long it will take for the population to become ten times as large.

Exponential Growth and Differential Eqns.

(1) $P_{n+1} = (1+r)P_n$

(2) $P_{n+1} - P_n = r P_n$

(3) $\dfrac{P_{n+1} - P_n}{\Delta t} = r P_n$

(4) $\dfrac{dP}{dt} = r P$

Exponential Growth and Differential Eqns.

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(2) $P_{n+1} - P_n = r P_n$

(3) $\dfrac{P_{n+1} - P_n}{\Delta t} = r P_n$

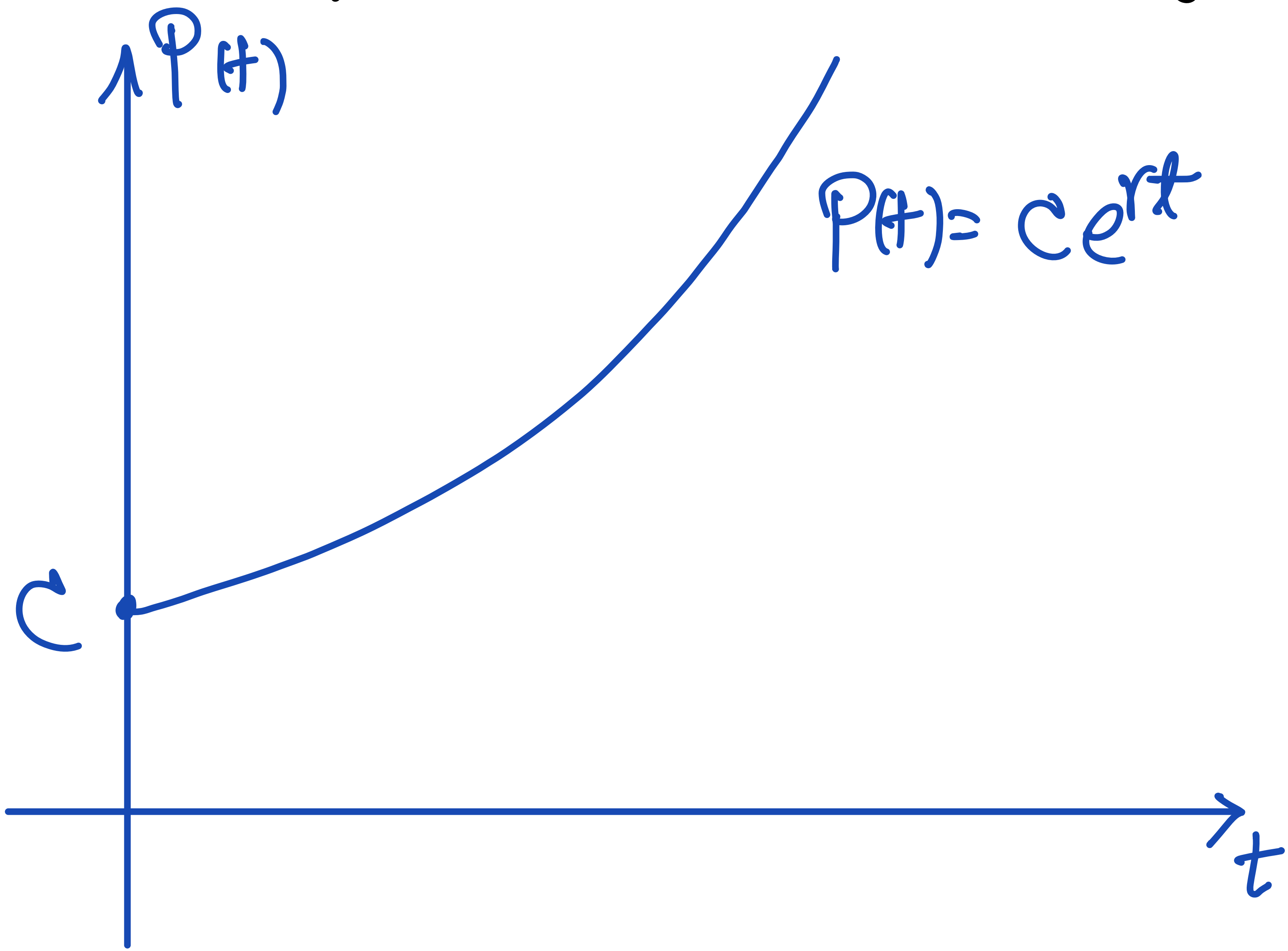
(4) $\dfrac{dP}{dt} = r P$

(5) $P(t) = Ce^{rt}$

$LHS = \dfrac{dP}{dt} = Cre^{rt}$

$RHS = rP = rCe^{rt}$

Exponential Growth and Differential Eqns.



Chapter 5
Malthus and the obstacles to geometric growth (1798)

In 1798 Malthus published anonymously a book entitled *An Essay on the Principle of Population, as It Affects the Future Improvement of Society, With Remarks on the Speculations of Mr Godwin, Mr Condorcet and Other Writers*. It came as

> [. . .] the power of population is indefinitely greater than the power in the earth to produce subsistence for man. Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio. A slight acquaintance with numbers will shew the immensity of the first power in comparison of the second. By that law of our nature which makes food necessary to the life of man, the effects of these two unequal powers must be kept equal. This implies a strong and constantly operating check on population from the difficulty of subsistence. This difficulty must fall somewhere; and must necessarily be severely felt by a large portion of mankind.

Chapter 5
Malthus and the obstacles to geometric growth (1798)

war, epidemics, economic factors. . . . For Malthus, delayed marriage was the best option to stabilize the population. Four other editions of the book followed in 1806,

Certainly Malthus' theses were not completely new. For example, the idea that population tends to grow geometrically is often attributed[1] to him, even though we saw in Chapter 3 that this idea was already familiar to Euler half a century earlier. However, Malthus gave it publicity by linking it in a polemic way to real legislative problems. Ironically it was in communist China that Malthus' suggestion to limit births would find its most striking application (see Chapter 25).

Chapter 6
Verhulst and the logistic equation (1838)

Exponential Growth:

$$\frac{dP}{dt} = rP$$

$$\frac{1}{P}\frac{dP}{dt} = r \quad \text{(growth rate per capita)}$$

Chapter 6
Verhulst and the logistic equation (1838)

Exponential Growth:

$$\frac{dP}{dt} = rP$$

$$\frac{1}{P}\frac{dP}{dt} = r \quad \text{(growth rate per capita)}$$

$$\frac{1}{P}\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)$$

Chapter 6
Verhulst and the logistic equation (1838)

Logistic Growth:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

$$\boxed{P(t) = \frac{P_0 e^{rt}}{1 + P_0\left(e^{rt} - 1\right)/K}}$$

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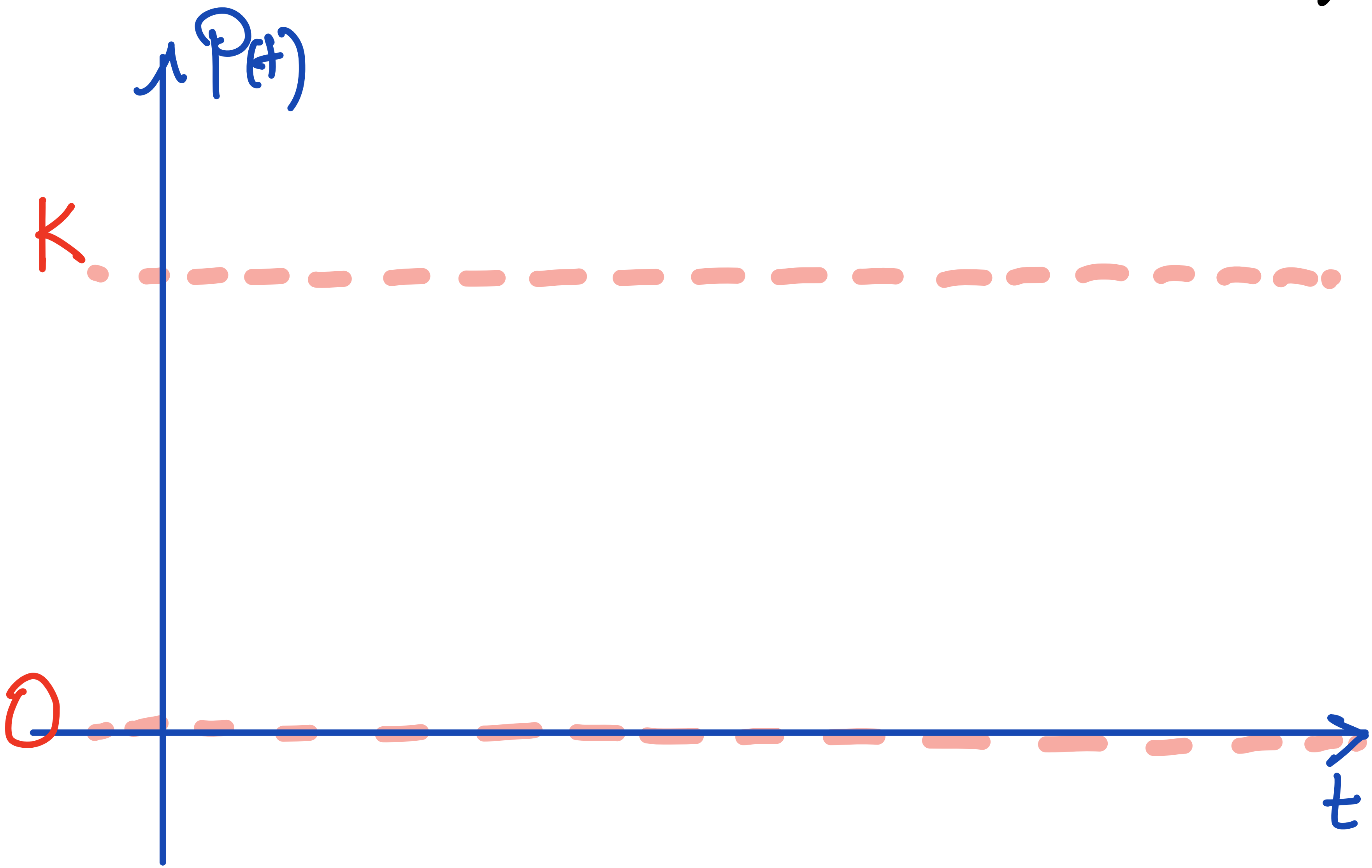
Carrying Capacity

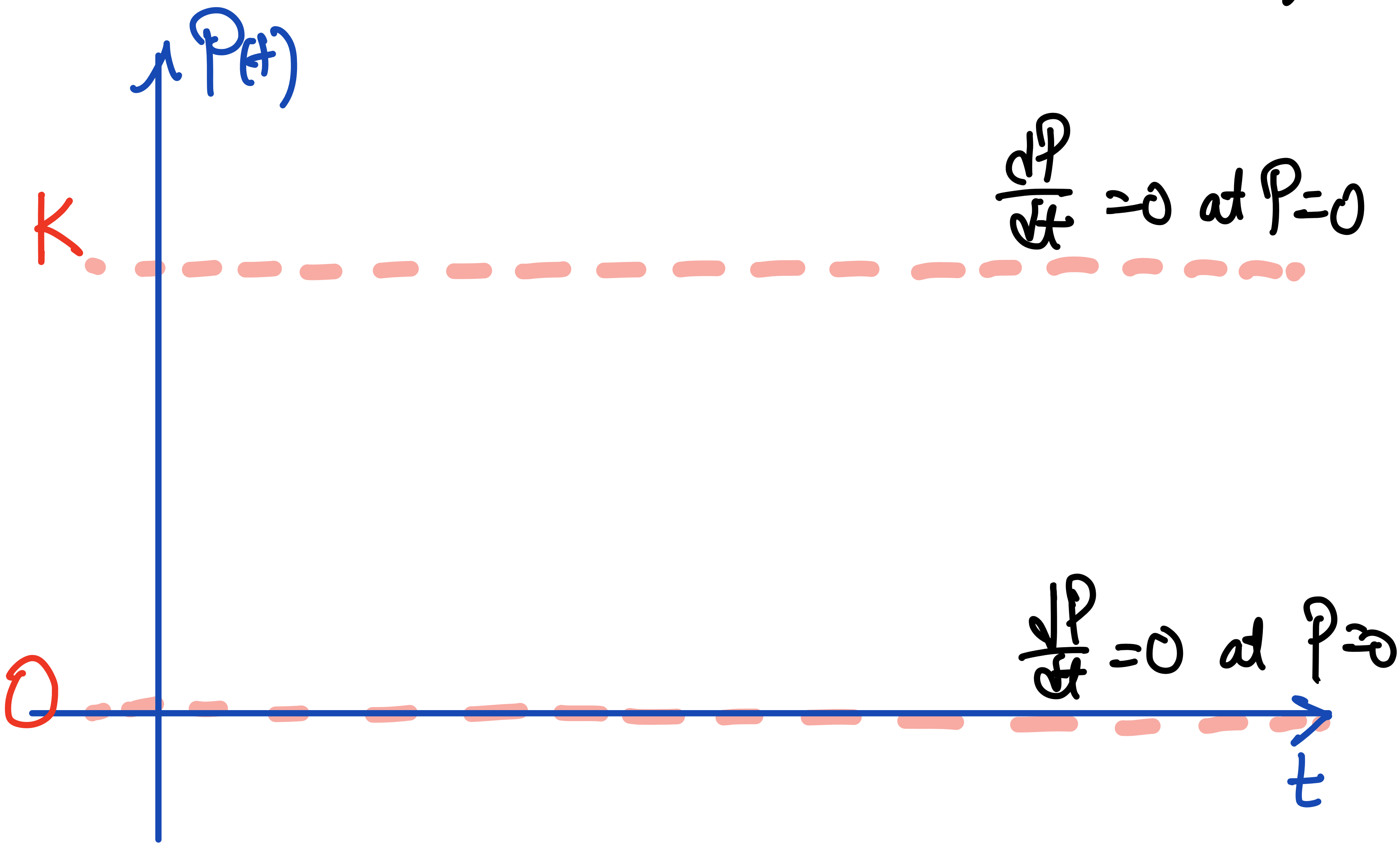
Logistic Growth:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$



Logistic Growth:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

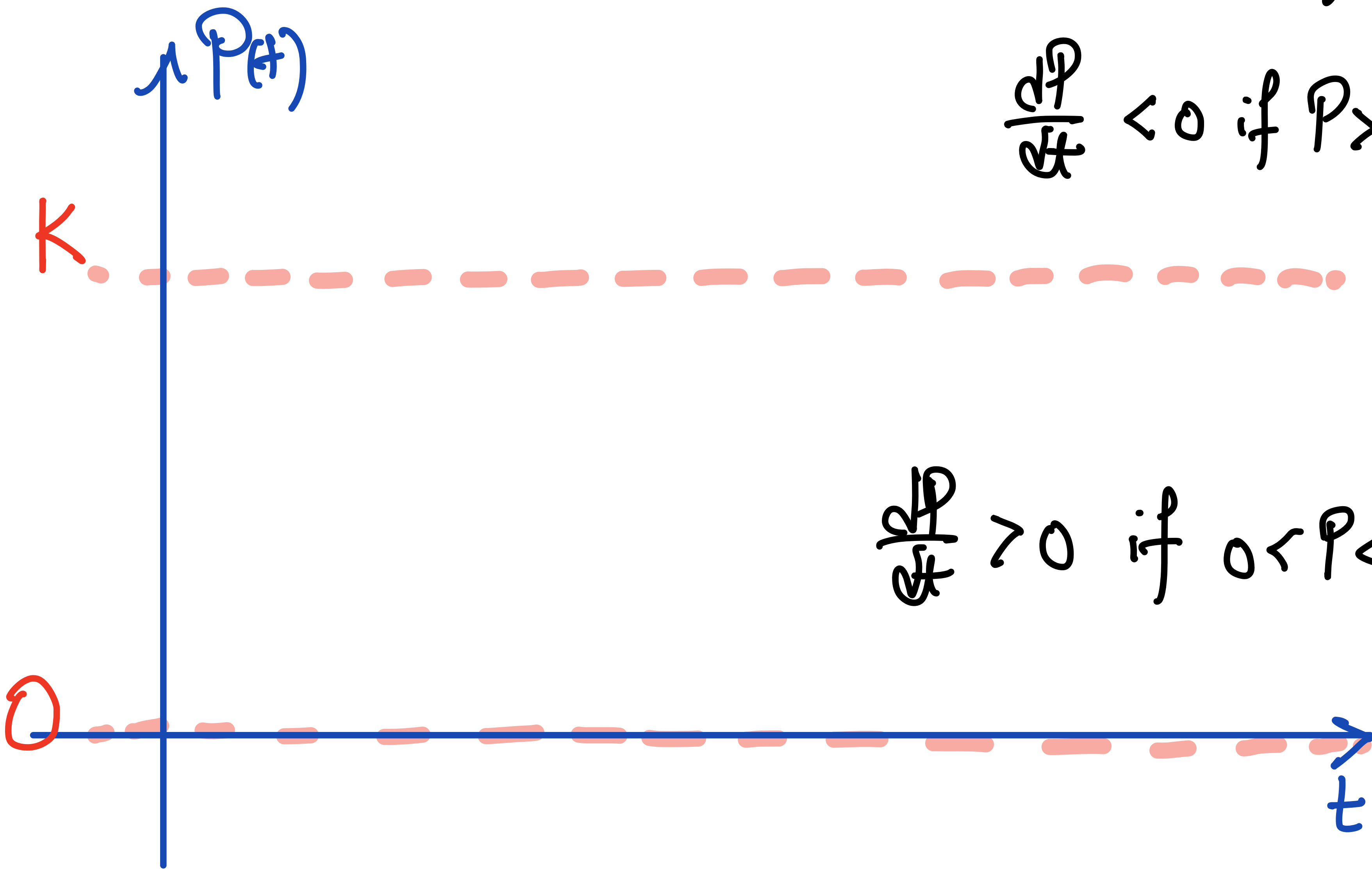


Logistic Growth:

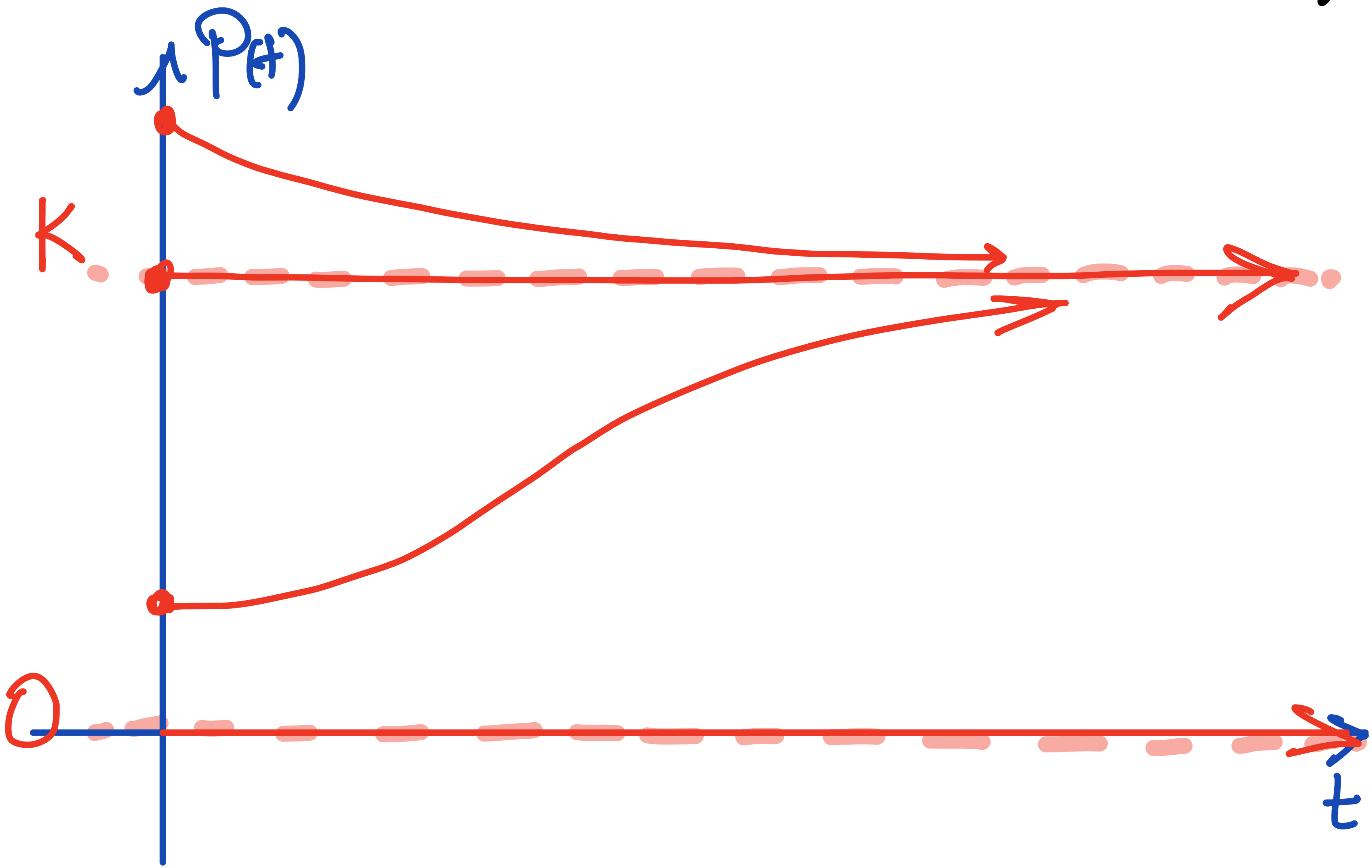
$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

$$\frac{dP}{dt} < 0 \text{ if } P > K$$

$$\frac{dP}{dt} > 0 \text{ if } 0 < P < K$$



Logistic Growth: $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$



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ON THE RATE OF GROWTH OF THE POPULATION OF THE UNITED STATES SINCE 1790 AND ITS MATHEMATICAL REPRESENTATION[1]

By Raymond Pearl and Lowell J. Reed

Department of Biometry and Vital Statistics, Johns Hopkins University

Read before the Academy, April 26, 1920

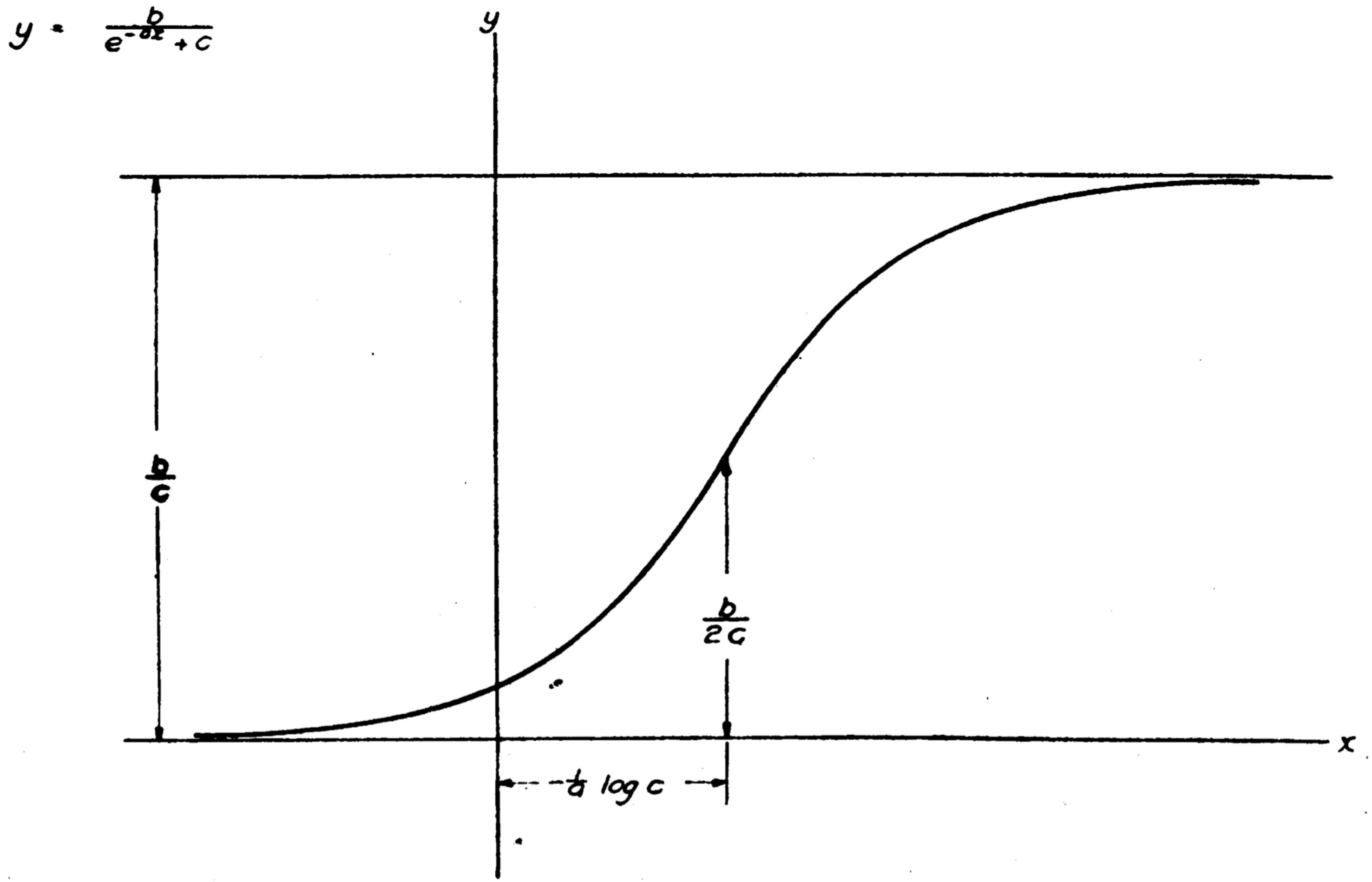


FIG. 2
General form of curve given by equation (ix).

Solution of Logistic Equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

$$\frac{dP}{P\left(1 - \frac{P}{K}\right)} = r\,dt$$

Solution of Logistic Equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

$$\int \frac{dP}{P\left(1 - \frac{P}{K}\right)} = \int r\,dt$$

Solution of Logistic Equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

$$\int \frac{dP}{P\left(1 - \frac{P}{K}\right)} = \int r\,dt$$

$$= rt + C$$

Solution of Logistic Equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

$$\int \frac{dP}{P\left(1 - \frac{P}{K}\right)} = \int r\,dt$$

$$= rt + C$$

$$\frac{1}{P} + \frac{\frac{1}{K}}{1 - \frac{P}{K}}$$

Solution of Logistic Equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

$$\int \frac{dP}{P\left(1 - \frac{P}{K}\right)} = \int r\,dt$$

$$= rt + C$$

$$\int \left(\frac{1}{P} + \frac{\frac{1}{K}}{1 - \frac{P}{K}}\right) dP = \ln P - \ln\left(1 - \frac{P}{K}\right)$$

$$= \ln\left(\frac{P}{1 - P/K}\right)$$

Solution of Logistic Equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

$$\int \frac{dP}{P\left(1 - \frac{P}{K}\right)} = \int r\,dt$$

$$\ln\left(\frac{P}{1 - \frac{P}{K}}\right) = rt + C \Rightarrow \text{solve for } P$$

(2 Species)

Chapter 13
Lotka, Volterra and the predator–prey system (1920–1926)

in chemical kinetics, let $x(t)$ be the total mass of plants and $y(t)$ the total mass of herbivores at time t. Lotka used as a model the following system of differential equations

$$\frac{dx}{dt} = ax - bxy, \tag{13.1}$$

$$\frac{dy}{dt} = -cy + dxy, \tag{13.2}$$

Chapter 13

(2 species)

Lotka, Volterra and the predator–prey system (1920–1926)

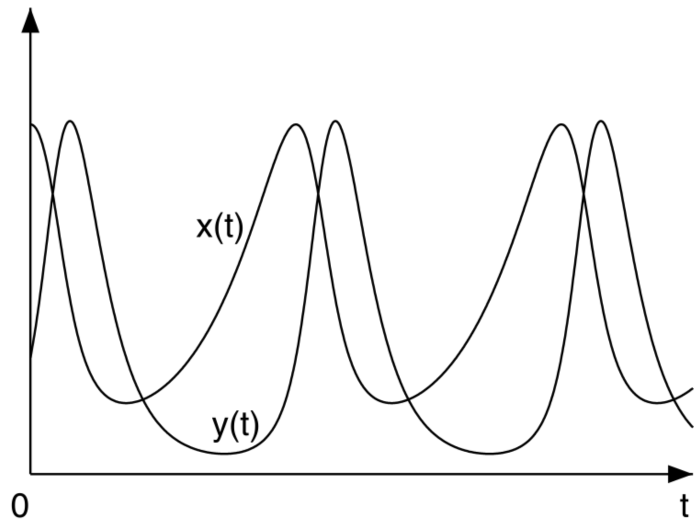


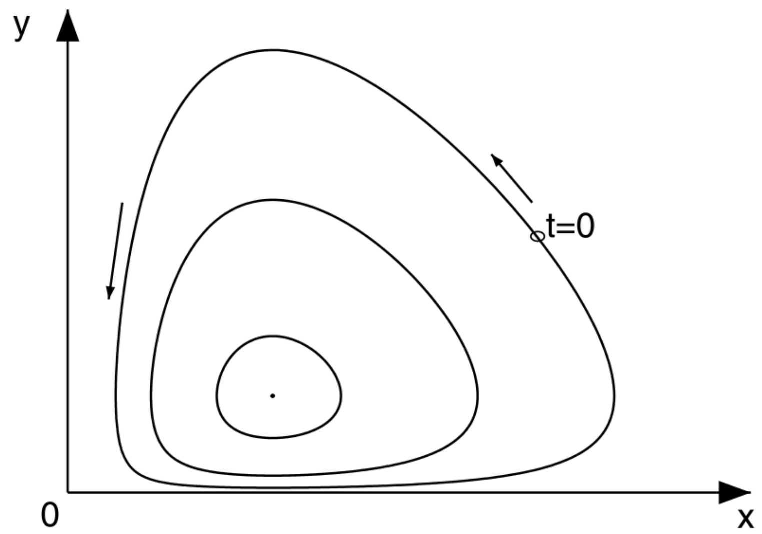
Fig. 13.1 Oscillations of the total mass of plants $x(t)$ and of the total mass of herbivores $y(t)$ as a function of time.

Chapter 13

(2 species)

Lotka, Volterra and the predator–prey system (1920–1926)

Fig. 13.2 Diagram with the total mass of plants $x(t)$ on the horizontal axis and the total mass of herbivores $y(t)$ on the vertical axis. The three closed curves around the steady state correspond to different initial conditions.



Chapter 16
McKendrick and Kermack on epidemic modelling (1926–1927)

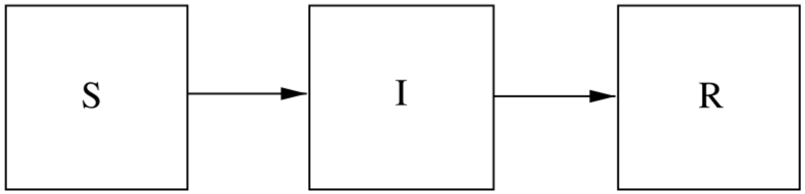


Fig. 16.2 Possible states: susceptible (S), infected (I), recovered (R).

$$\frac{dS}{dt} = -aSI,$$
$$\frac{dI}{dt} = aSI - bI,$$
$$\frac{dR}{dt} = bI.$$

Chapter 12
Ross and malaria (1911)

(Mosquitos and Human)

Solution. John Murray, London (1923). www.archive.org

5. Rowland, J.: *The Mosquito Man, The Story of Sir Ronald Ross*. Roy Publishers, New York (1958)