Dynamics on Inhomogeneous Media

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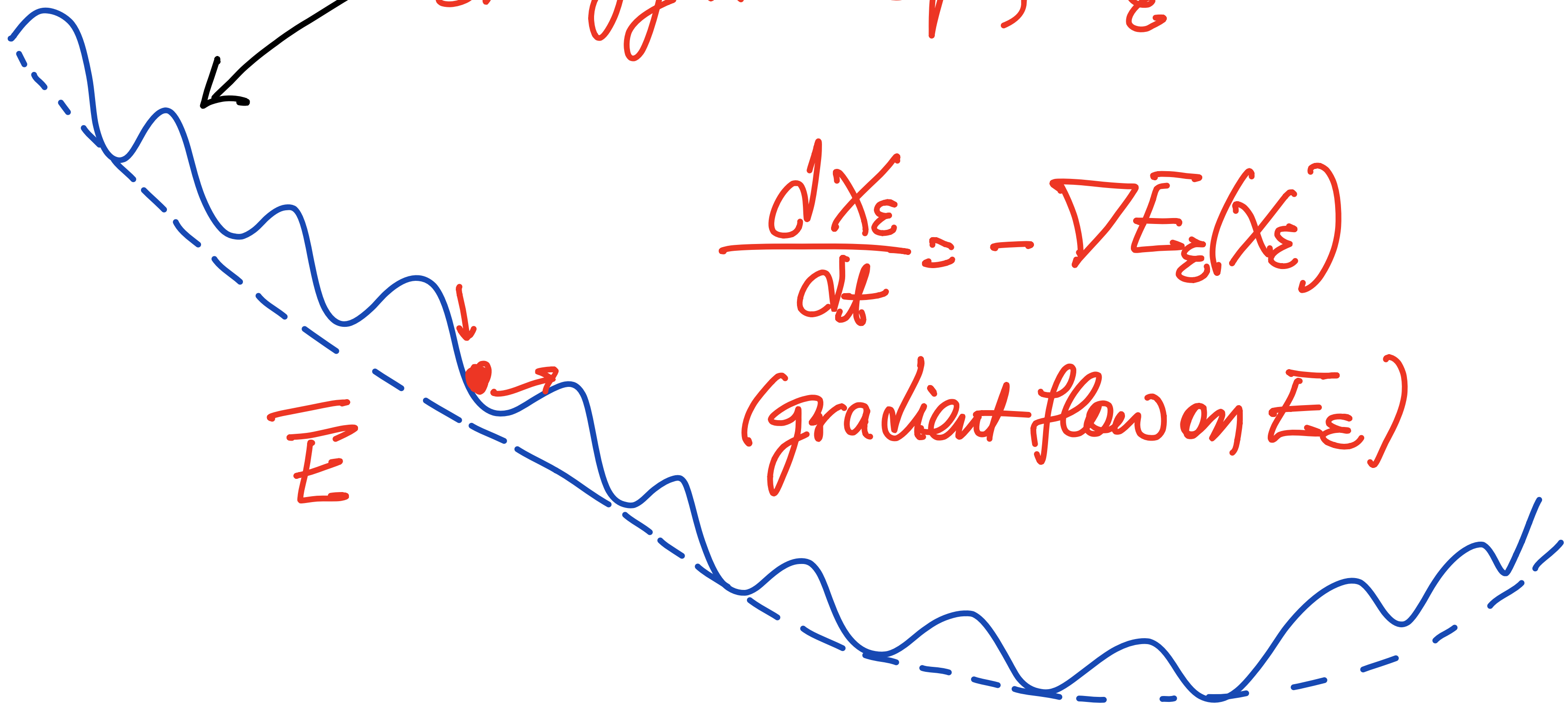
Energy Landscape, E_ε



$$\frac{dX_\varepsilon}{dt} = -\nabla E_\varepsilon(X_\varepsilon)$$

(gradient flow on E_ε)

If E_ε "$\longrightarrow$" $\bar{E}$, does $X_\varepsilon(\cdot) \longrightarrow \bar{X}(\cdot)$ that solves $\dfrac{d\bar{X}}{dt} = -\nabla \bar{E}(\bar{X})$?

Outline

(1) ODE (Dynamics of a particle)

$$\boxed{m\ddot{x} = -E'(x) - \gamma\dot{x} + F + \sqrt{2\gamma\beta^{-1}}\,\dot{W}}$$

(2) PDE (Motion by Mean Curvature of a surface)

$$\boxed{V = \kappa + f(x)}$$

(3) Energetic approach and interpretation of convergence of gradient flow

Dynamics of a Particle on a Tilted Periodic Potential

(Cheng-Y.)

Langevin equation

$$\boxed{m\ddot{x} = -E'(x) - \gamma\dot{x} + F + \sqrt{2\gamma\beta^{-1}}\,\dot{W}}$$

(ma = Force + noise, stochastic Newton's 2nd Law)

Dynamics of a Particle on a Tilted Periodic Potential

($E(x) = \sin x$)

(Cheng-Y.)

$$\boxed{m\ddot{x} = -E'(x) - \gamma\dot{x} + F + \sqrt{2\gamma\beta^{-1}}\,\dot{W}}$$

mass (inertia)

force

friction (dissipation)

tilt

temp. (β^{-1})

white noise

Convergence regimes:

(1) $m \to 0$ (vanishing inertia)

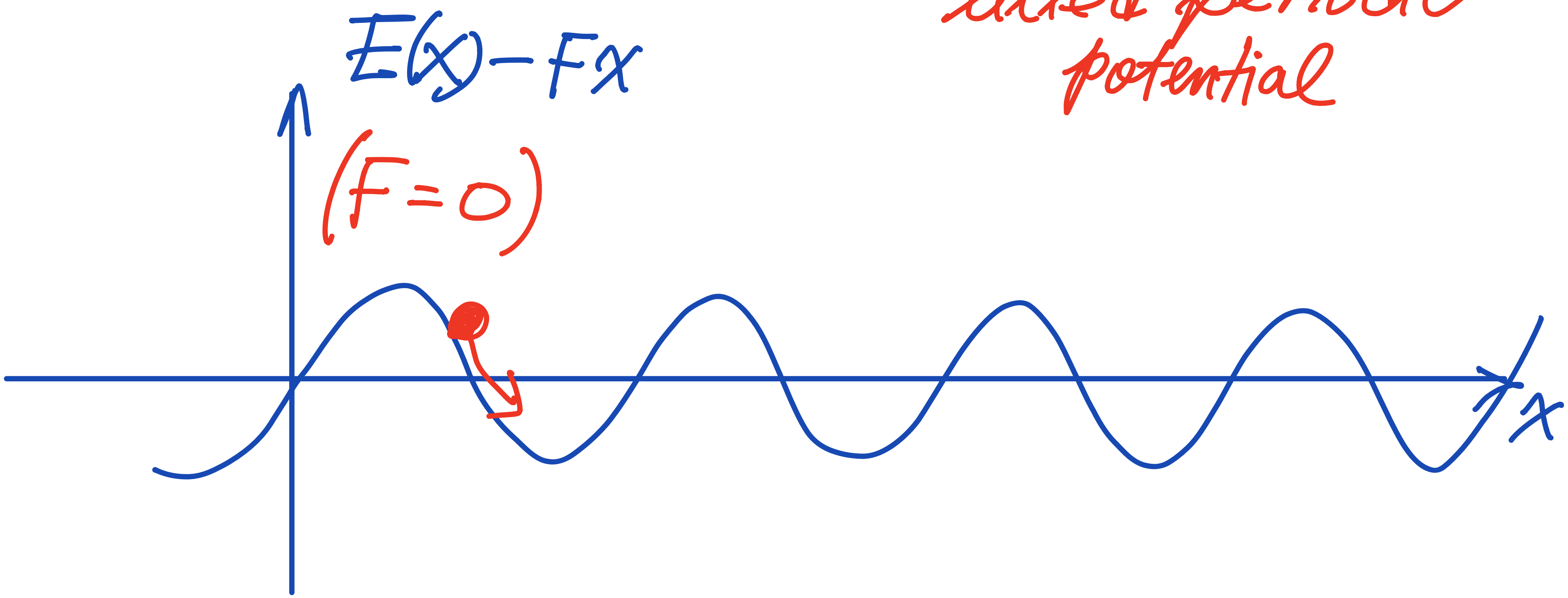
(2) $\gamma \to 0$ (vanishing dissipation)

(3) $\beta \to +\infty$ (zero noise limit)

Deterministic Case

$E(x) = \sin x$

$$\dot{x} = -\cos x + F = -(\underbrace{\sin x - Fx}_{\text{tilted periodic potential}})$$



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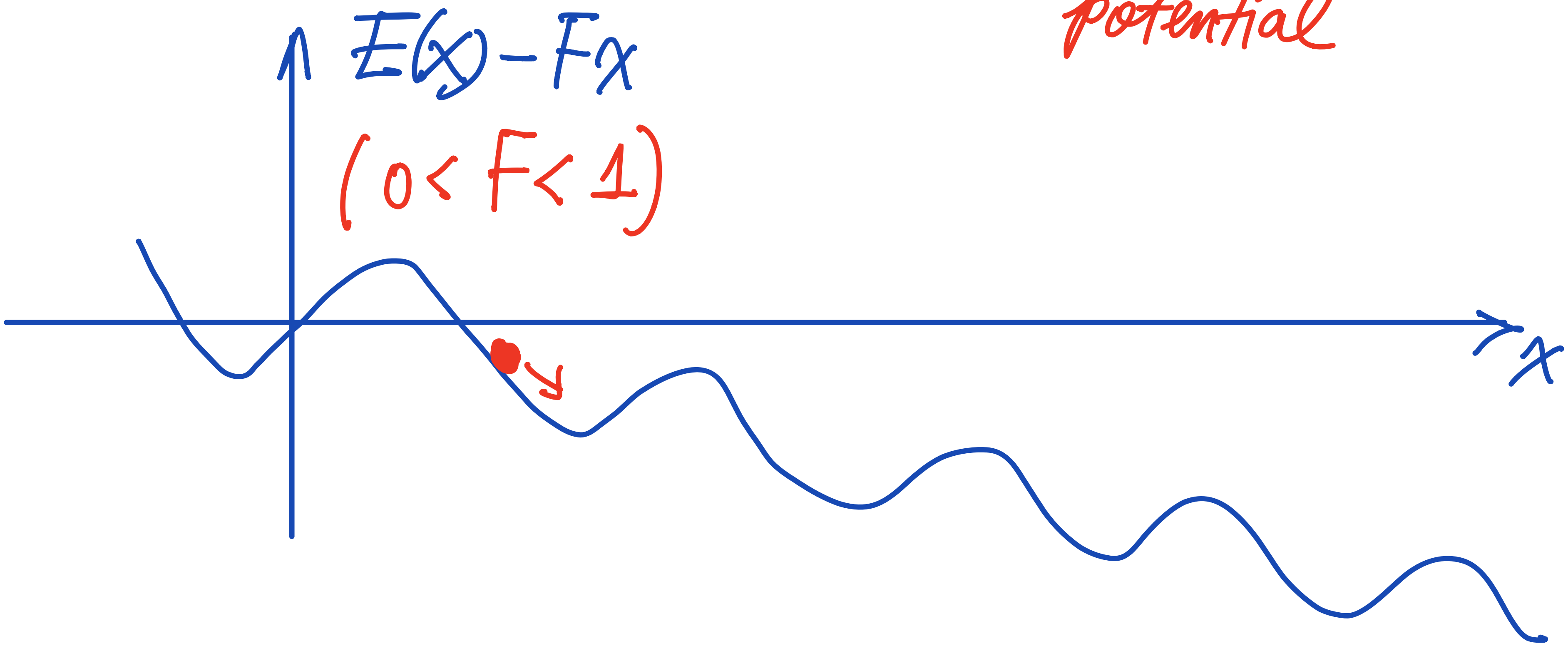

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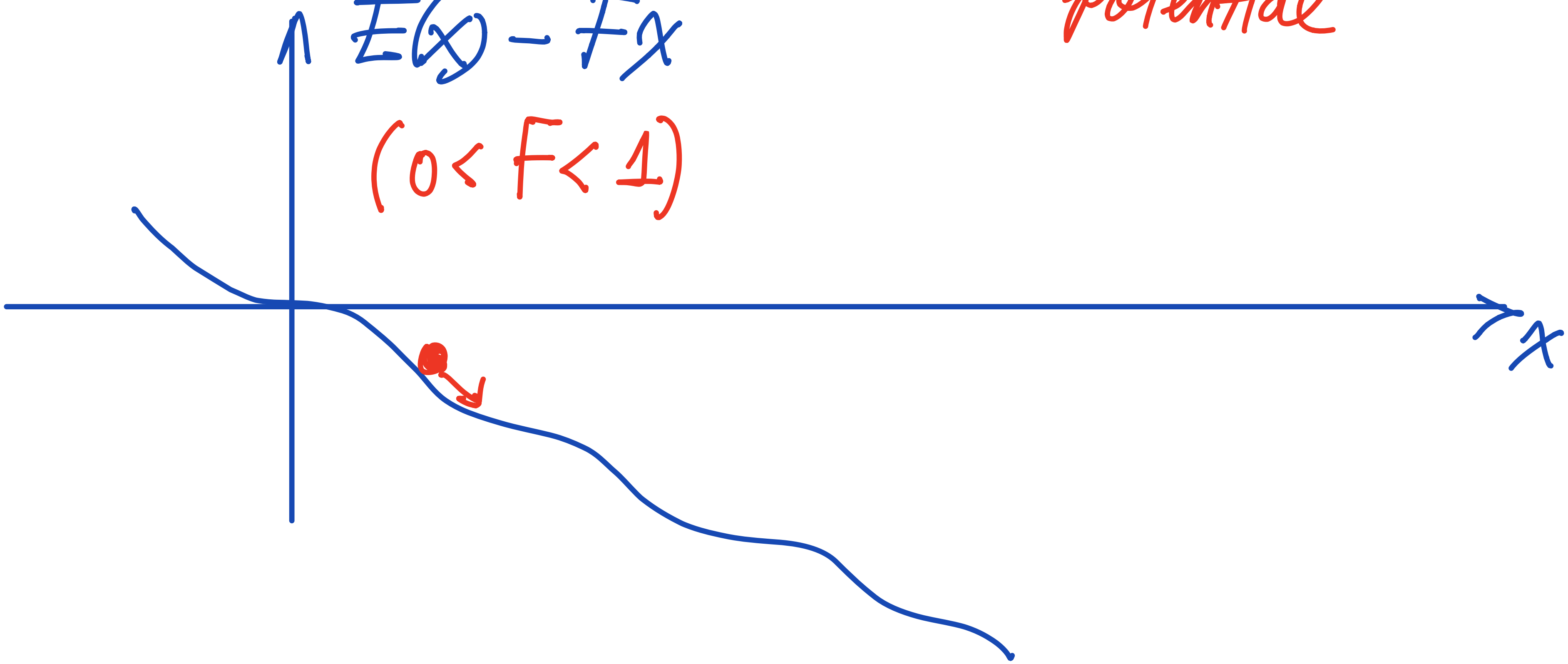
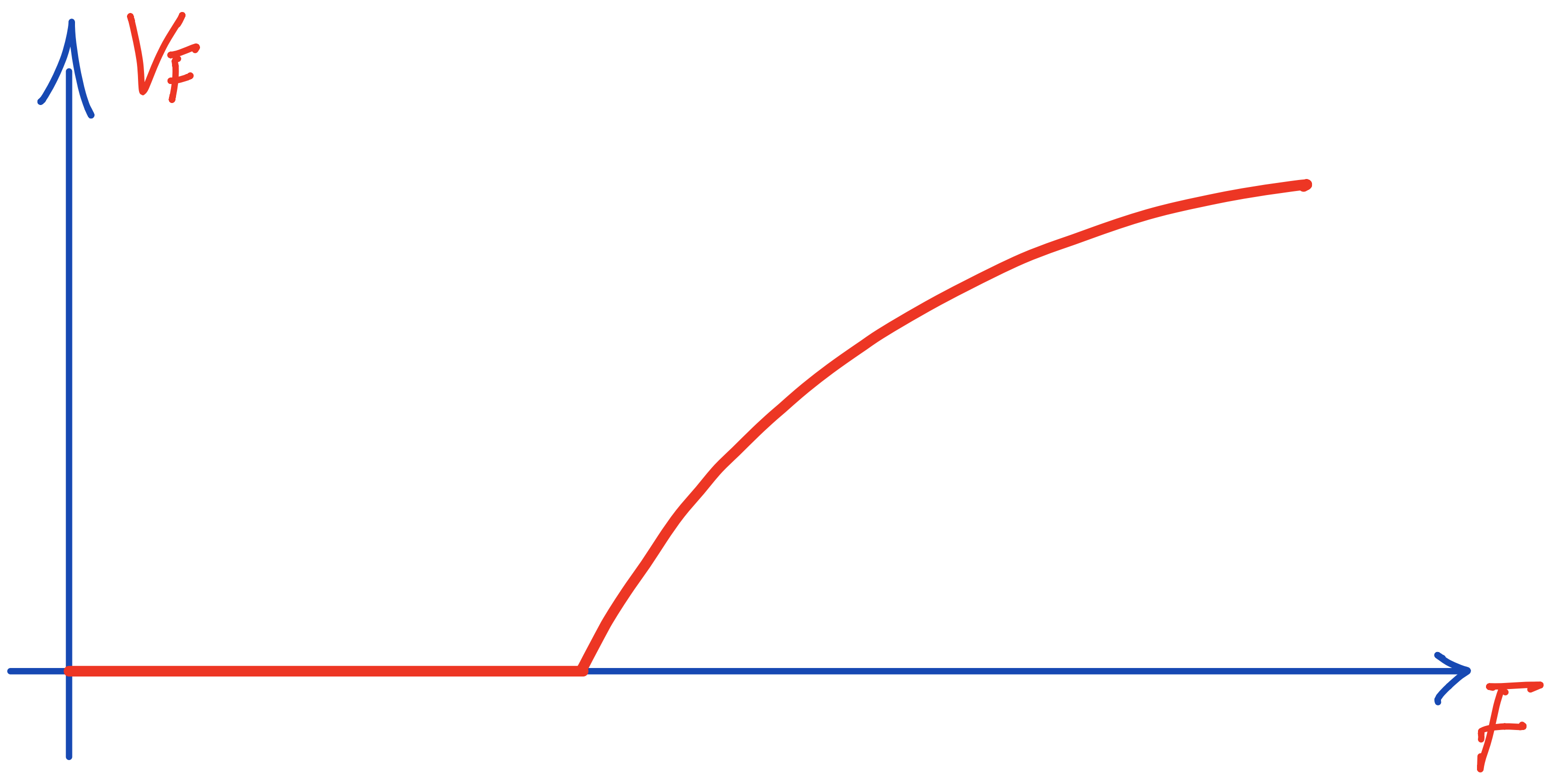


Long Time (Average) Velocity

$$\boxed{V_F = \lim_{t\to\infty} \frac{x(t)}{t}}$$ (as a function of F)



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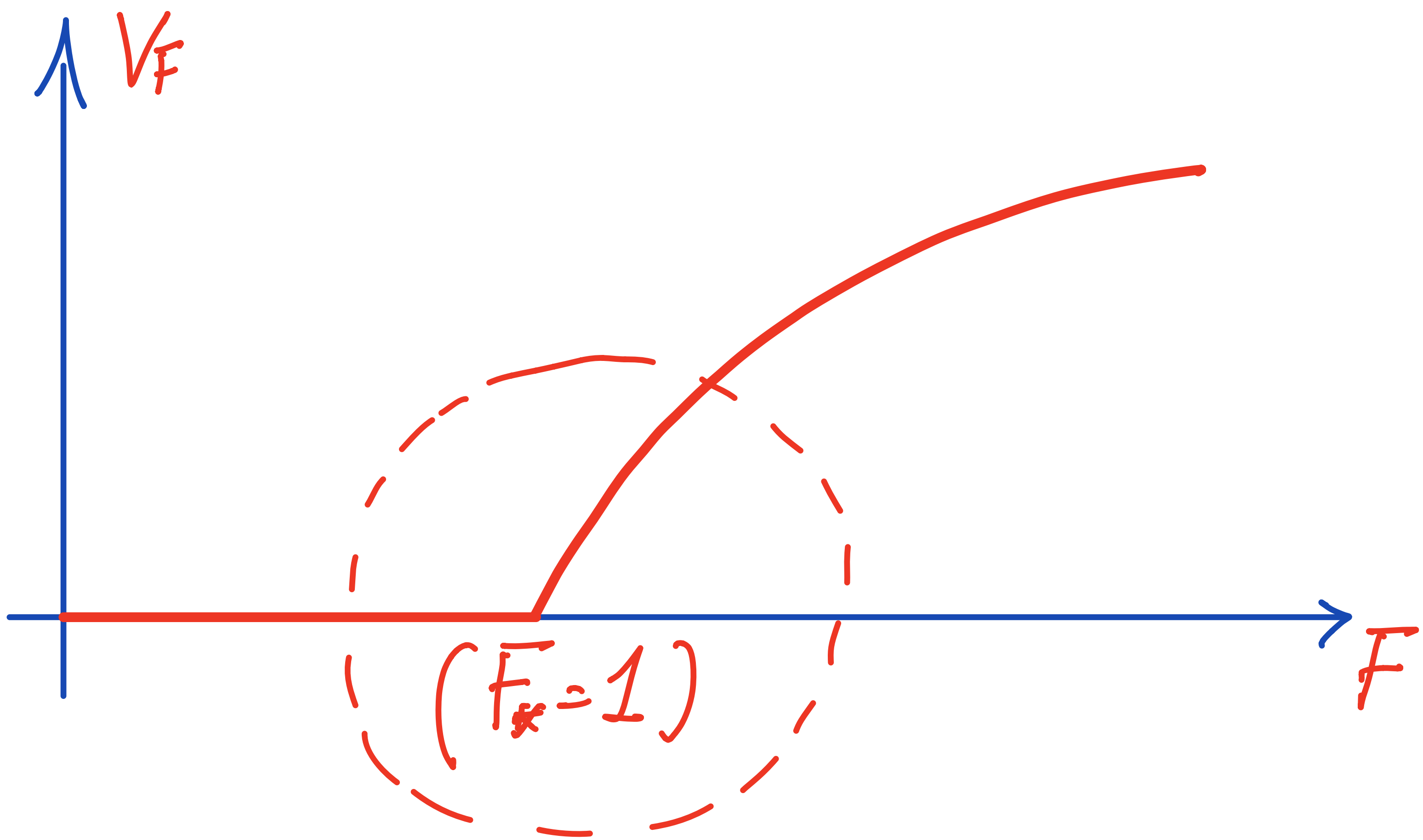


Long Time (Average) Velocity

$$\boxed{V_F = \lim_{t\to\infty} \frac{x(t)}{t}}$$ (as a function of F)

$$\begin{aligned}\frac{dx}{dt} &= -\cos x + F \\ &= -\cos x + 1 + F - 1 \\ &= 1 - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots + (F-1) \\ &\cong \frac{x^2}{2} + \delta \end{aligned}$$

$(F - 1 = \delta)$

Long Time (Average) Velocity

$$\boxed{V_F = \lim_{t\to\infty} \frac{x(t)}{t}}$$ (as a function of F)

$$\begin{aligned}
\frac{dx}{dt} &= -\cos x + F \\
&= -\cos x + 1 + F - 1 \\
&= 1 - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots + (F-1) \\
&\cong \boxed{\frac{x^2}{2} + \delta}
\end{aligned}$$

(normal form) $(F - 1 = \delta)$

Long Time (Average) Velocity

$$\boxed{V_F = \lim_{t\to\infty} \frac{x(t)}{t}}$$ (as a function of F)

$$\int_0^{2\pi} \frac{dx}{\frac{x^2}{2} + \delta} = \int_0^T dt \; (= T)$$

$$T \cong C\delta^{-1/2}$$

$$\boxed{V_F \cong C(F - F_*)^{1/2}, \quad \text{for } 0 < F - F_* \ll 1}$$

Long Time (Average) Velocity

$$\boxed{V_F = \lim_{t\to\infty} \frac{x(t)}{t}}$$ (as a function of F)

$$\int_0^{2\pi} \frac{dx}{\frac{x^2}{2} + \delta} = \int_0^T dt \, (= T)$$

$$T \simeq C\delta^{-1/2}$$

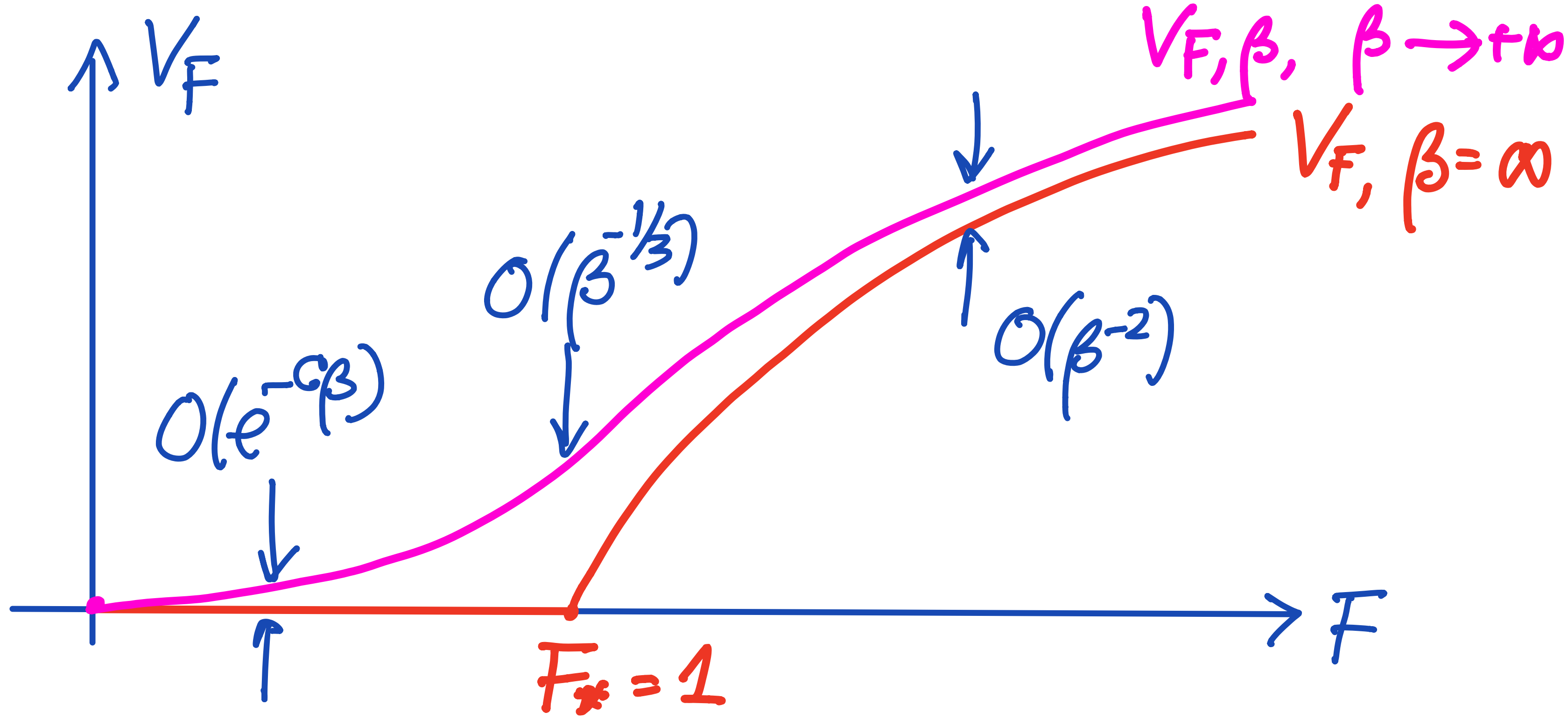
"very easy to change"

$$\boxed{V_F \simeq C(F - F_*)^{1/2}, \quad \text{for } 0 < F - F_* \ll 1}$$

Strong Dissipation ($m=0$, $m \ll 1$, $\gamma = O(1)$)

Smoluchowski Equation ($m=0$)

$$\boxed{\gamma \dot{x} = -E'(x) + F + \sqrt{2\gamma\beta^{-1}}\, \dot{w}}$$



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Smoluchowski Equation ($m=0$)

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(1) Explicit formula using exit time

$$\boxed{V_{F,\beta} = \frac{1}{\gamma}\,\frac{2\pi\beta^{-1}\left(1-e^{-2\pi F\beta}\right)}{\int_0^{2\pi}\int_{x-2\pi}^{x} e^{-\beta(E(x')-E(x))}\,dx\,dx'}}$$

Strong Dissipation ($m=0$, $m \ll 1$, $\gamma = O(1)$)

Smoluchowski Equation ($m=0$)

$$\boxed{\gamma \dot{x} = -E'(x) + F + \sqrt{2\gamma\beta^{-1}}\,\dot{w}}$$

(2) (Reimann, et. al.) Giant Diffusion

Diffusion coefficient (D) $= \infty$ at $F = F_*$

$$D = \lim_{t\to+\infty} \frac{\langle x^2(t)\rangle - \langle x(t)\rangle^2}{2t}$$

Strong Dissipation ($m=0$, $m \ll 1$, $\gamma = O(1)$)

Overdamped Limit ($0 < m \ll 1$)

$$m\ddot{x} = -E'(x) - \gamma \dot{x} + F + \sqrt{2\gamma\beta^{-1}}\,\dot{W}$$

$m \to 0$

$$\gamma \dot{x} = -E'(x) + F + \sqrt{2\gamma\beta^{-1}}\,\dot{W}$$

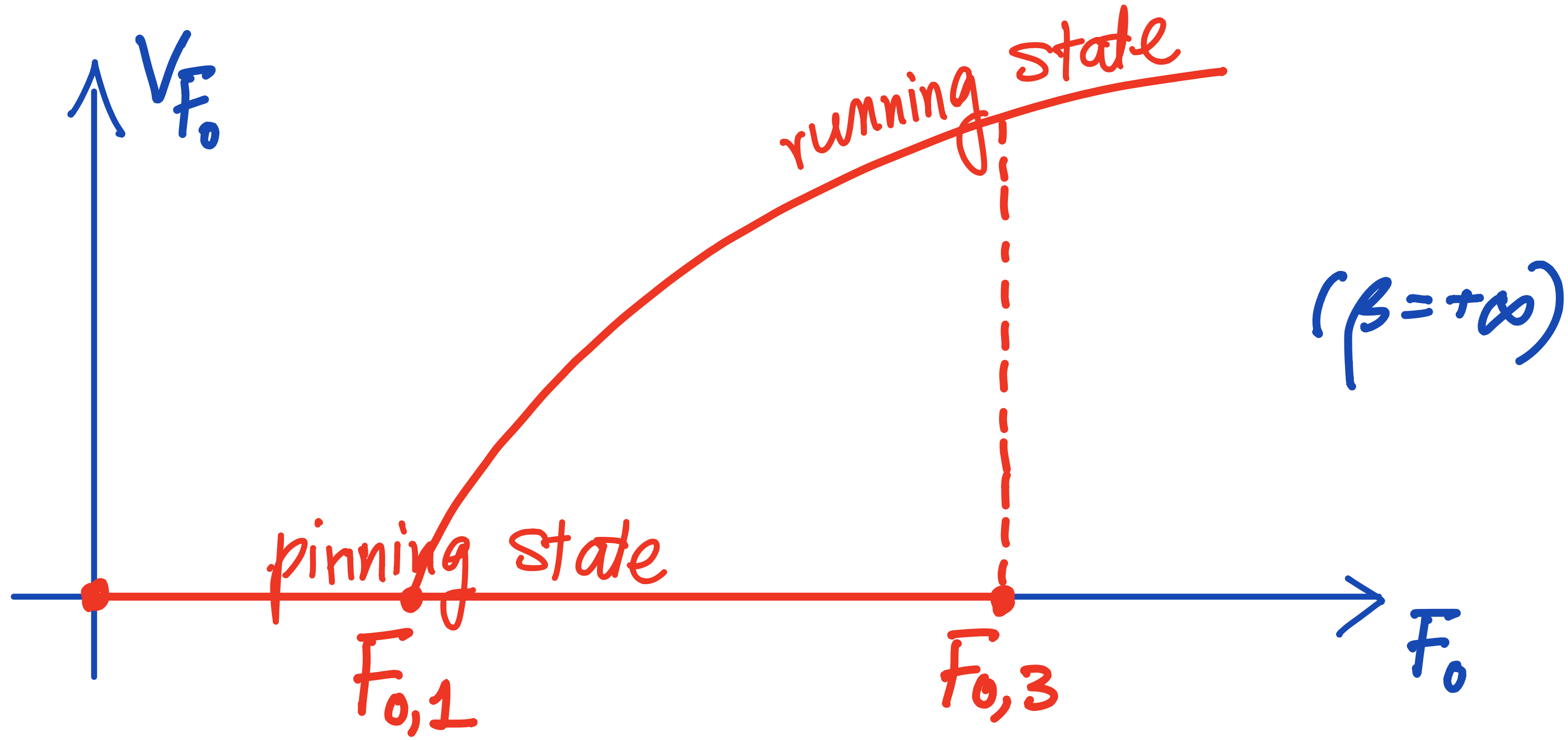
(Kramer-Smoluchowski limit, Freidlin)

$$\boxed{V_F(m>0, \beta) \xrightarrow{m \to 0} V_F(m=0, \beta)}$$

Weak (Vanishing) Dissipation ($\gamma \to 0^+$)

Deterministic Case ($\beta = +\infty$) (Rescale F: $F = F_0 \gamma$)

$$\boxed{m\ddot{x} = -E'(x) - \gamma\dot{x} + F_0\gamma}$$



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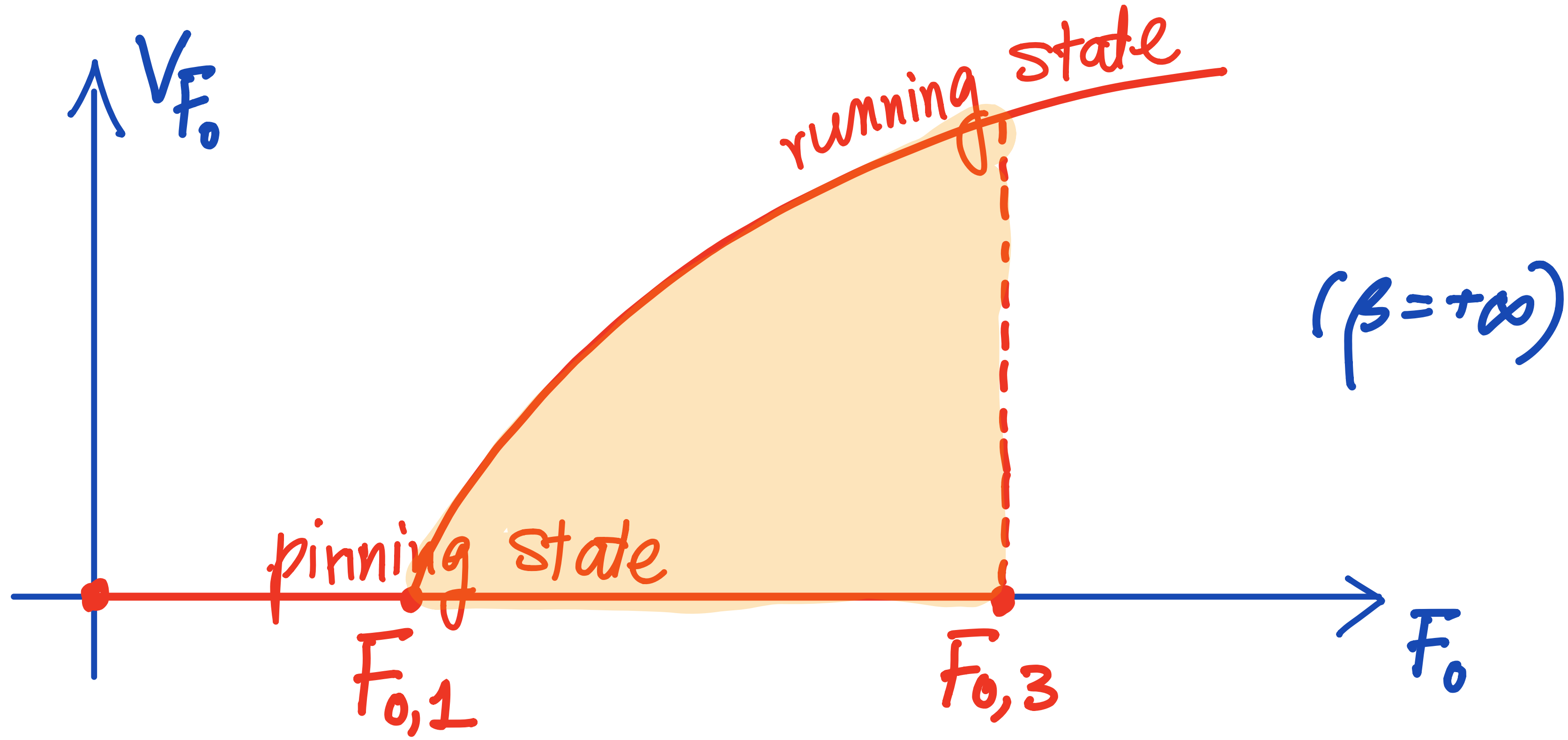

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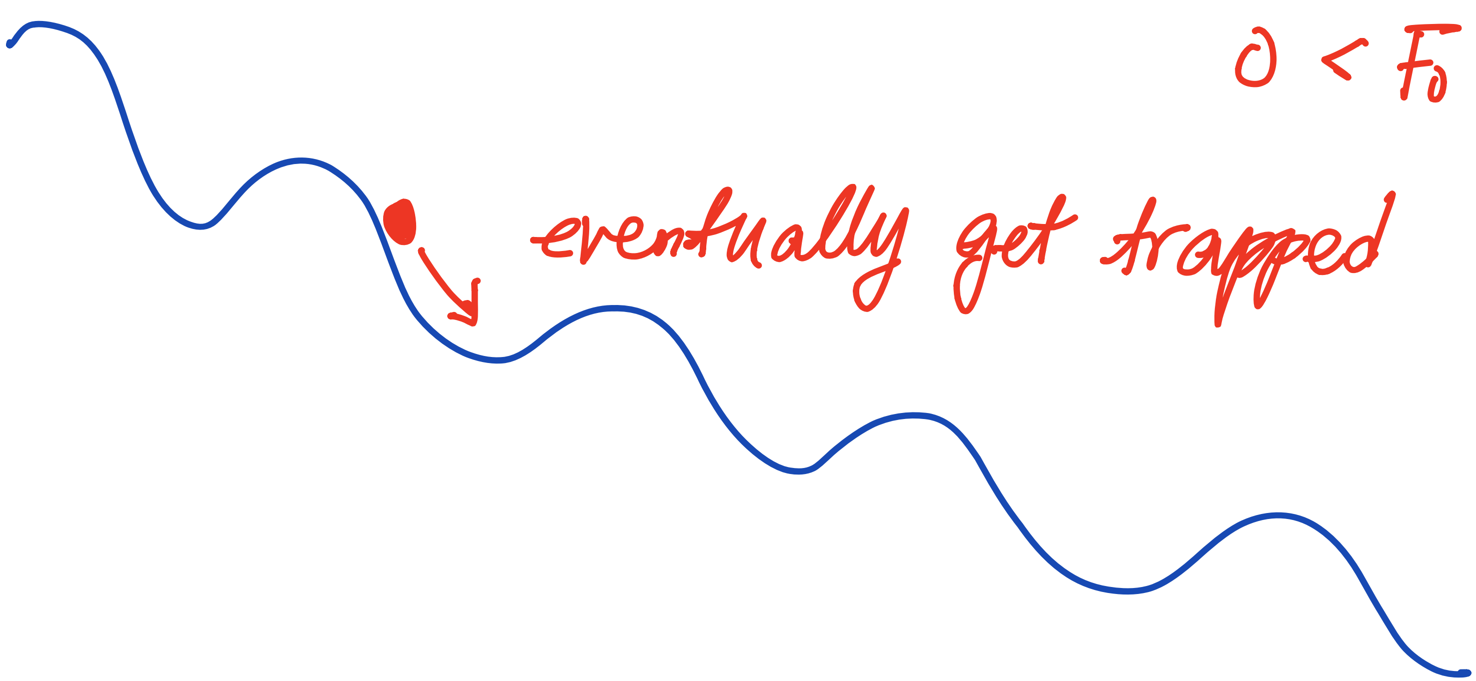
$$\boxed{m\ddot{x} = -E'(x) - \gamma\dot{x} + F_0\gamma}$$

$0 < F_0 < F_{0,1}$



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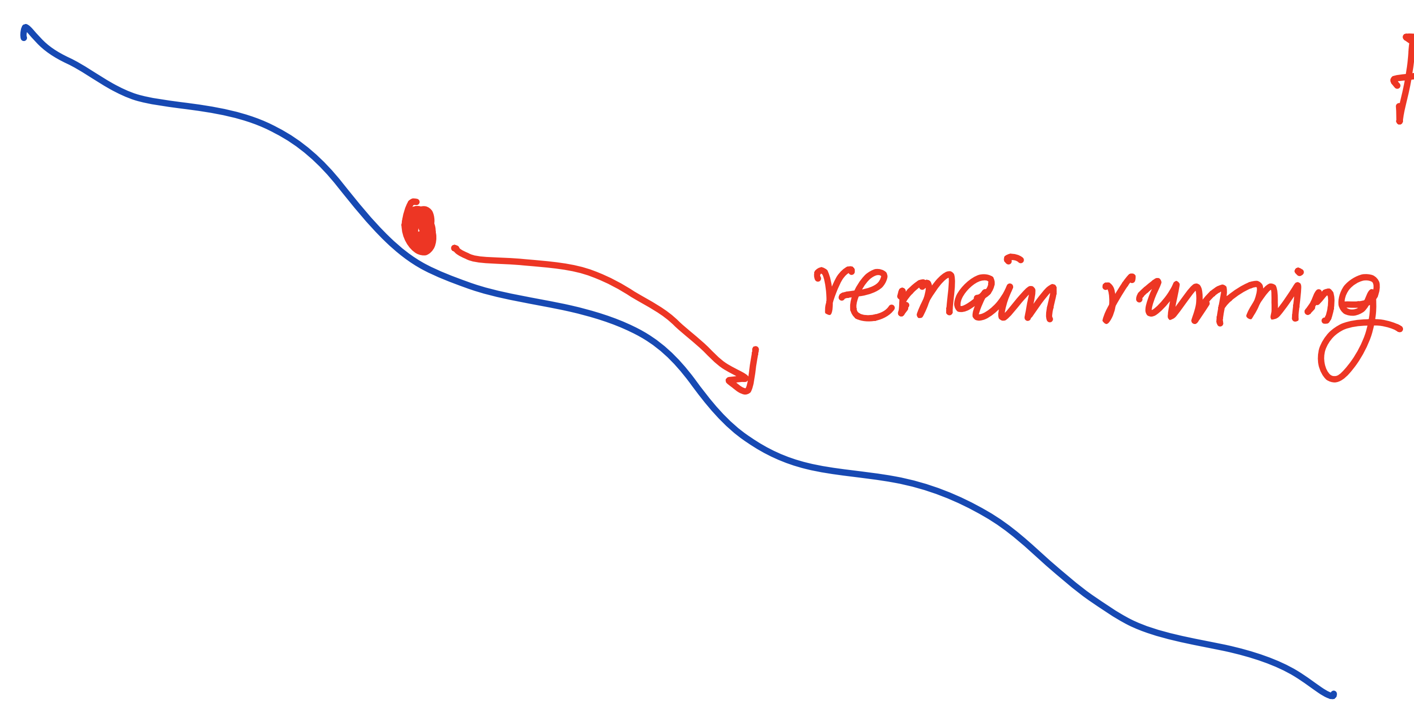
$$\boxed{m\ddot{x} = -E'(x) - \gamma\dot{x} + F_0\gamma}$$

$F_0 > F_{0,3}$



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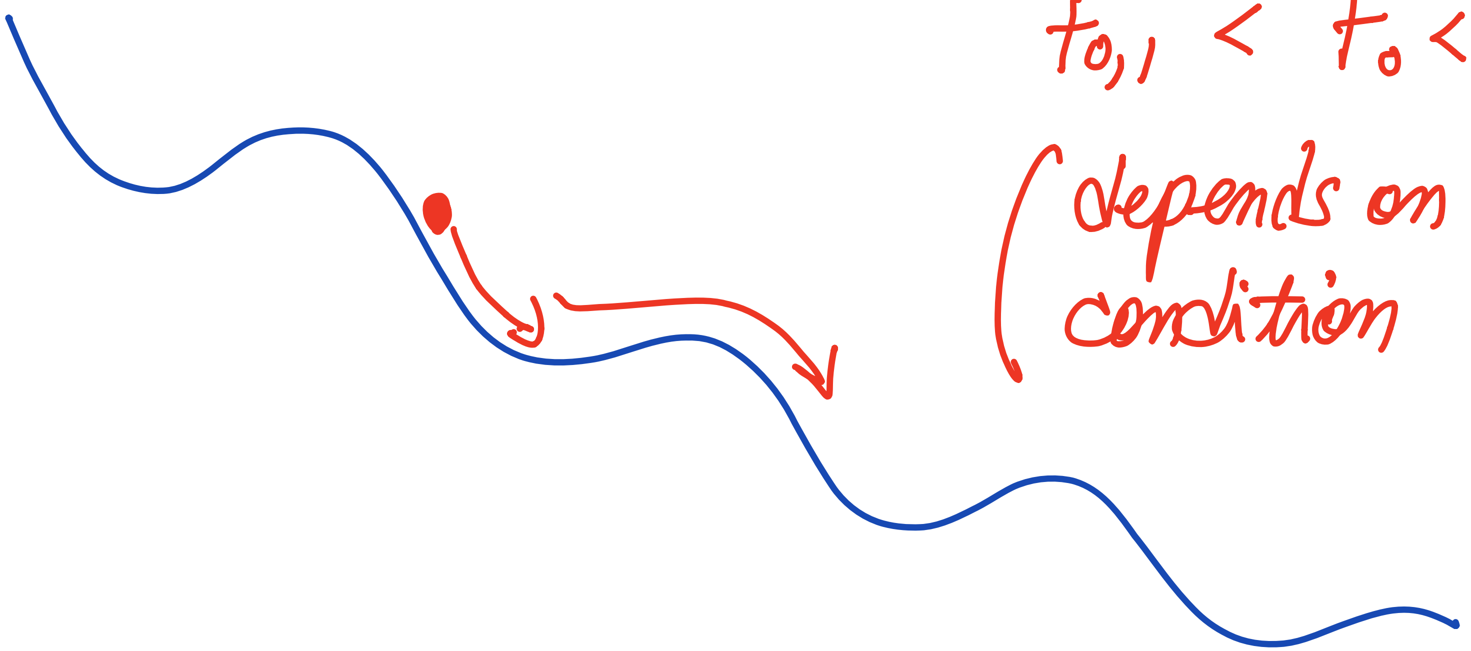

$F_{0,1} < F_0 < F_{0,3}$

(depends on initial condition)

Weak (Vanishing) Dissipation ($\gamma \to 0^+$)

Stochastic Case ($\beta < \infty$) (Rescale F: $F = F_0 \gamma$)

$$\boxed{m\ddot{x} = -E'(x) - \gamma \dot{x} + F_0 \gamma + \sqrt{2\gamma\beta^{-1}}\, \dot{W}}$$

Weak (Vanishing) Dissipation ($\gamma \to 0^+$)

Stochastic Case ($\beta < \infty$)

(Rescale F: $F = F_0 \gamma$)

$$\boxed{m\ddot{x} = -E'(x) - \gamma \dot{x} + F_0 \gamma + \sqrt{2\gamma\beta^{-1}}\, \dot{W}}$$

$\gamma \to 0$ in finite time

$$\boxed{m\ddot{x} = -E'(x)}$$

(Simple Hamiltonian System, 1D)

Weak (Vanishing) Dissipation ($\gamma \to 0^+$)

Stochastic Case ($\beta < \infty$) (Rescale F: $F = F_0 \gamma$)

$$\boxed{m\ddot{x} = -E'(x) - \gamma \dot{x} + F_0 \gamma + \sqrt{2\gamma\beta^{-1}}\, \dot{W}}$$

$\gamma \to 0$

in appropriate long time regime

$$\boxed{\text{Diffusion on a ``Hamiltonian'' graph}}$$

(based on Freidlin–Wentzell Theory)

Weak (Vanishing) Dissipation ($\gamma \to 0^+$)

Stochastic Case ($\beta < \infty$) (Rescale F: $F = F_0 \gamma$)

$$\boxed{m\ddot{x} = -E'(x) - \gamma\dot{x} + F_0\gamma + \sqrt{2\gamma\beta^{-1}}\,\dot{W}}$$

$$t \quad \begin{cases} \dot{x} = y \\ \dot{y} = -\gamma y + F_0\gamma - E'(x) + \sqrt{2\gamma\beta^{-1}}\,\dot{W} \end{cases}$$

$$\downarrow$$

$$\frac{t}{\gamma} \quad \begin{cases} \dot{x} = \frac{1}{\gamma} y \\ \dot{y} = -\frac{1}{\gamma}E'(x) + (-y + F_0) + \sqrt{2\beta^{-1}}\,\dot{W}) \end{cases}$$

Weak (Vanishing) Dissipation ($\gamma \to 0^+$)

Stochastic Case ($\beta < \infty$)

(Rescale F: $F = F_0 \gamma$)

$$\boxed{m\ddot{x} = -E'(x) - \gamma \dot{x} + F_0 \gamma + \sqrt{2\gamma\beta^{-1}}\,\dot{w}}$$

$$\begin{cases} \dot{x} = H_y(x,y) \\ \dot{y} = -H_x(x,y) - \gamma y + F_0\gamma + \sqrt{2\gamma\beta^{-1}}\,\dot{w} \end{cases}$$

$t \downarrow \dfrac{t}{\gamma}$

$$\begin{cases} \dot{x} = \frac{1}{\gamma} H_y(x,y) \\ \dot{y} = -\frac{1}{\gamma} H_x(x,y) + (-y + F_0) + \sqrt{2\beta^{-1}}\,\dot{w} \end{cases}$$

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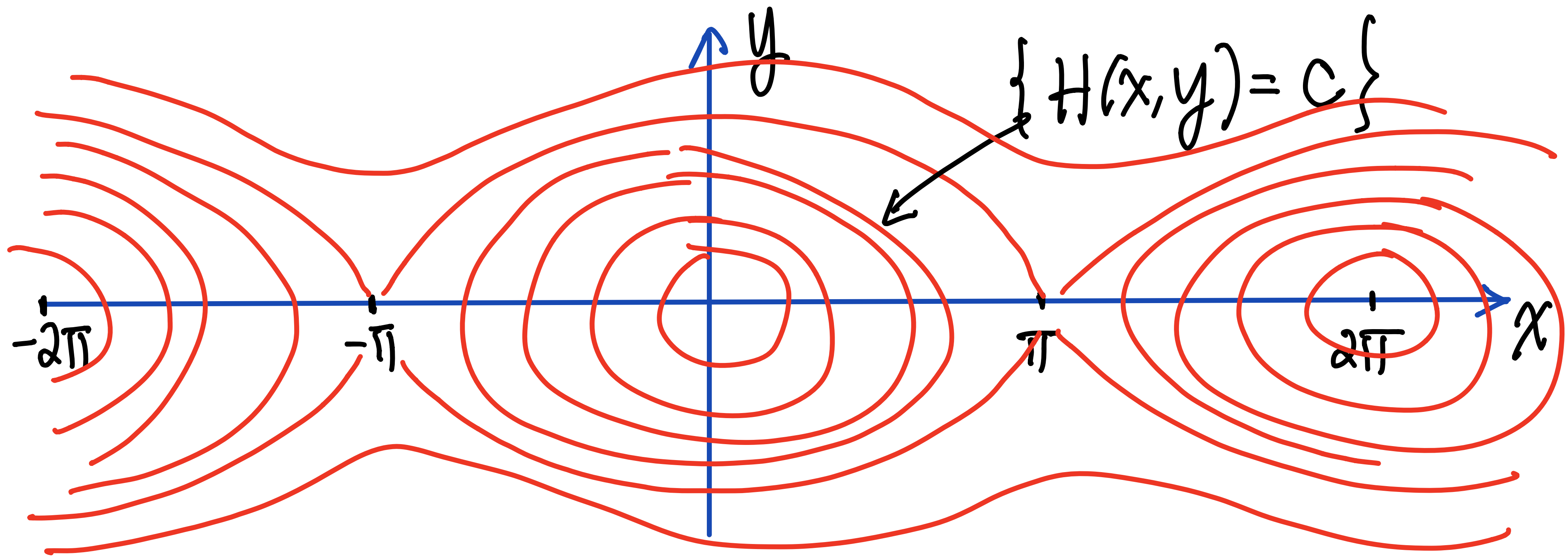

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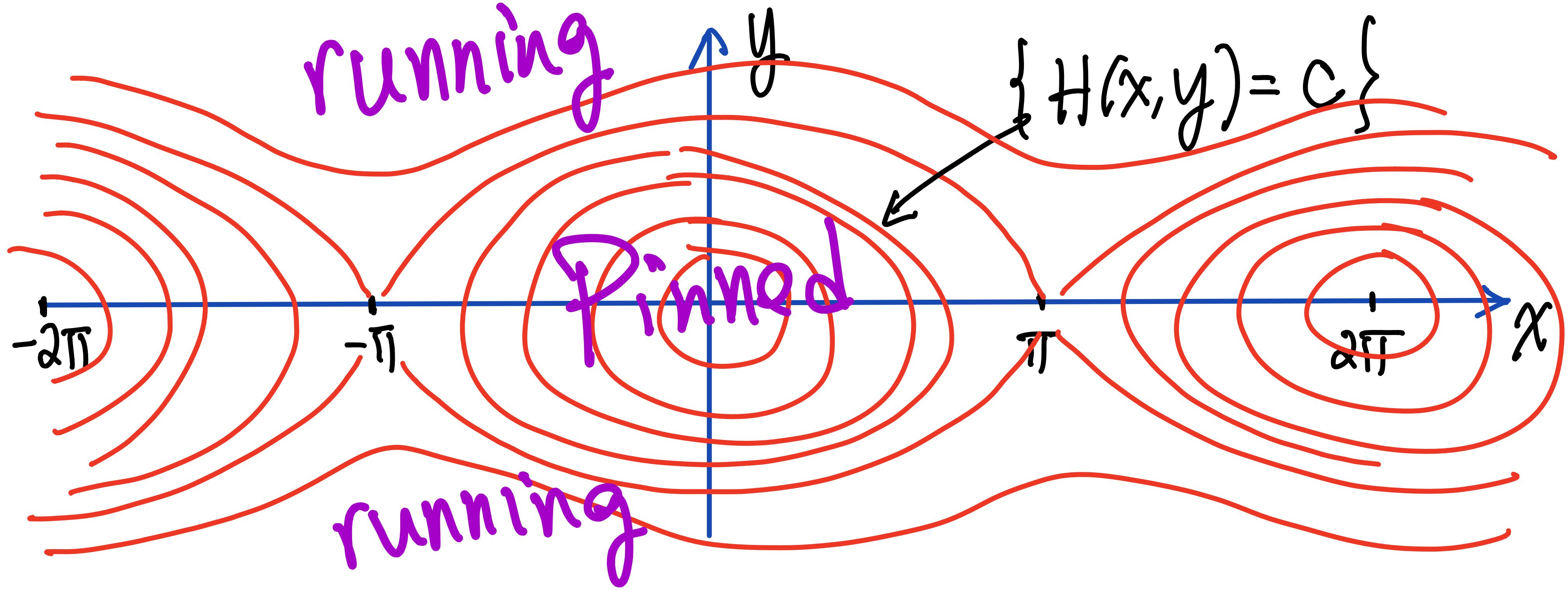
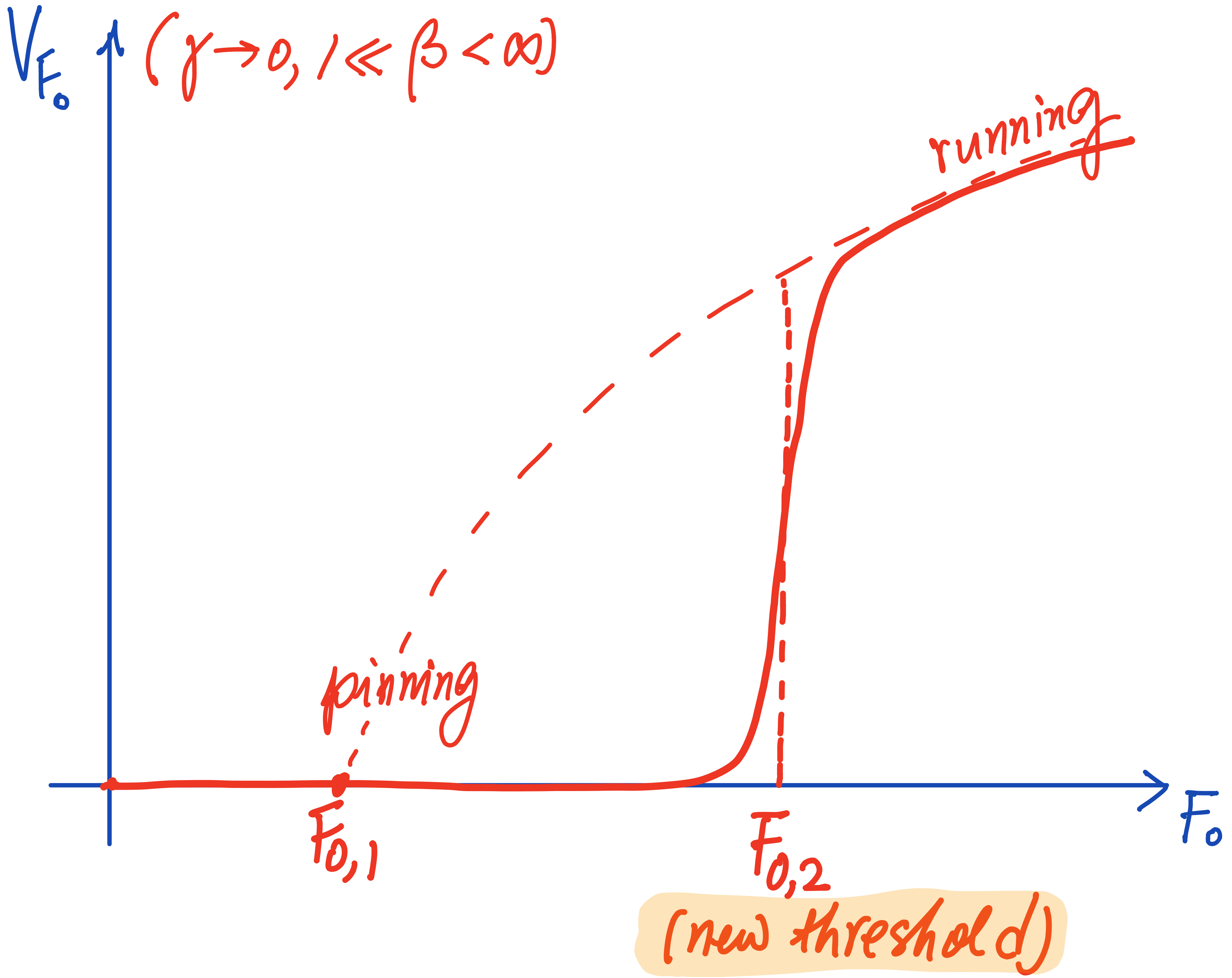
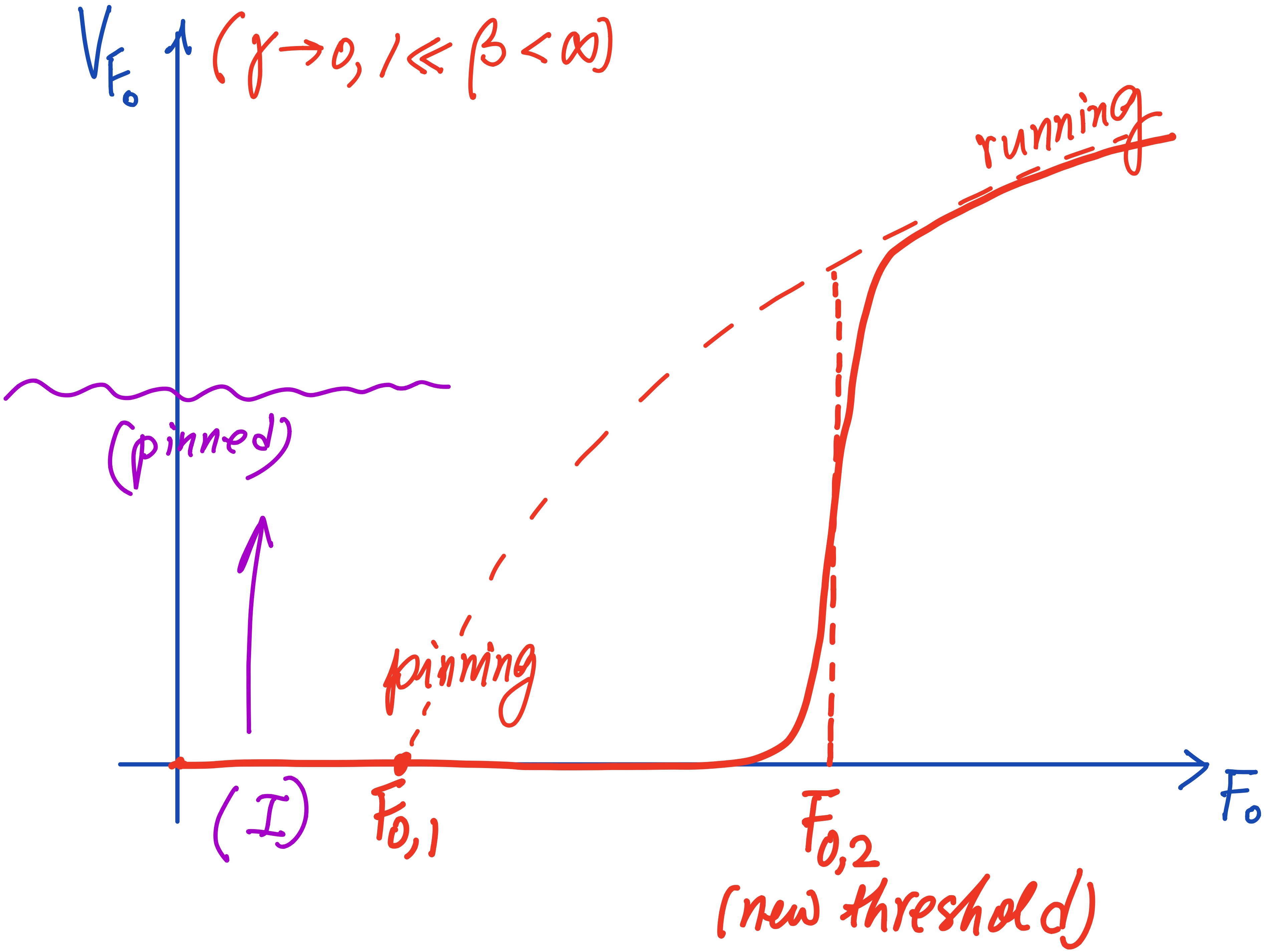
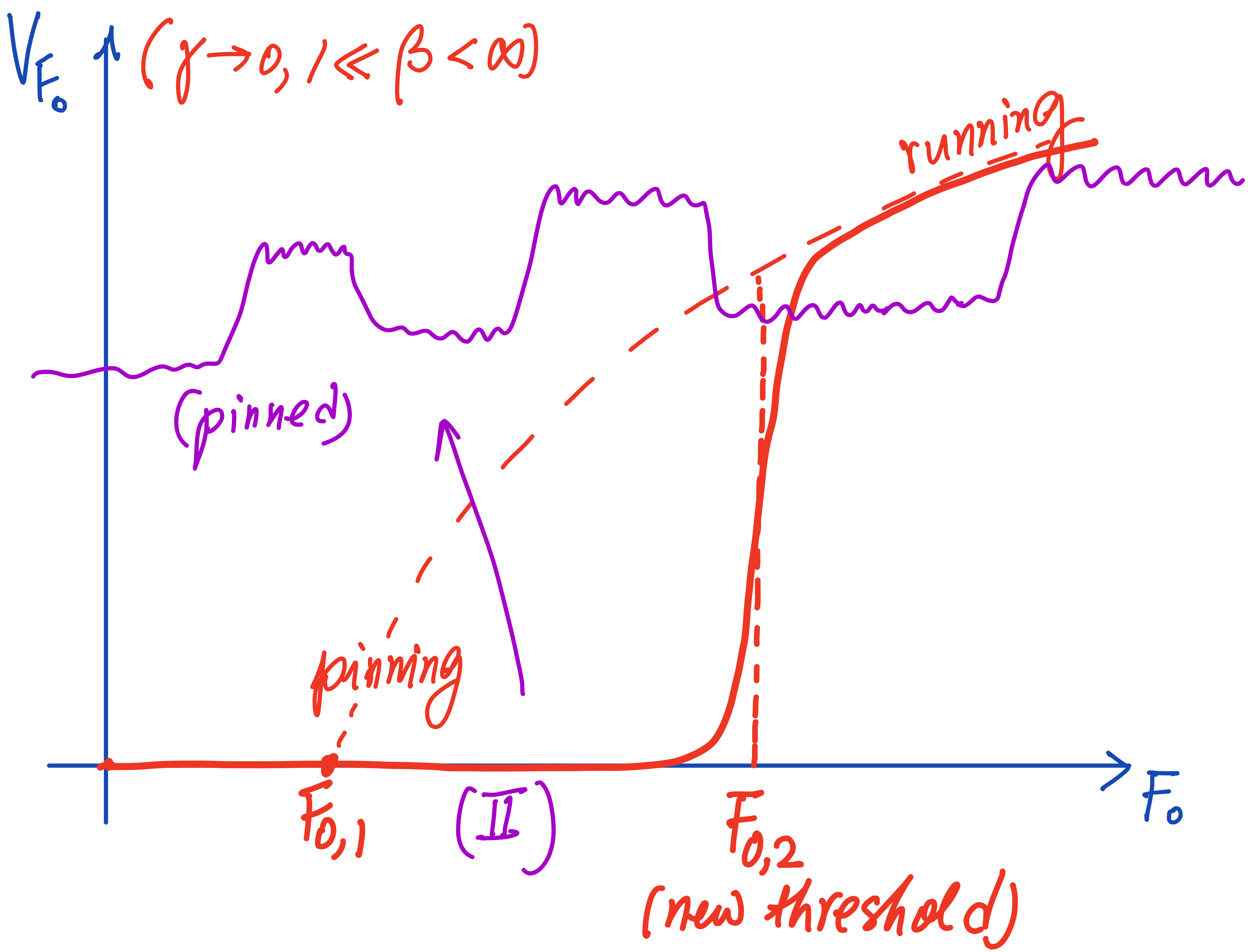



V_{F_0}
$(\gamma \to 0, 1 \ll \beta < \infty)$
running
pinning
$F_{0,1}$
$F_{0,2}$
(new threshold)
F_0



V_{F_0}
$(\gamma \to 0, 1 \ll \beta < \infty)$
running
(pinned)
pinning
(I)
$F_{0,1}$
$F_{0,2}$
(new threshold)
F_0


V_{F_0}
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(pinned)
running
pinning
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(II)
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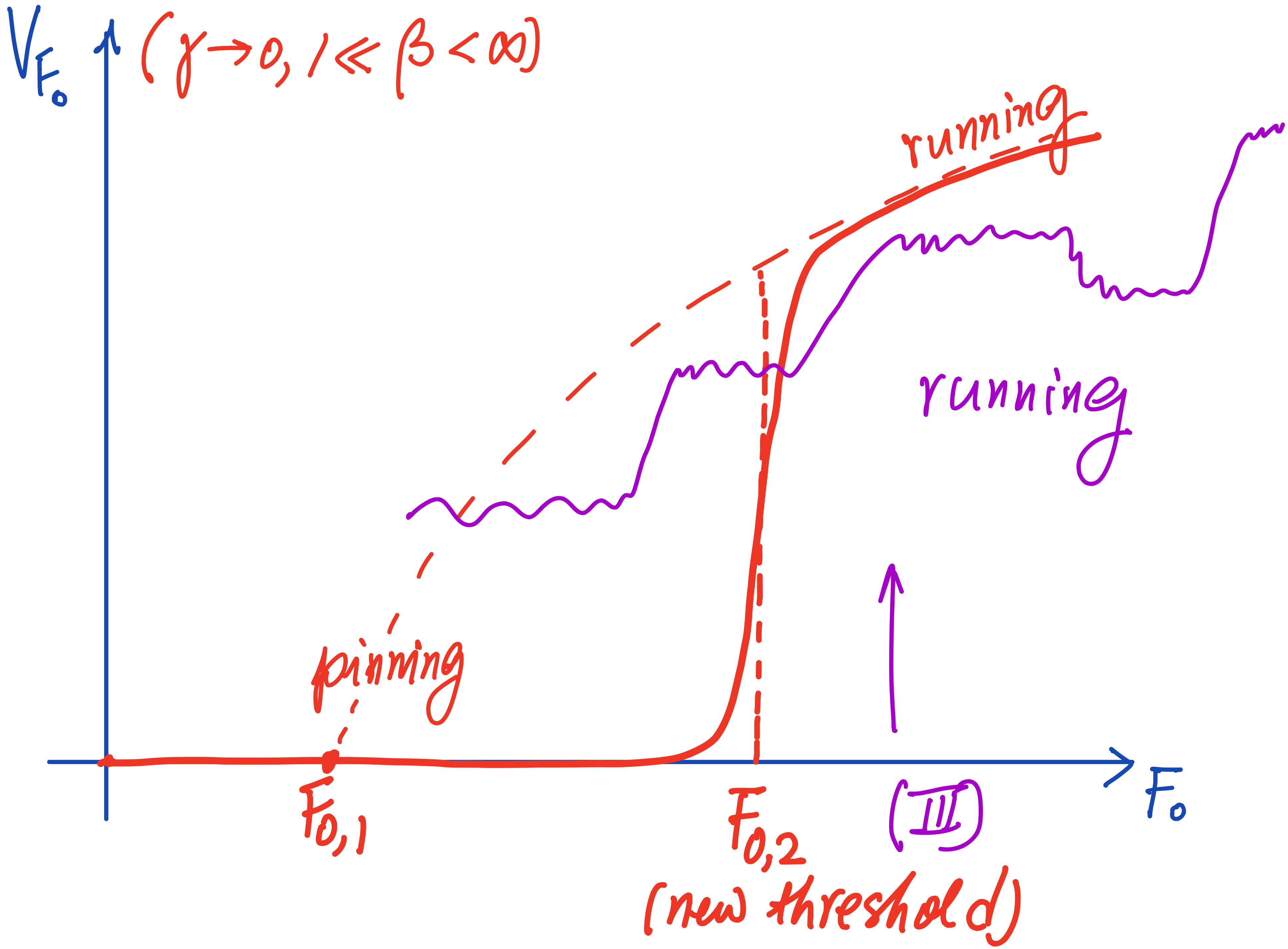
F_0


V_{F_0}
$(\gamma \to 0,\ 1 \ll \beta < \infty)$
running
running
pinning
$F_{0,1}$
$F_{0,2}$
(new threshold)
(III)
F_0

An Example with a Random Media (1D)

(Kawazu-Tanaka)

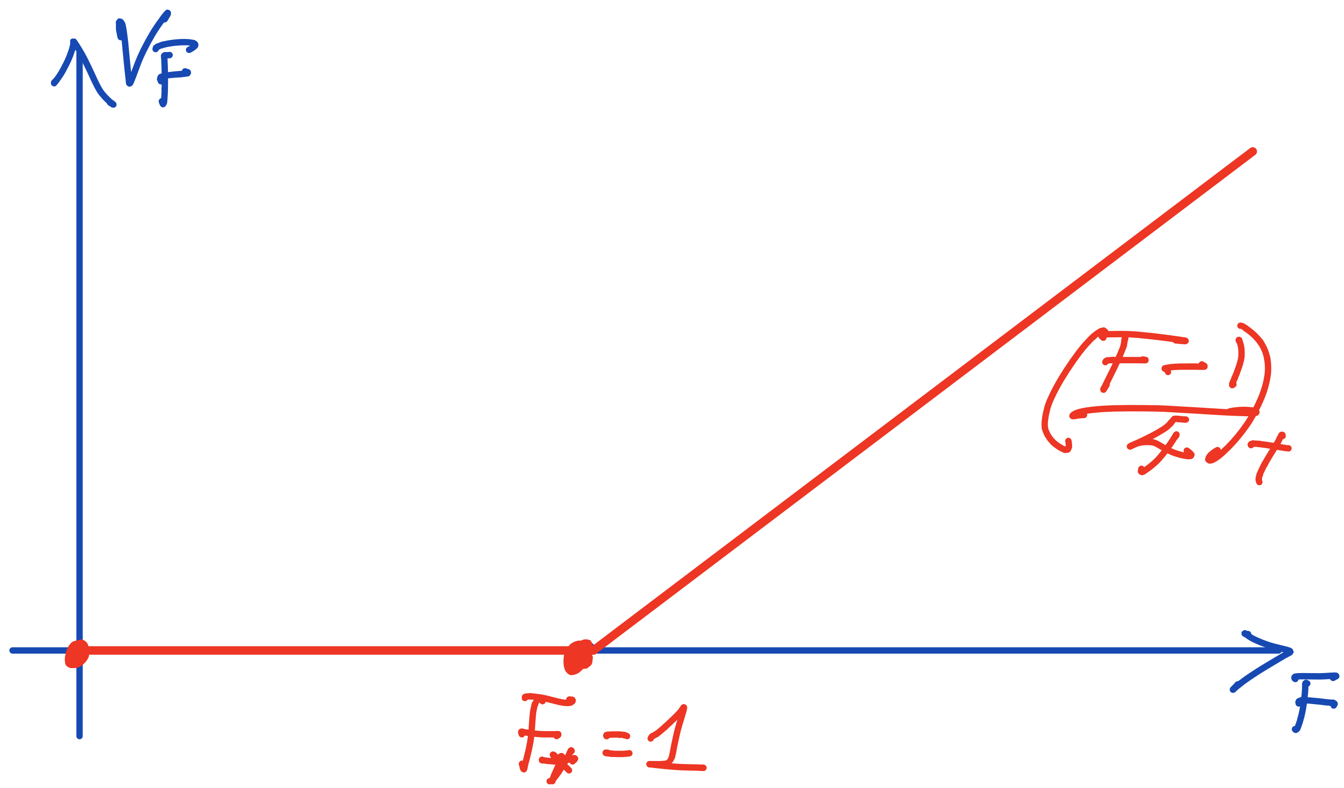
$$\boxed{dx_t = -\frac{1}{2}\Big(\underbrace{B'(x)}_{\text{Brownian potential}} - \underbrace{\frac{1}{2}F}_{\text{tilt}}\Big) + \underbrace{dW}_{\text{white noise}}}$$

$$\boxed{V_F = \lim_{t\to\infty} \frac{X(t)}{t}}$$

An Example with a Random Media (1D)

(Kawazu–Tanaka)

$$\boxed{dx_t = -\frac{1}{2}\left(B'(x) - \frac{1}{2}F\right) + dW}$$



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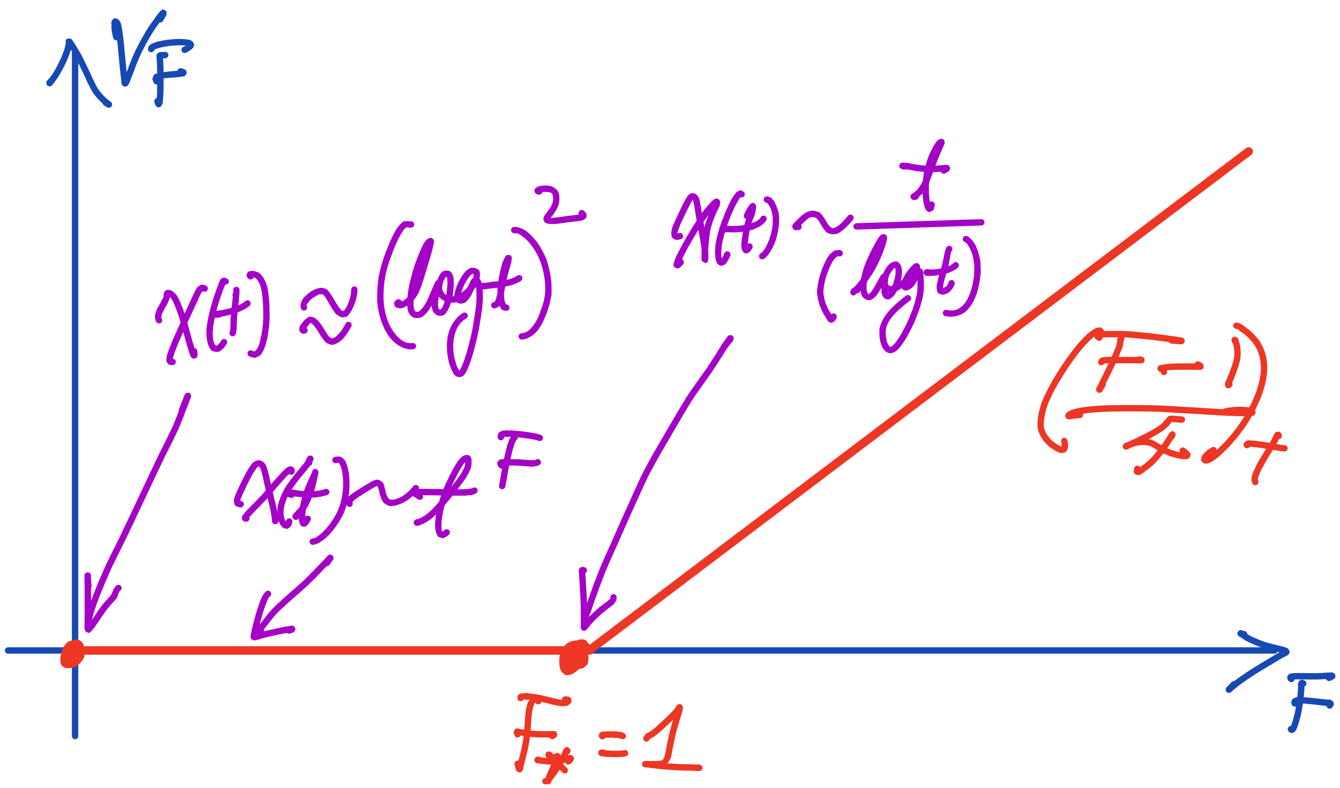
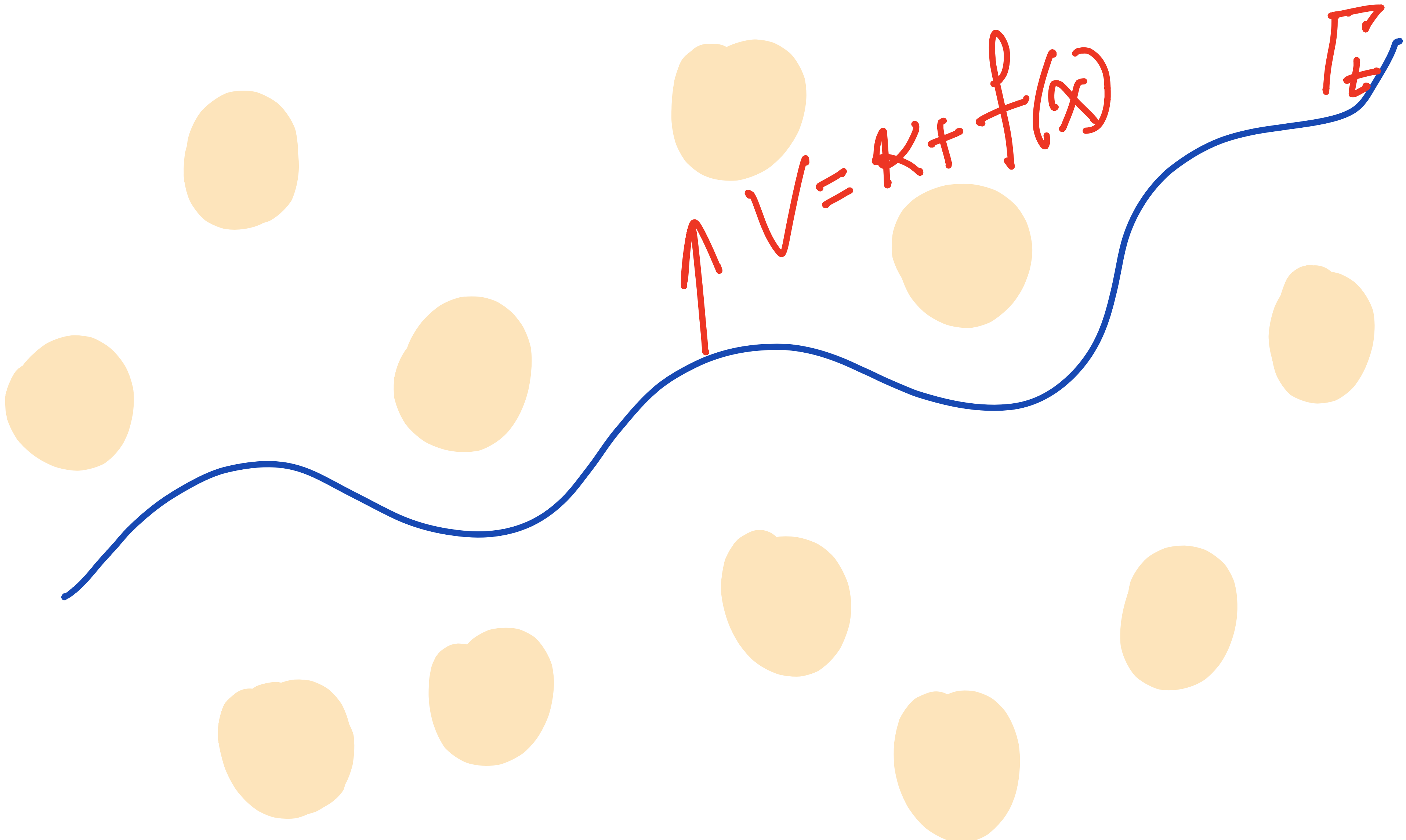
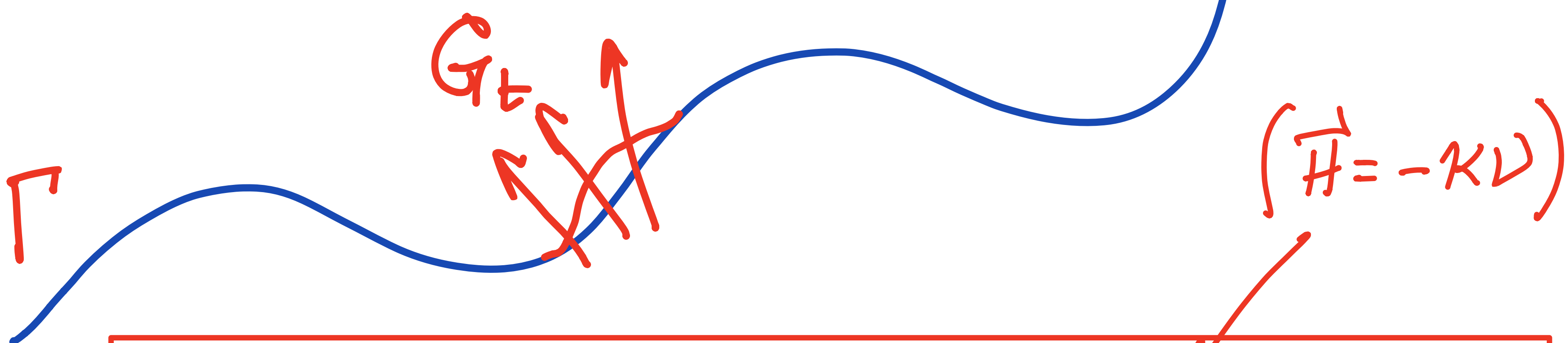


Motion by Mean Curv. in Inhomog. Media



Motion by Mean Curv. in Inhomog. Media

$$\boxed{\begin{aligned} &K \text{ (mean curv.)} \\ &= \text{First Variation of Area Fct.} \end{aligned}}$$



$(\vec{H} = -K\nu)$

$$\boxed{\frac{d}{dt}\left(\mathrm{Area}(G_t(\Gamma))\right)\Big|_{t=0} = \int_\Gamma \langle \vec{H}, \vec{X} \rangle \, da}$$

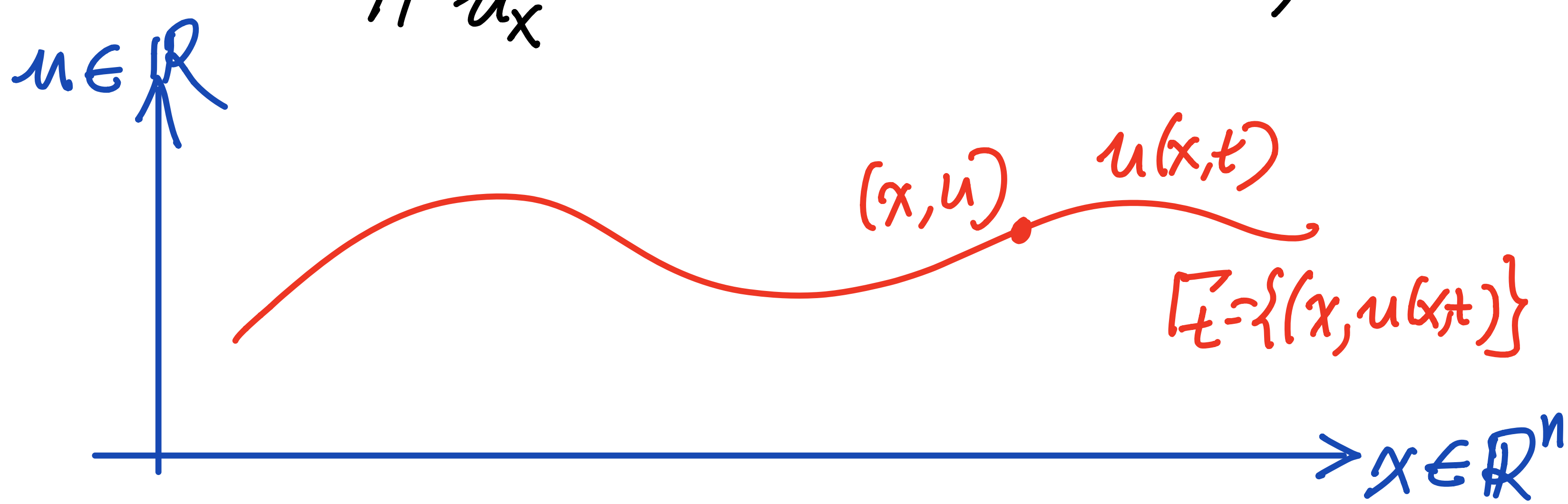
$\vec{X} = \dot{G}_t\big|_{t=0}$

Motion by Mean Curv. in Inhomog. Media

Written in graph form: $u(x,t)$

$$u_t = \sqrt{1+|\nabla u|^2}\,\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + \sqrt{1+|\nabla u|^2}\, f(x,u)$$

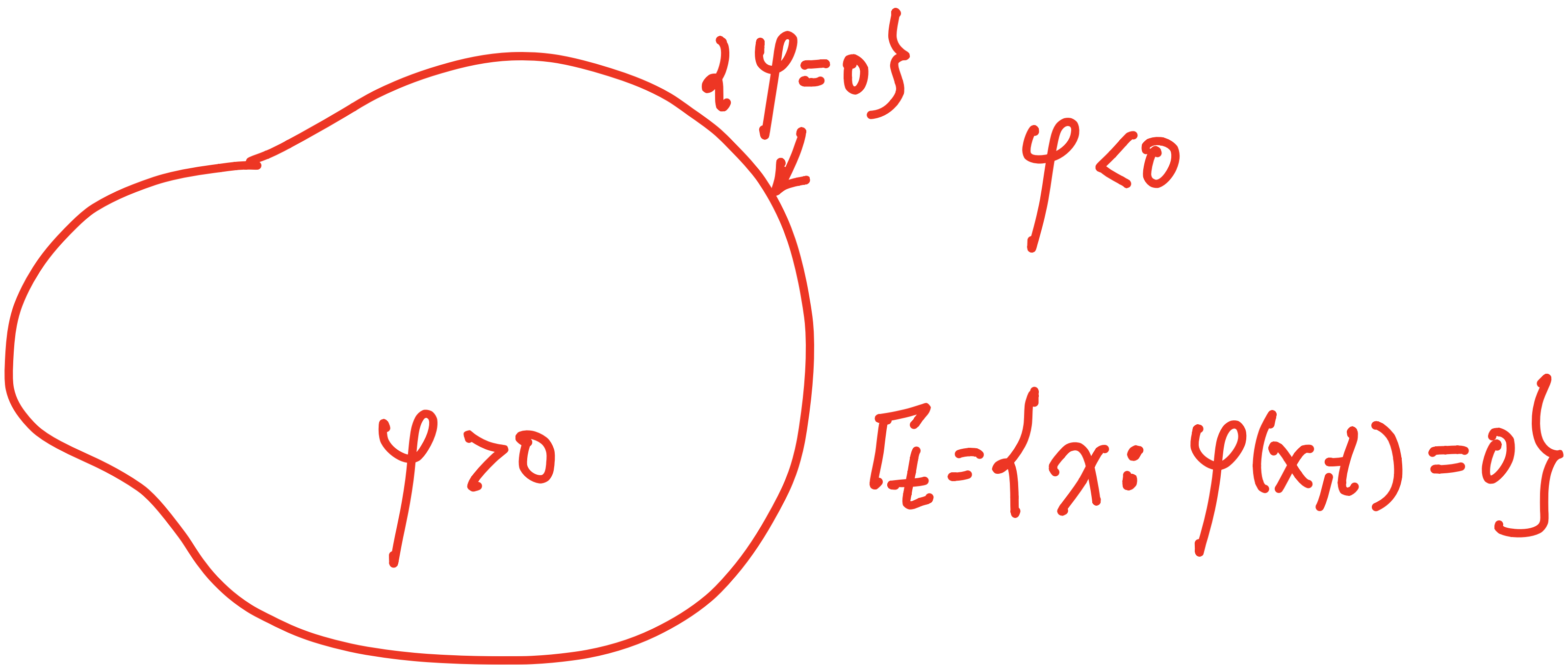
$$\left(1D: \quad u_t = \frac{u_{xx}}{1+u_x^2} + \sqrt{1+u_x^2}\, f(x,u)\right)$$



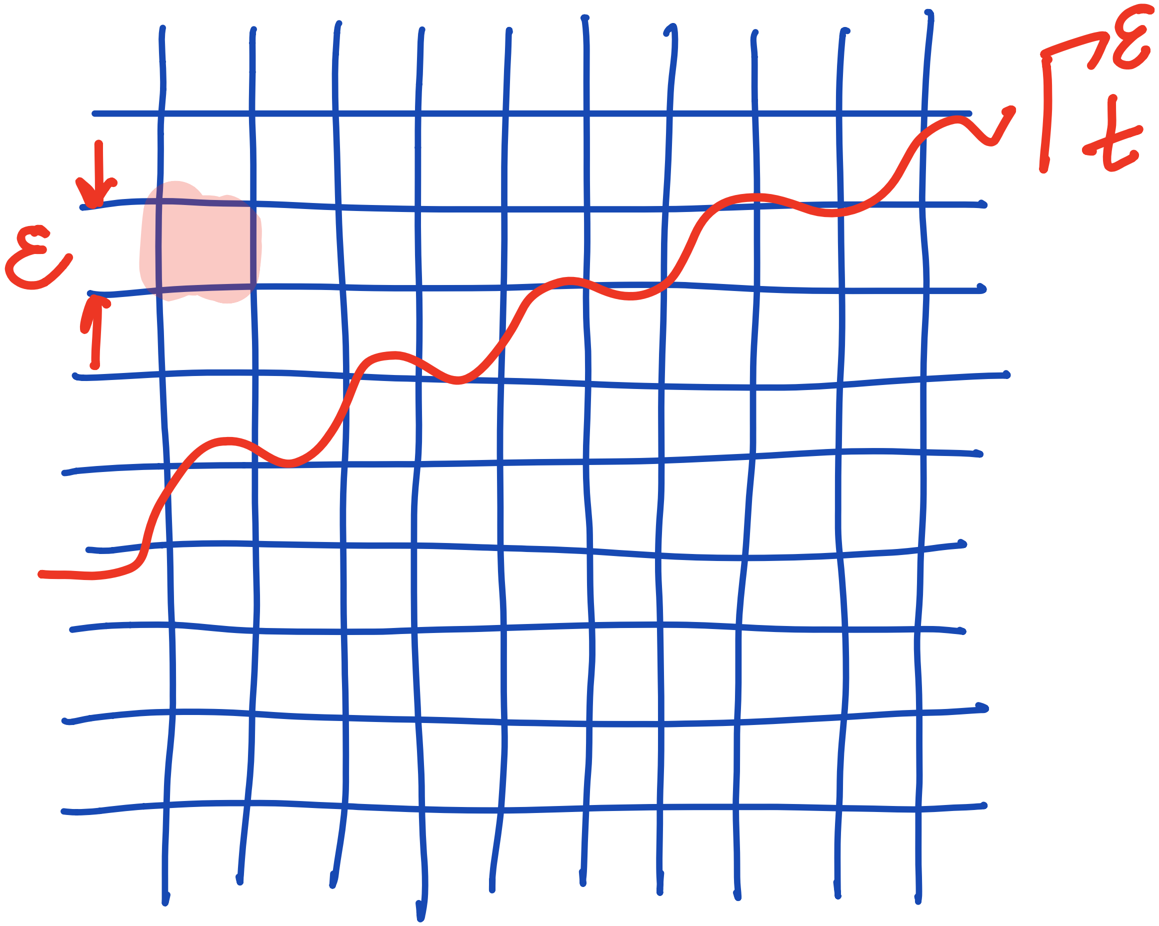
Motion by Mean Curv. in Inhomog. Media

Written in level set form:

$$\partial_t \varphi = |\nabla \varphi| \operatorname{div}\left(\frac{\nabla \varphi}{|\nabla \varphi|}\right) + |\nabla \varphi| f(x)$$



Homogenization of MMC in Inhomog. Media



$\Gamma_\varepsilon \longrightarrow ?\quad \varepsilon \to 0$

① $$\boxed{V = \varepsilon\kappa + f\left(\frac{x}{\varepsilon}\right)}$$

$$\left(u_t = \frac{\varepsilon u_{xx}}{1+u_x^2} + f\left(\frac{x}{\varepsilon}, \frac{u}{\varepsilon}\right)\right)$$

or

② $$\boxed{V = \kappa + \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}\right)}$$

$$\left(u_t = \frac{u_{xx}}{1+u_x^2} + \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}, \frac{u}{\varepsilon}\right)\right)$$

Homogenization of $V = \varepsilon\kappa + f(\frac{x}{\varepsilon})$

$$u_t = \frac{\varepsilon u_{xx}}{1+u_x^2} + \sqrt{1+u_x^2}\, f\left(\frac{x}{\varepsilon}, \frac{u}{\varepsilon}\right)$$

$\varepsilon \to 0$

$$\boxed{\bar{u}_t = c(\hat{n})\sqrt{1+\bar{u}_x^2}}$$

$$\boxed{V = c(\hat{n})} \quad \text{HJE}$$

$$u_t = \varepsilon |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + |\nabla u| f\left(\frac{x}{\varepsilon}\right)$$

$\varepsilon \to 0$

$$\boxed{\bar{u}_t = c(\widehat{\nabla \bar{u}})|\nabla \bar{u}|}$$

(Bhattacharya - Craciun
Cardaliaguet - Lions - Souganidis)

Homogenization of $V = \varepsilon K + f\left(\frac{x}{\varepsilon}\right)$

$$u_t = \frac{\varepsilon u_{xx}}{1+u_x^2} + \sqrt{1+u_x^2}\, f\left(\frac{x}{\varepsilon}, \frac{u}{\varepsilon}\right)$$

Equation for the cell problem

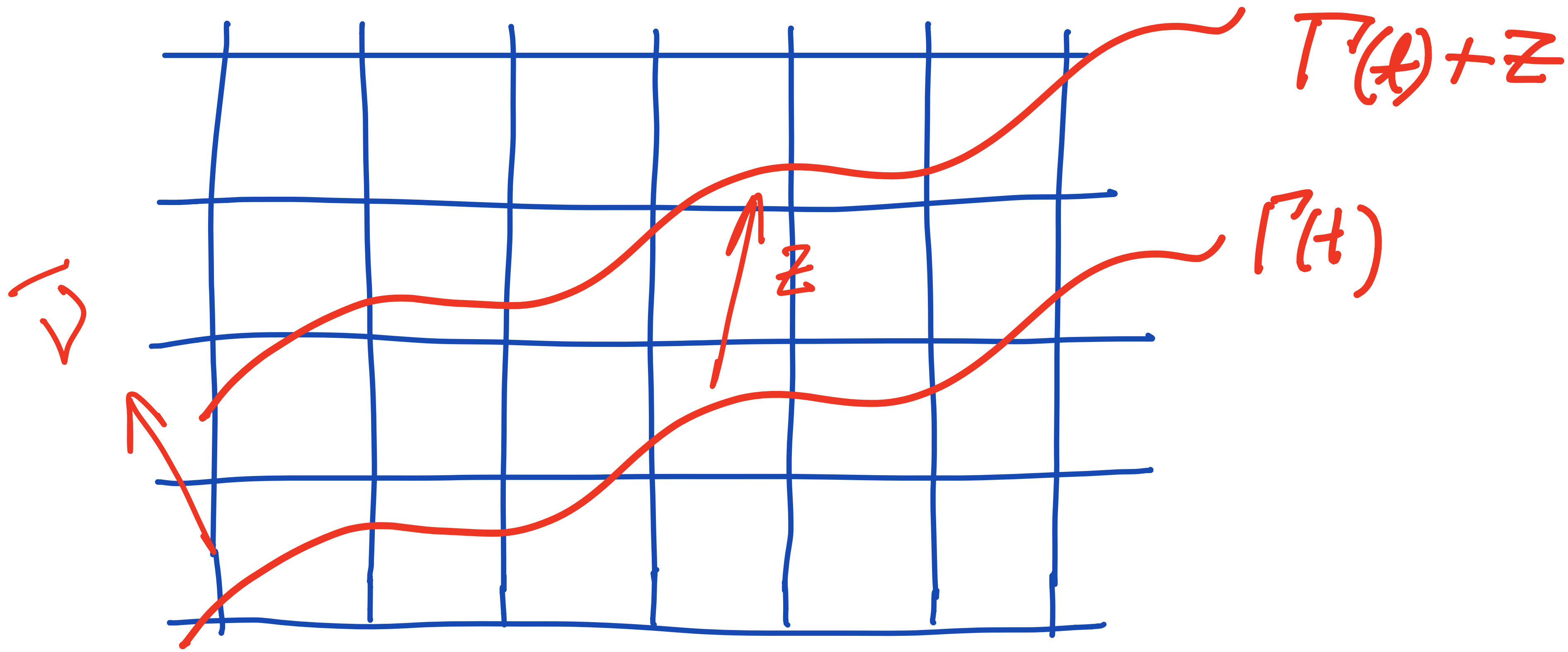
$$\begin{cases} v(y,\tau) = \frac{1}{\varepsilon} u(\varepsilon y, \varepsilon \tau) \\ v_\tau = \frac{v_{yy}}{1+v_y^2} + \sqrt{1+v_y^2}\, f(y,v) \end{cases}$$

$x = \varepsilon y, \; t = \varepsilon \tau$

Look for pulsating wave solutions.

periodic in space + time

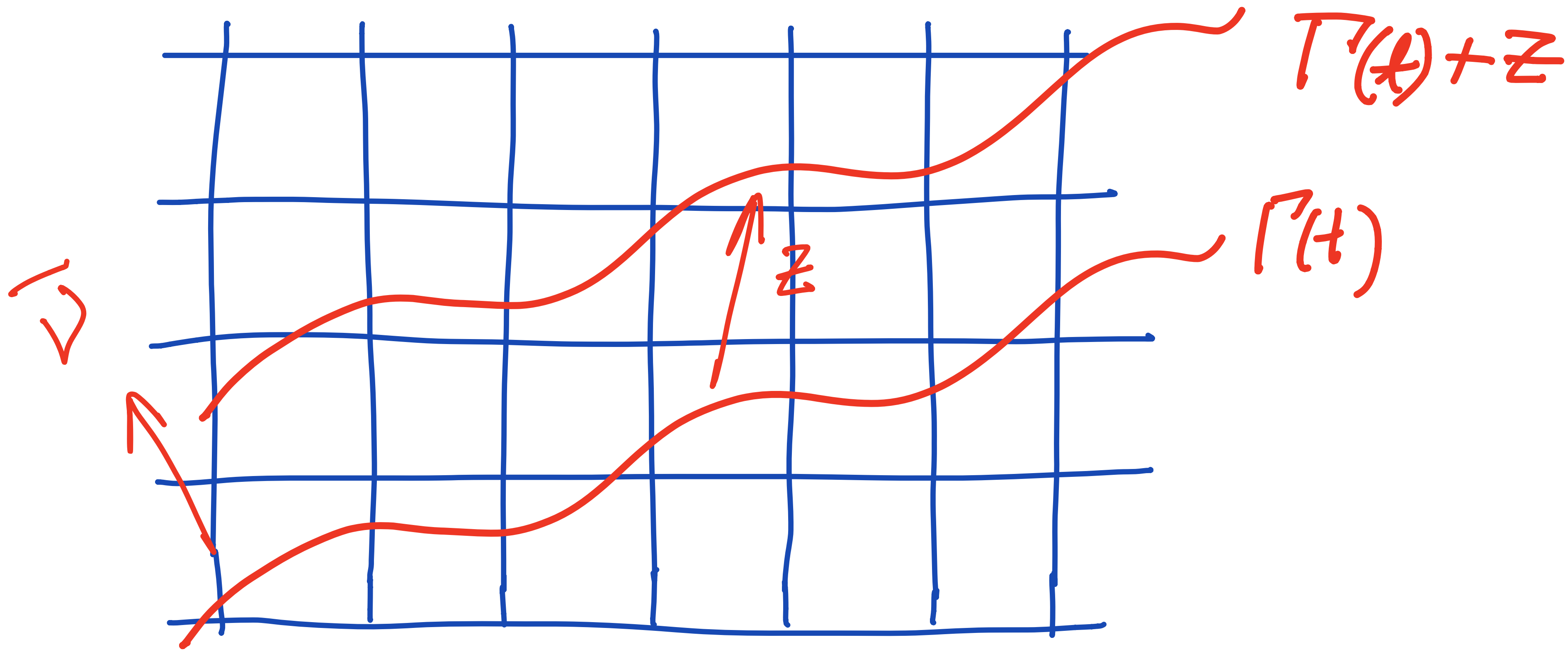
Existence of Pulsating Wave Solution



$$\boxed{\exists\, c_{\bar{\nu}} \text{ s.t. } \quad \Gamma(t)+z=\Gamma\left(t+\frac{\bar{\nu}\cdot z}{c_{\bar{\nu}}}\right)}$$

(space-time periodic solution)

Existence of Pulsating Wave Solution



(1) Cardaliaguet - Lions - Souganidis $\boxed{(f>0)}$

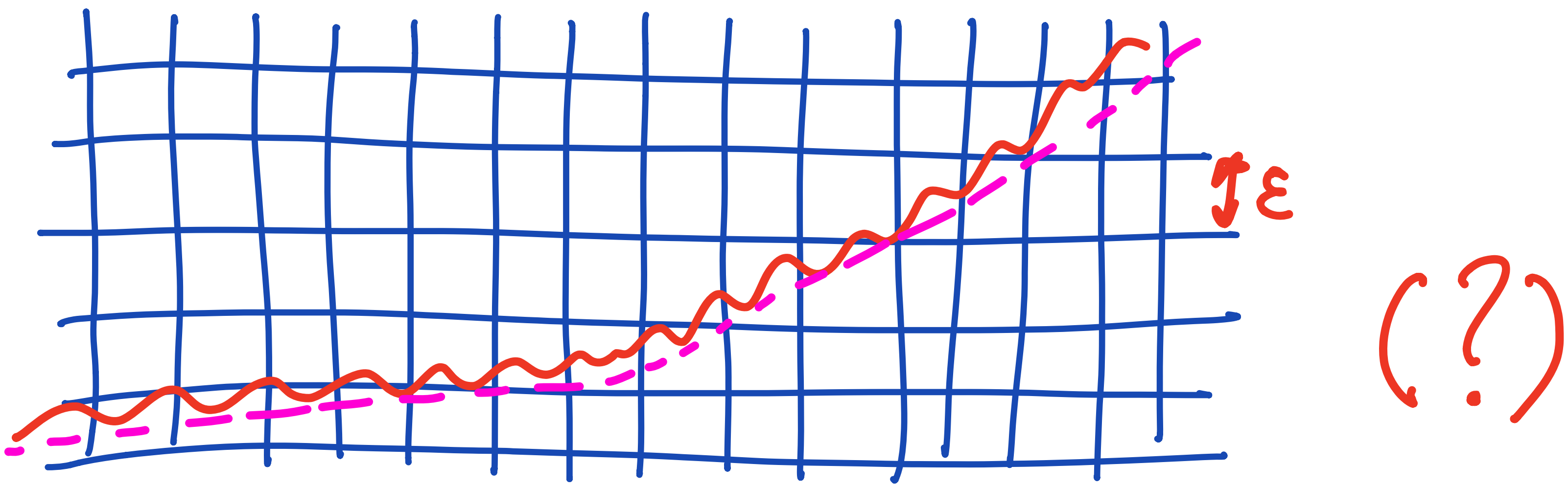
(2) Dirr - Karali - Y. $\boxed{(f \text{ is small})}$

Homogenization of $V = K + \frac{1}{\varepsilon} f(x)$

If $c_{\bar{D}} = 0$, look at longer time scale:

$$\boxed{V = \varepsilon K + f\left(\frac{x}{\varepsilon}\right) \longrightarrow \tilde{V} = K + \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}\right)}$$

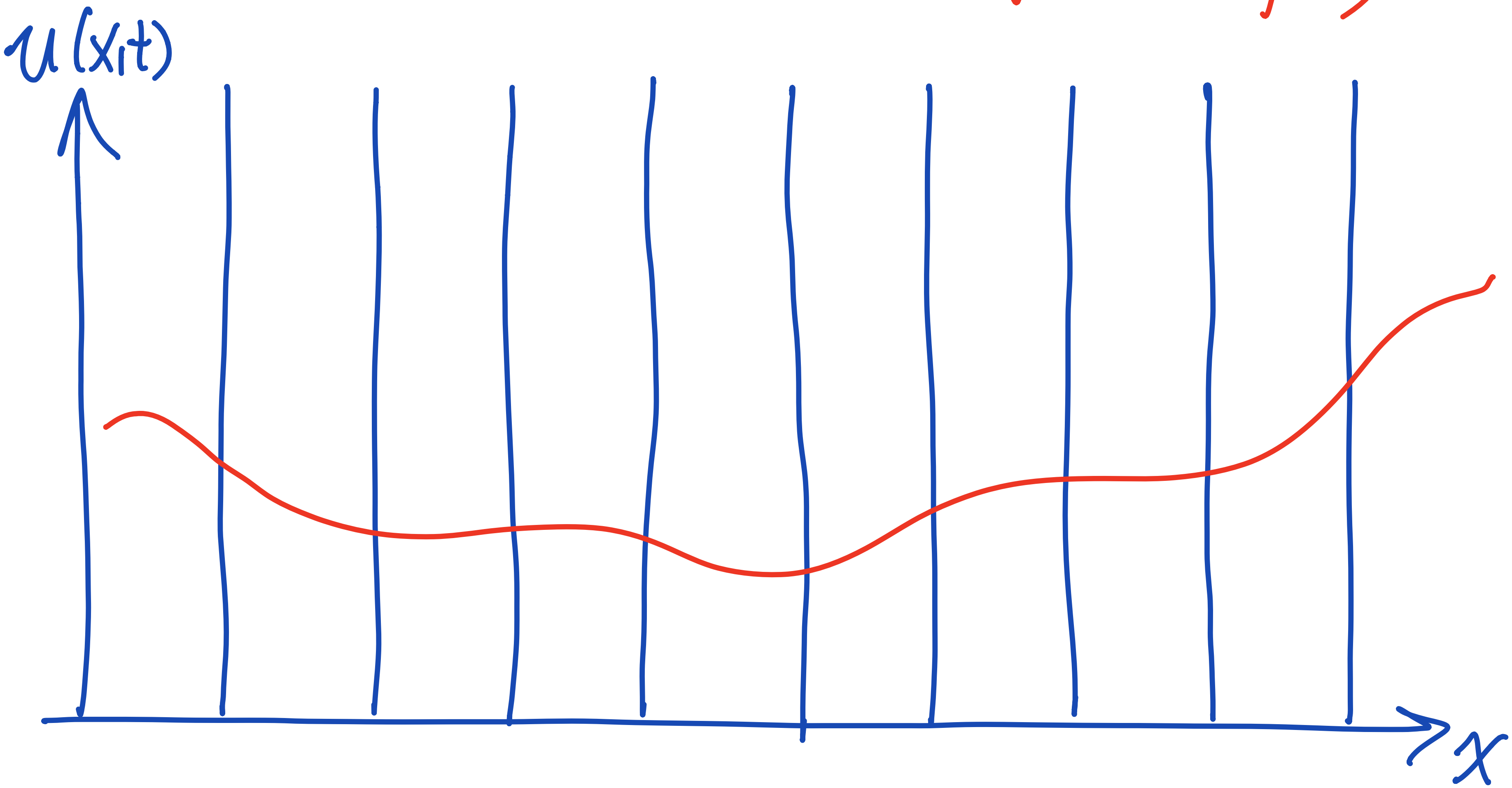
(t) $\qquad\qquad$ $\tilde{t} = \varepsilon t$



Homogenization of $V = K + \frac{1}{\varepsilon} f(x)$

In the case of graph and laminate env.

$f(x,u) = f(x)$



Homogenization of $V = \kappa + \frac{1}{\varepsilon} f(x)$

In the case of graph and laminate env.

$f(x,u) = f(x)$

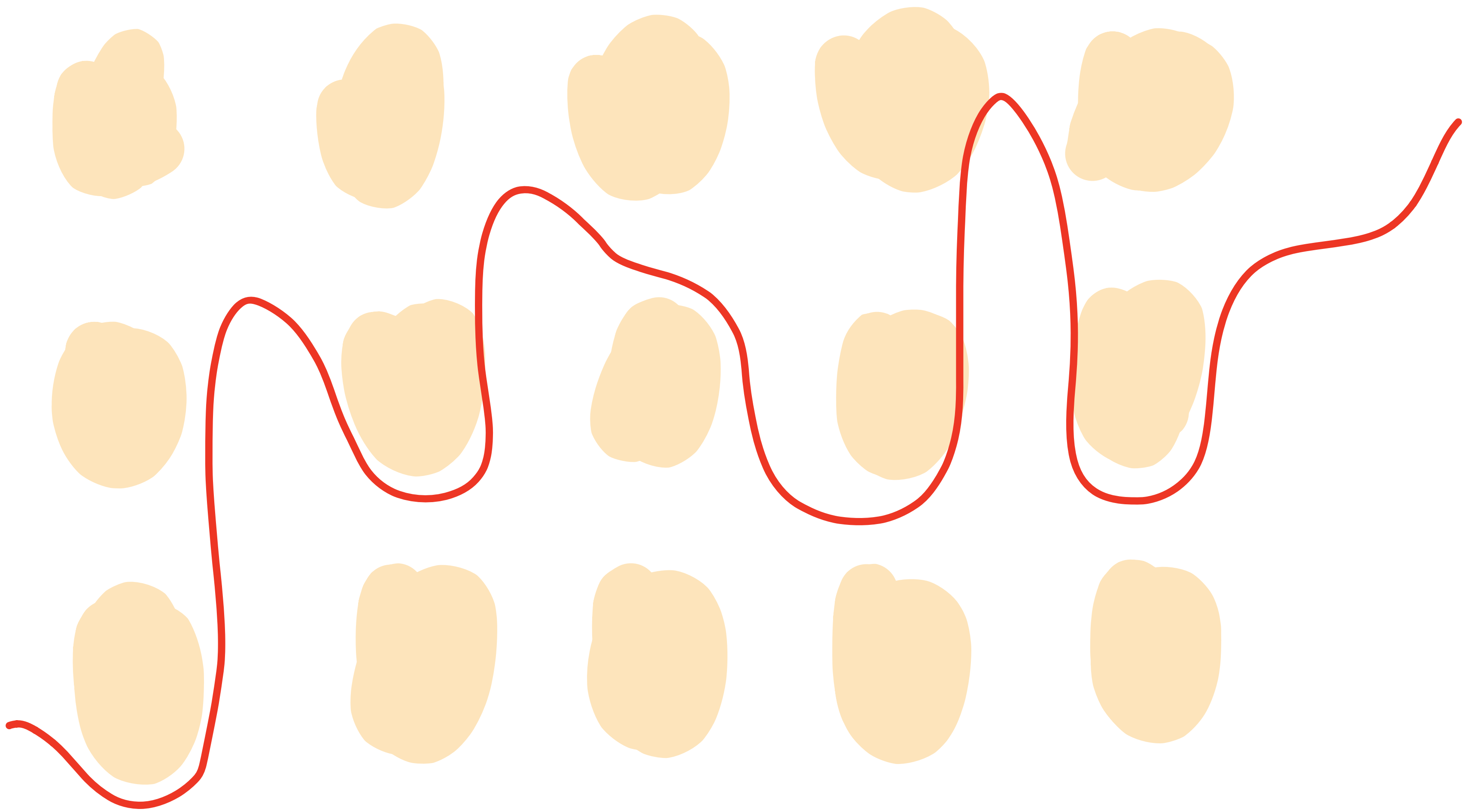
$$\boxed{u_t = \sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + \sqrt{1+|\nabla u|^2}\,\frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}\right)}$$

$\Big\downarrow \varepsilon \to 0$ Barles-Cesaroni-Novaga

$$\boxed{\overline{u}_t = \mathrm{tr}\left(A(Du) D^2 u\right)}$$

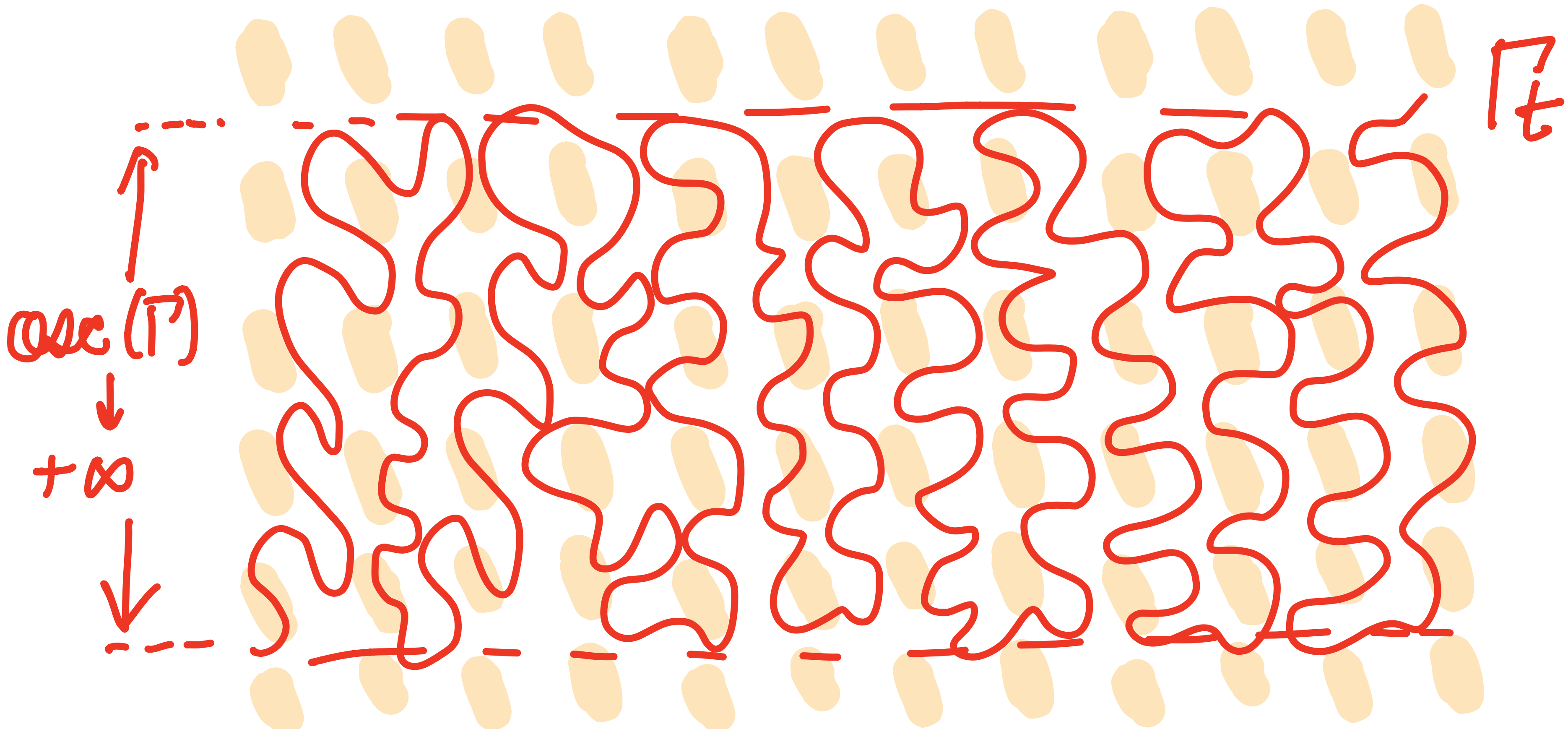
Smallness Condition for f in :

$$\boxed{V = \varepsilon K + f\left(\frac{x}{\varepsilon}\right)}$$



Smallness Condition for f in :

$$\boxed{V = \varepsilon K + f\left(\frac{x}{\varepsilon}\right)}$$



Smallness Condition for f in :

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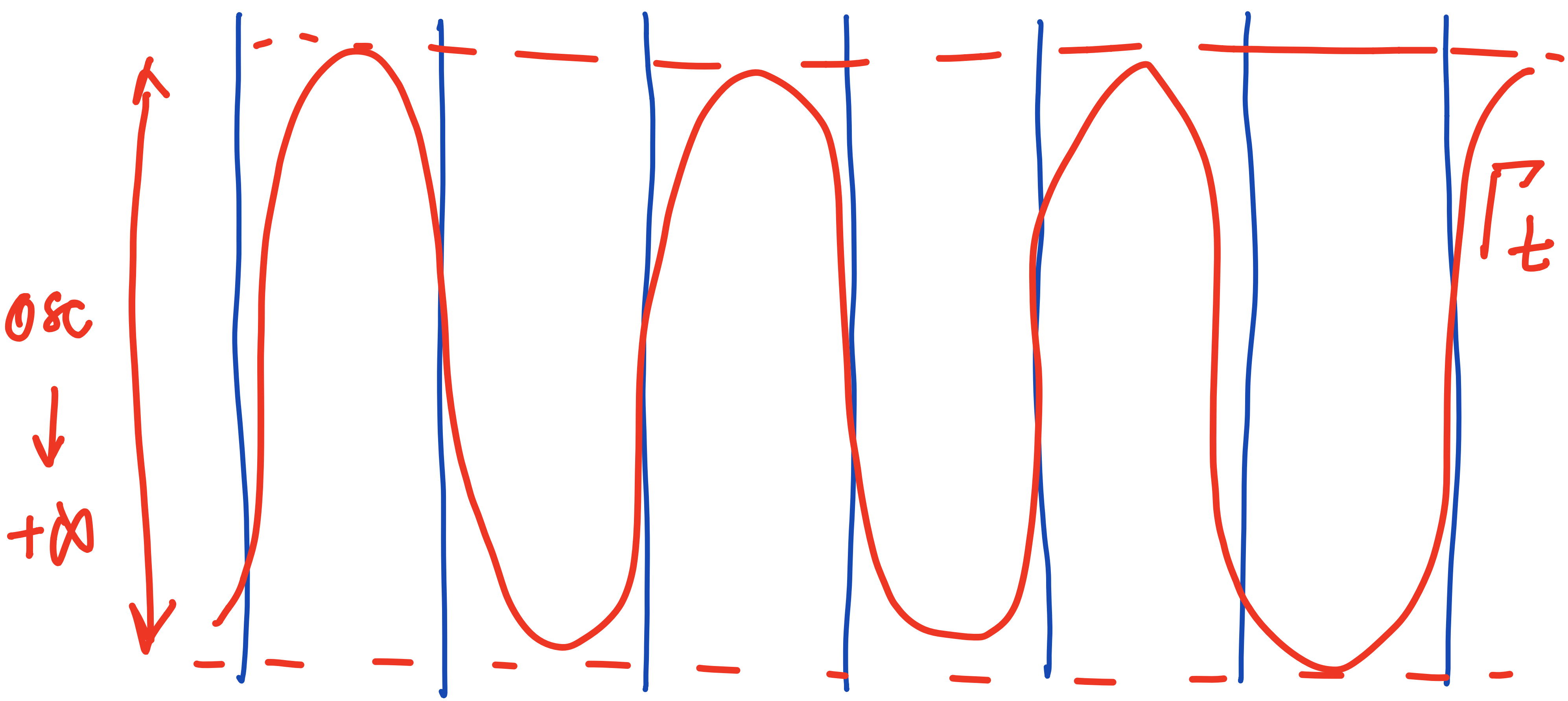
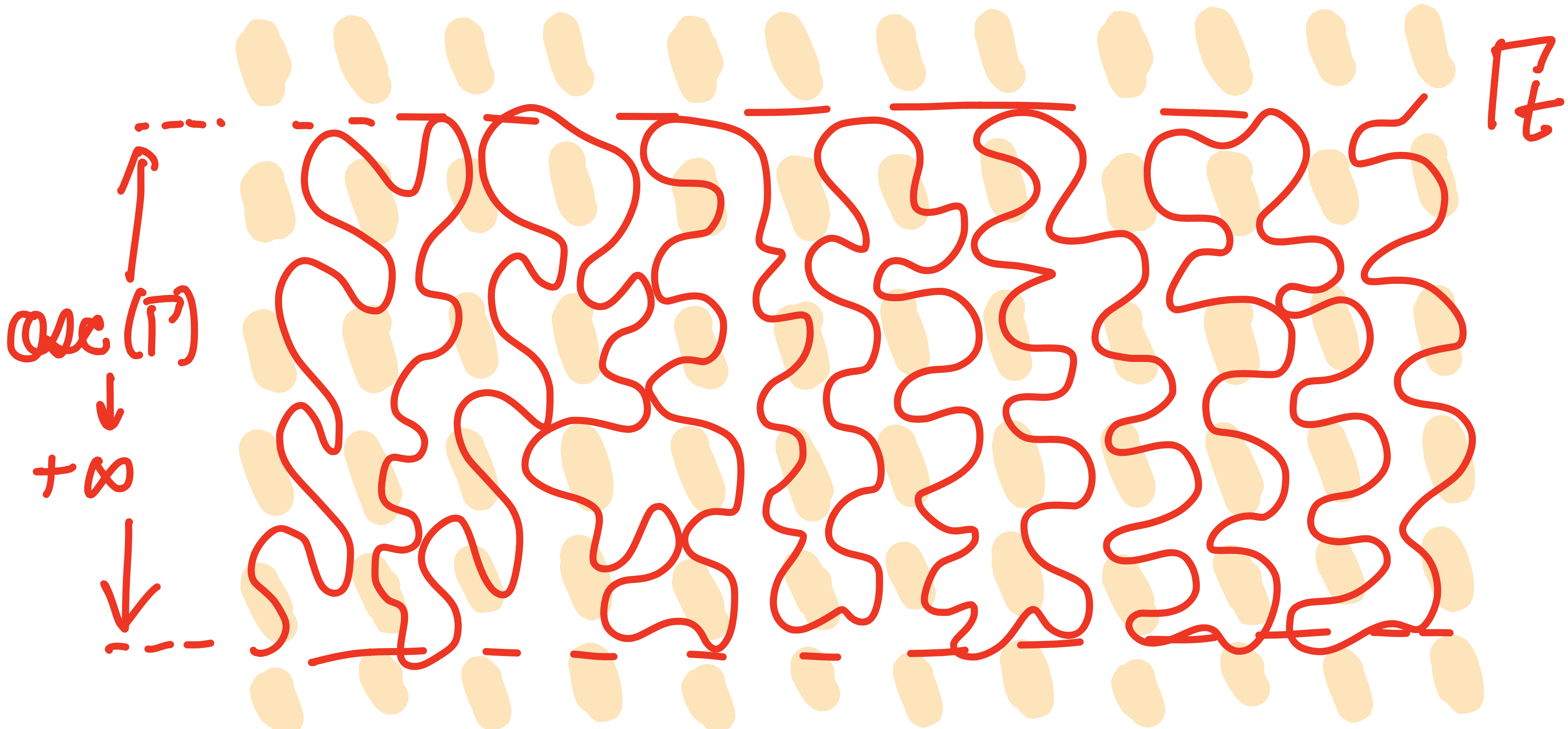
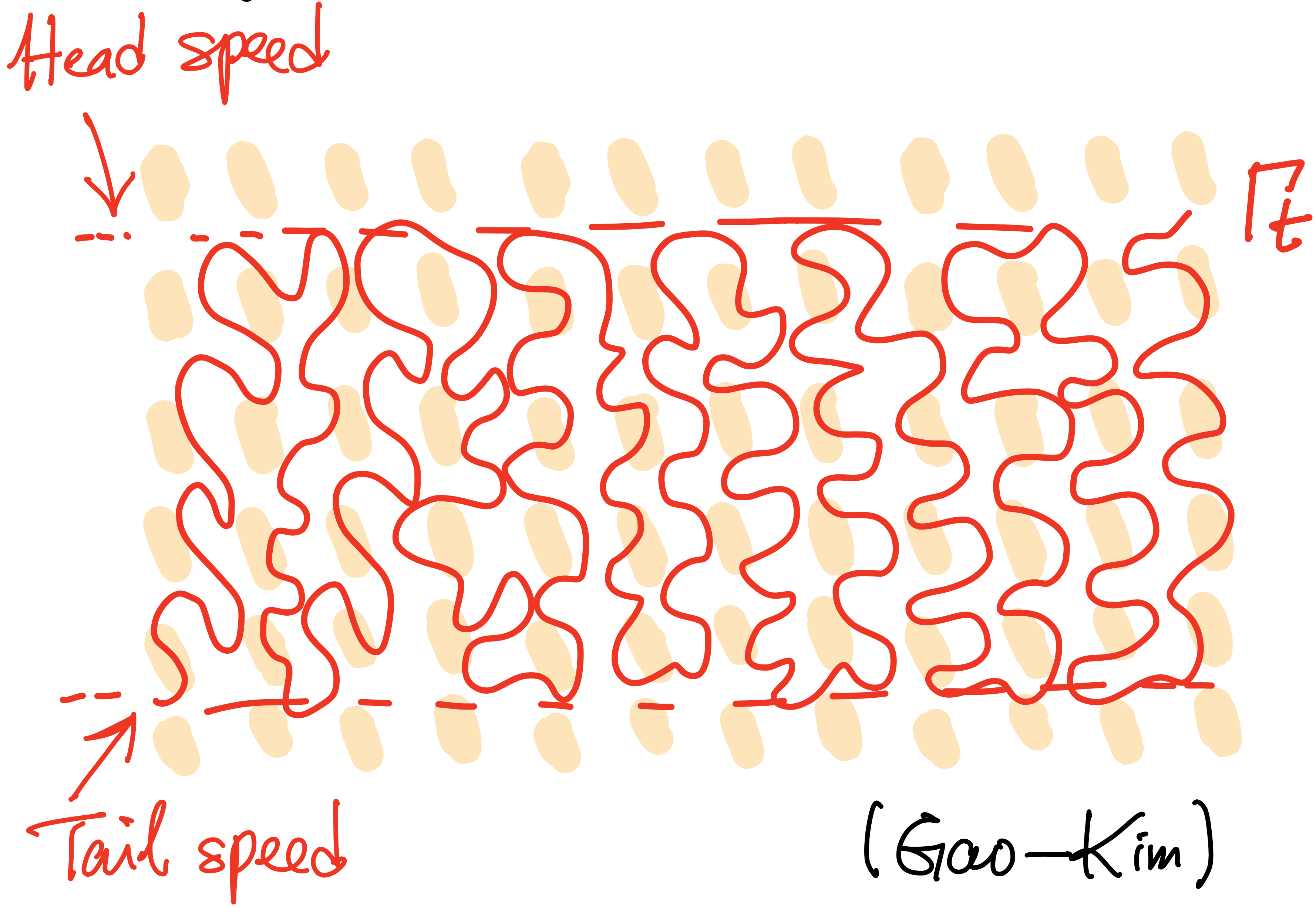


Propagation of Pattens

$$\boxed{V = \varepsilon K + f\left(\frac{x}{\varepsilon}\right)}$$



Propagation of Patterns



(Gao–Kim)

Propagation of Discrete Patterns

(Braides – Cicalese – Y.)

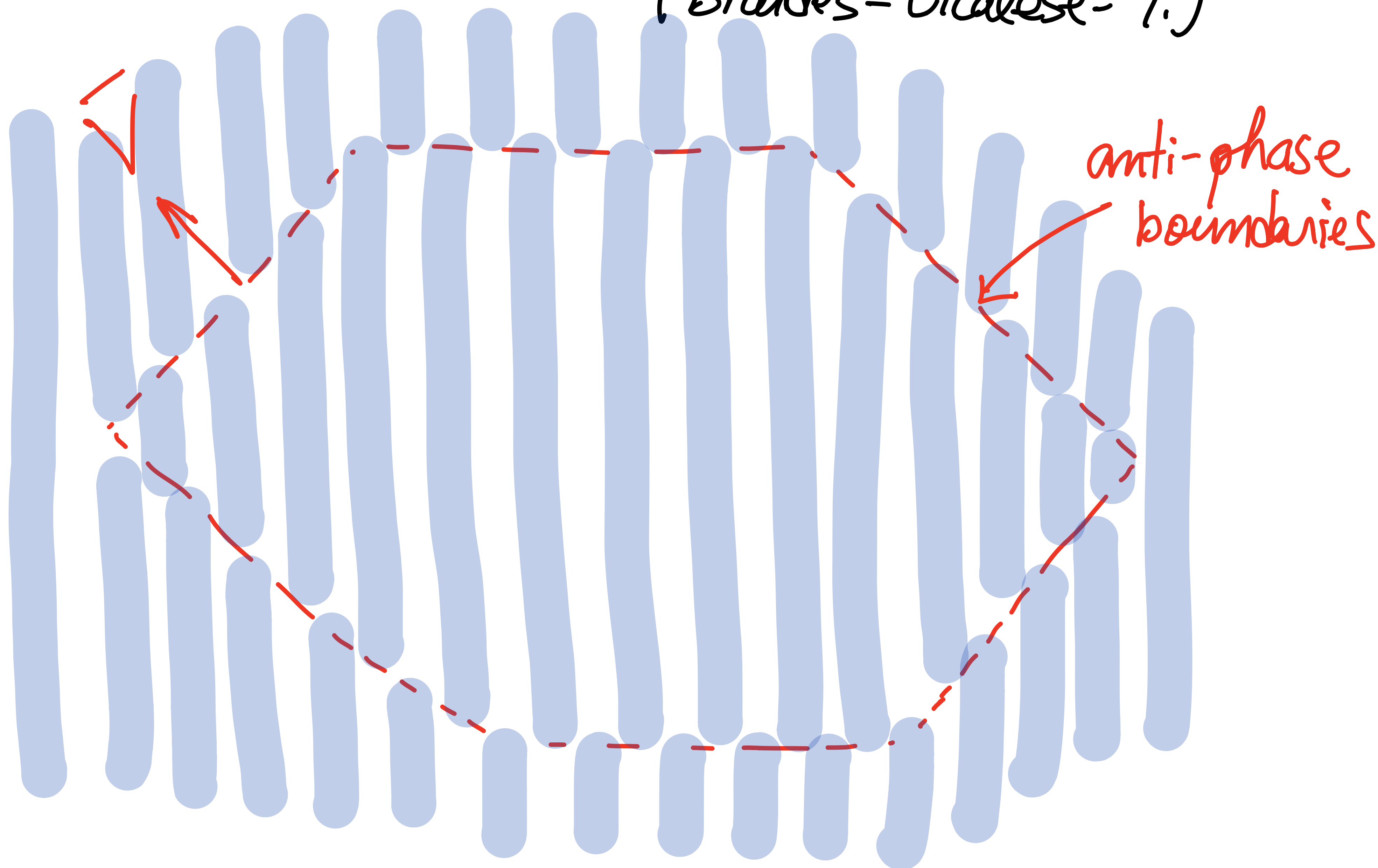
$$\boxed{E_\varepsilon(u) = c_1 \sum_{N.N.} \varepsilon^2 u_i u_j + c_2 \sum_{N.N.N.} \varepsilon^2 u_i u_j}$$

u is a "spin" function : $\varepsilon^2 \mathbb{Z}^2 \longrightarrow \{-1, 1\}$

Ground States:

Propagation of Discrete Patterns

(Braides - Cicalese - Y.)



Energetic Approach / Interpretation of Convergence of Gradient Flow

(Sandier-Serfaty)

$$\boxed{E_\varepsilon : \quad \frac{dX_\varepsilon(t)}{dt} = -\nabla E_\varepsilon(X_\varepsilon(t))}$$

$$\varepsilon \to 0 \ \Big\downarrow \ ?$$

$$\boxed{\overline{E} : \quad \frac{d\overline{X}(t)}{dt} = -\nabla \overline{E}(\overline{X}(t))}$$

Energetic Approach / Interpretation of Convergence of Gradient Flow

Look at energy dissipation:

$$\frac{d}{dt} E_\varepsilon(X_\varepsilon) = \nabla E_\varepsilon(X_\varepsilon) \cdot (X_\varepsilon)_t$$

$$= -|\nabla E_\varepsilon(X_\varepsilon)|\,|(X_\varepsilon)_t|$$

$\left(X_{\varepsilon_t} = -\nabla E_\varepsilon(X_\varepsilon) \right) \rightarrow$

$$= -\frac{1}{2}\left[|\nabla E_\varepsilon(X_\varepsilon)|^2 + |X_{\varepsilon_t}|^2 \right]$$

Energetic Approach / Interpretation of Convergence of Gradient Flow

Look at energy dissipation:

$$\frac{d}{dt} E_\varepsilon(X_\varepsilon) = \nabla E_\varepsilon(X_\varepsilon) \cdot (X_\varepsilon)_t$$

$$\leq -|\nabla E_\varepsilon(X_\varepsilon)|\,|(X_\varepsilon)_t|$$

$$\left(X_{\varepsilon_t} = -\nabla E_\varepsilon(X_\varepsilon) \right) \longrightarrow$$

$$\leq -\frac{1}{2}\left[|\nabla E_\varepsilon(X_\varepsilon)|^2 + |X_{\varepsilon_t}|^2 \right]$$

Energetic Approach / Interpretation of Convergence of Gradient Flow

Look at energy dissipation:

$$E_\varepsilon(X_\varepsilon(t)) + \int_0^t \frac{1}{2}\left(|\nabla E_\varepsilon(X_\varepsilon)|^2 + |X_{\varepsilon s}|^2\right) ds \le E_\varepsilon(X_\varepsilon(0))$$

(The above in fact is equiv. to $\dot{X}_\varepsilon = -\nabla E_\varepsilon(X_\varepsilon)$)

Energetic Approach / Interpretation of Convergence of Gradient Flow

Look at energy dissipation:

$$E_\varepsilon(X_\varepsilon(t)) + \int_0^t \frac{1}{2}\left(|\nabla E_\varepsilon(X_\varepsilon)|^2 + |X_{\varepsilon s}|^2\right) ds \leq E_\varepsilon(X_\varepsilon(0))$$

$$\left(\vec{a}\cdot\vec{b} \leq -\frac{1}{2}|\vec{a}|^2 - \frac{1}{2}|\vec{b}|^2 \iff \vec{a} = -\vec{b}\right)$$

Energetic Approach / Interpretation of Convergence of Gradient Flow

Look at energy dissipation:

If

$$X_\varepsilon(\cdot) \longrightarrow \bar{X}(\cdot)$$

$$E_\varepsilon(X_\varepsilon(\cdot)) \longrightarrow \bar{E}(\bar{X}(\cdot))$$

$$\nabla E_\varepsilon(X_\varepsilon(\cdot)) \longrightarrow \nabla \bar{E}(\bar{X}(\cdot))$$

Energetic Approach / Interpretation of Convergence of Gradient Flow

Look at energy dissipation:

If

$$X_\varepsilon(\cdot) \longrightarrow \bar{X}(\cdot) \qquad \text{Compactness}$$

$$E_\varepsilon(X_\varepsilon(\cdot)) \longrightarrow \bar{E}(\bar{X}(\cdot)) \qquad \Gamma\text{- Convergence}$$

$$\nabla E_\varepsilon(X_\varepsilon(\cdot)) \longrightarrow \nabla \bar{E}(\bar{X}(\cdot)) \qquad \text{"}C^1\text{"} - \Gamma\text{-conv}$$

Energetic Approach / Interpretation of Convergence of Gradient Flow

Look at energy dissipation:

then, by LSC,

$$\underline{\lim} \, |\dot{X}_\varepsilon|^2 \geq |\dot{\bar{X}}|^2$$

$$\underline{\lim} \, |\nabla E_\varepsilon (X_\varepsilon(\cdot))|^2 \geq |\nabla \bar{E}(\bar{X}(\cdot))|^2$$

Energetic Approach / Interpretation of Convergence of Gradient Flow

Look at energy dissipation:

and hence:

$$\bar{E}(\bar{X}(t)) + \int_0^t \frac{1}{2}\left(|\nabla \bar{E}(\bar{X}(s))|^2 + |\dot{\bar{X}}|^2\right) ds \leq \bar{E}(\bar{X}(0))$$

(which implies: $\dot{\bar{X}} = -\nabla \bar{E}(\bar{X})$)

Thank You