How Can Random Noise Help Us

Global Transports from Thermal Fluctuations

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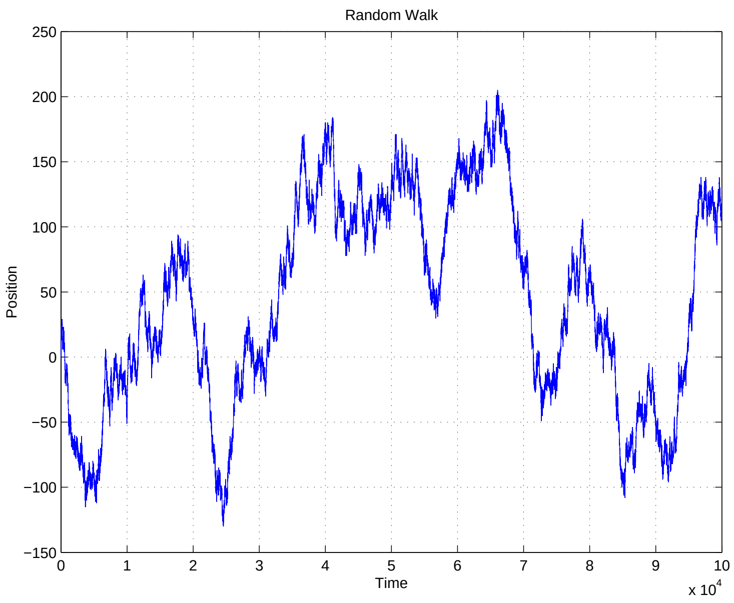
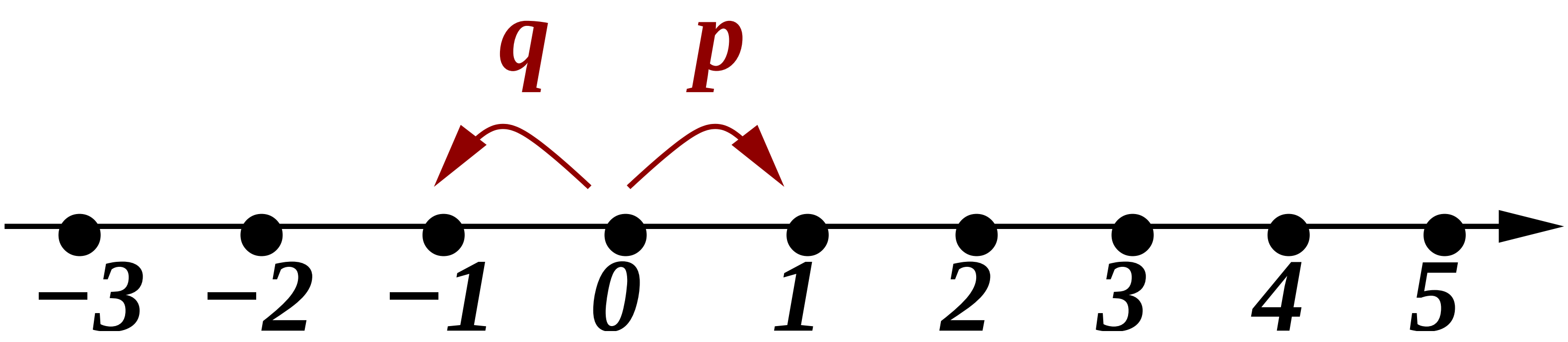


Basics of Random Walk

i.e. Drunken Walker

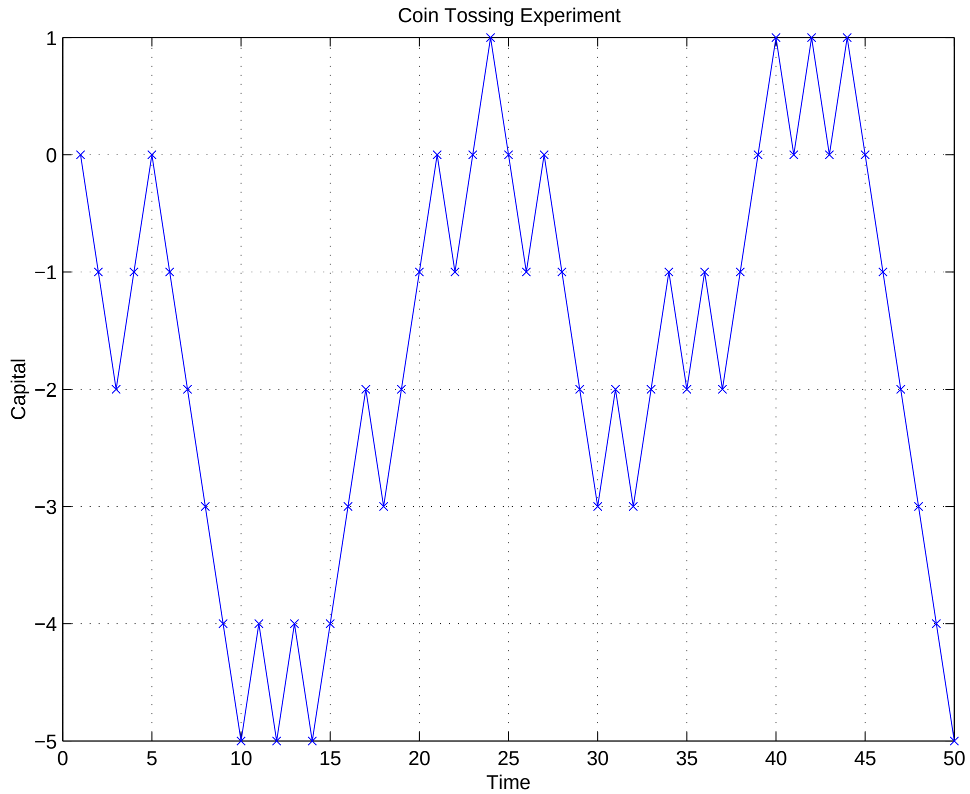
Start from the origin, to decide where to go next, you **toss a coin**, if the outcome is a **head**, you move one step to the **right**, if it is a **tail**, you move to the **left**.



Let $X_i = 1$ or -1 if the i-th toss is a head or tail. Then your **position** after time n is given by:

$$S_n = X_1 + X_2 + \cdots X_n$$

Basics of Random Walk – 2



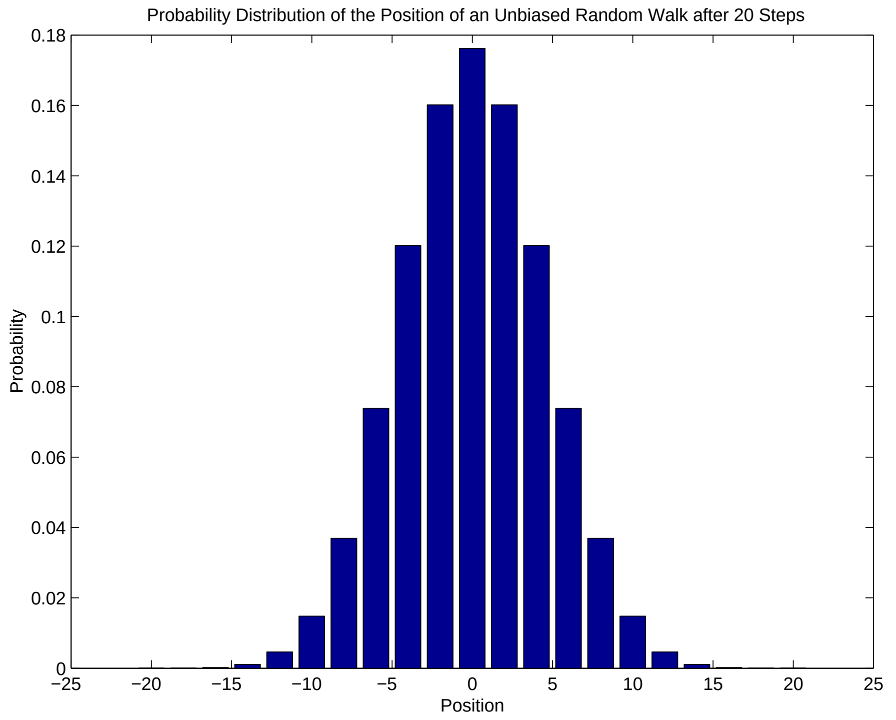
Suppose **head** comes up with **probability** p and **tail** with **probability** $q = 1 - p$. (If $p = q = \frac{1}{2}$, the coin is called **fair**.)

What is the **statistics** of your position, X_n (at time n)?

Binomial Distribution

$$\text{Prob}(S_n = j) = \binom{n}{\frac{n+j}{2}} p^{\frac{n+j}{2}} q^{\frac{n-j}{2}}$$

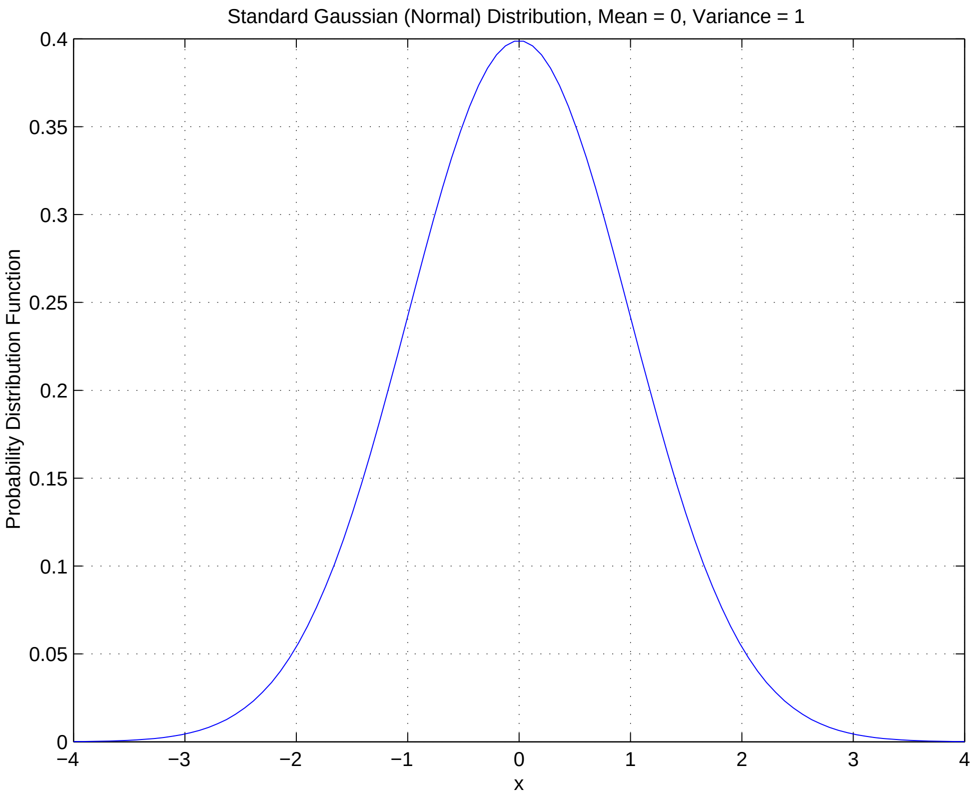
where $\binom{n}{m} = \frac{n(n-1)(n-2)\cdots(n-m+1)}{m(m-1)(m-2)\cdots(2)(1)} = \frac{n!}{m!(n-m)!}$



Gaussian (Normal) Distribution

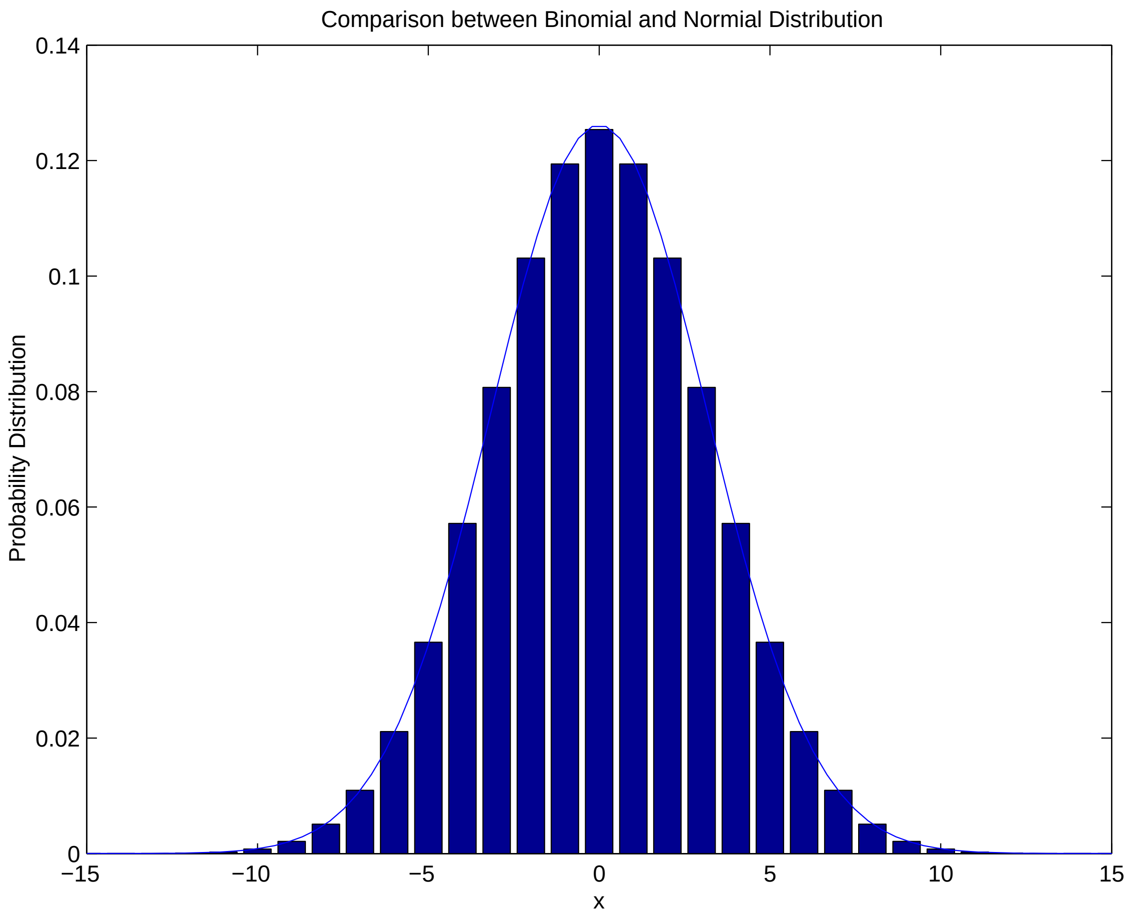
$$G_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Mean $= \mu$, **Variance** $= \sigma^2$



Binomial converges to Gaussian Distribution

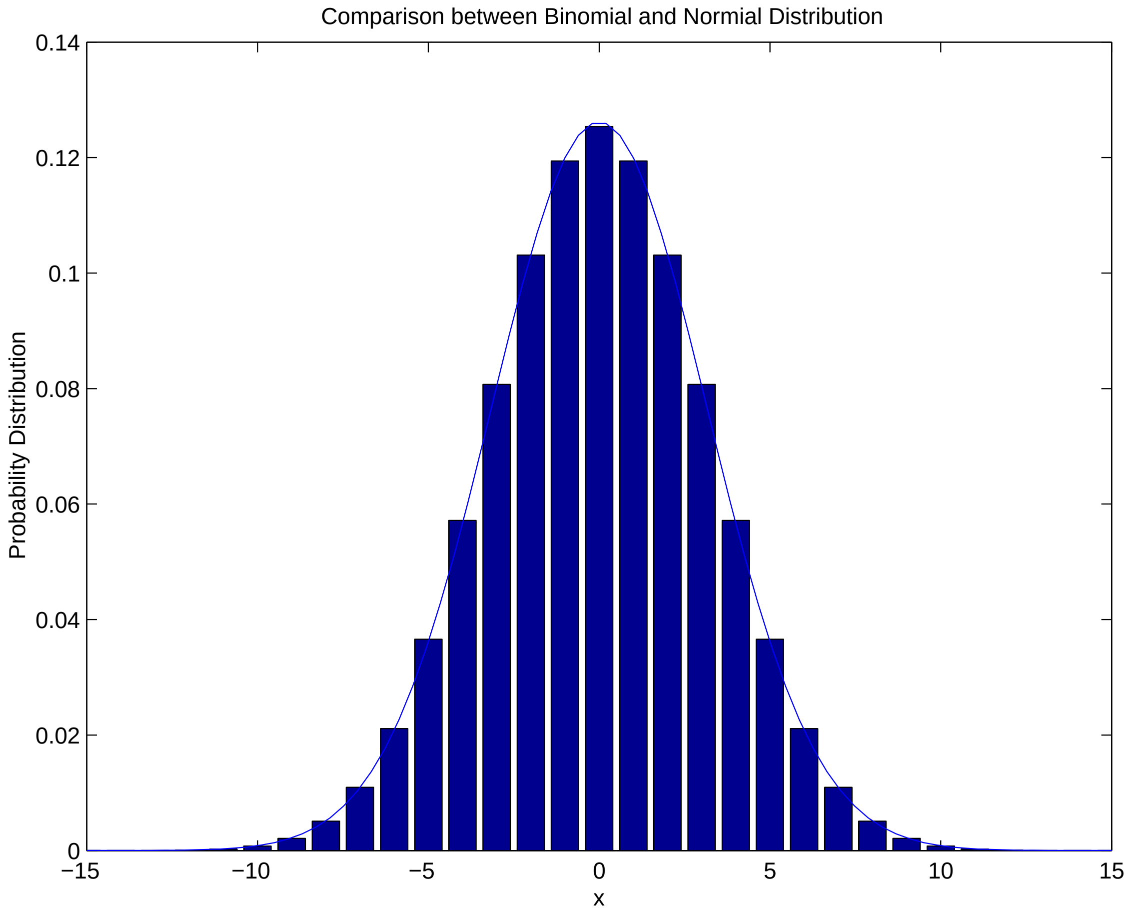
As $n \to \infty$,



What is this **Theorem** called?

Binomial converges to Gaussian Distribution

As $n \to \infty$,



Central Limit Theorem

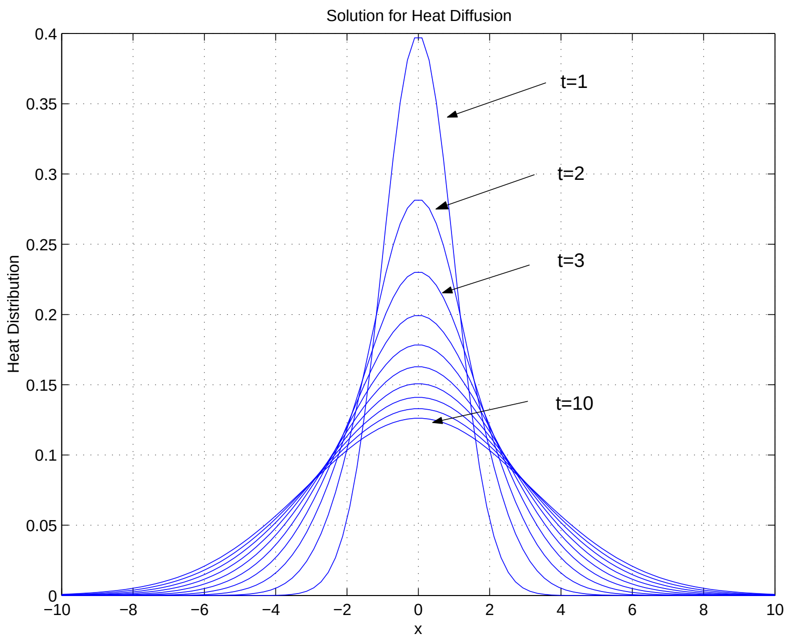
Gaussian Distribution and Heat Diffusion

Heat Equation – describes the **diffusion of heat**:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}; \qquad u(x,0) = \delta(x)$$

The solution is given by: $u(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$

— **Gaussian Distribution, mean zero and variance $\sqrt{t}$**

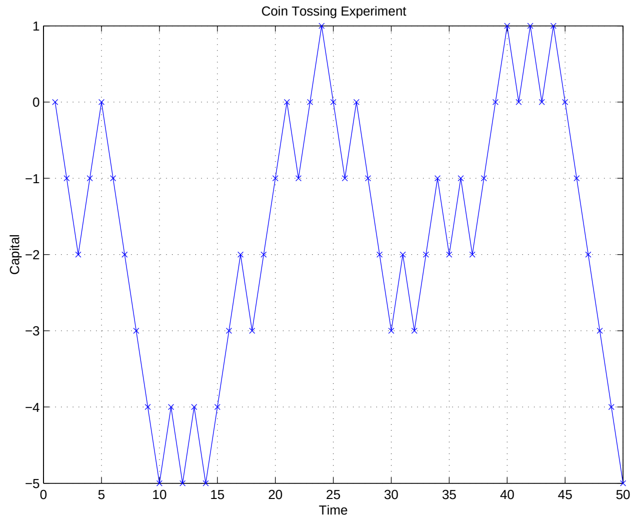


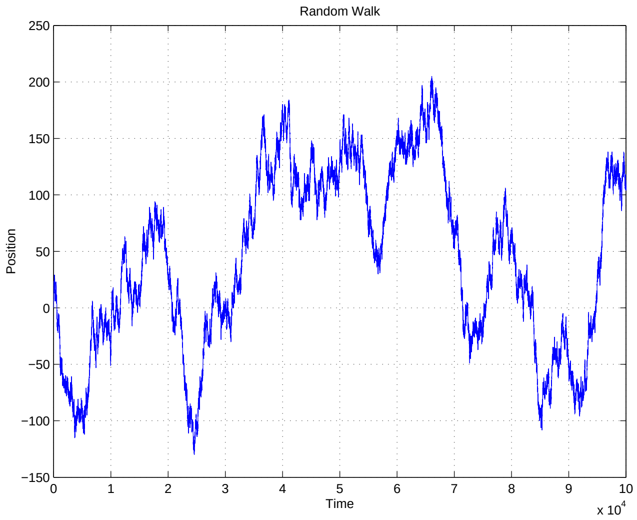
Random Walk and Brownian Motion

Let the **time step** $\triangle t$ and **spatial step** $\triangle x$ converge to zero in such a way that

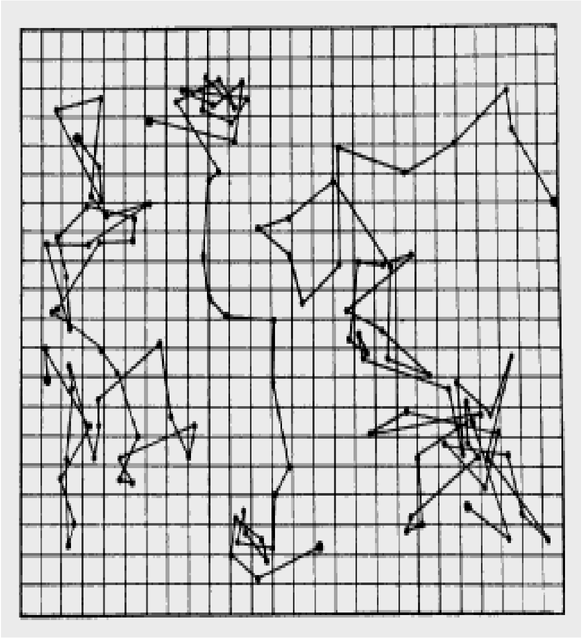
$$\frac{(\triangle x)^2}{\triangle t} \longrightarrow \text{Constant}$$

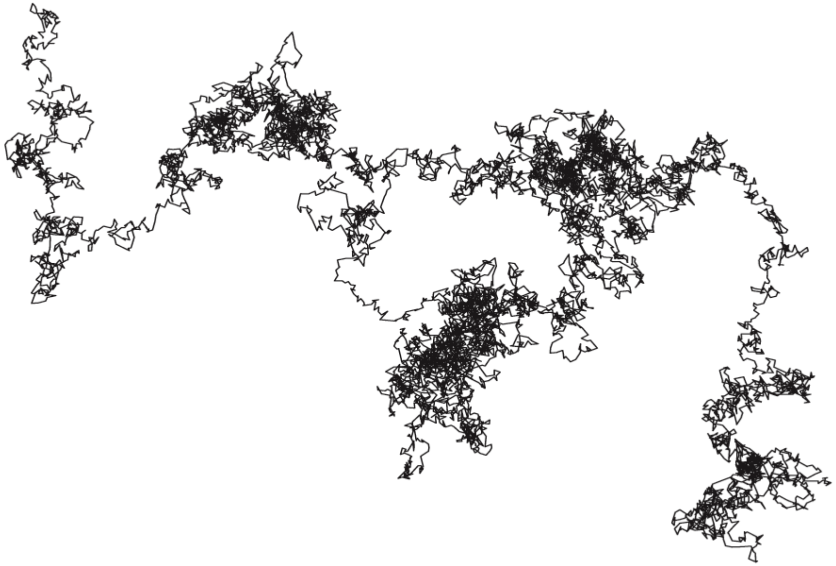
then **Random Walk** can be described by **Brownian Motion**.





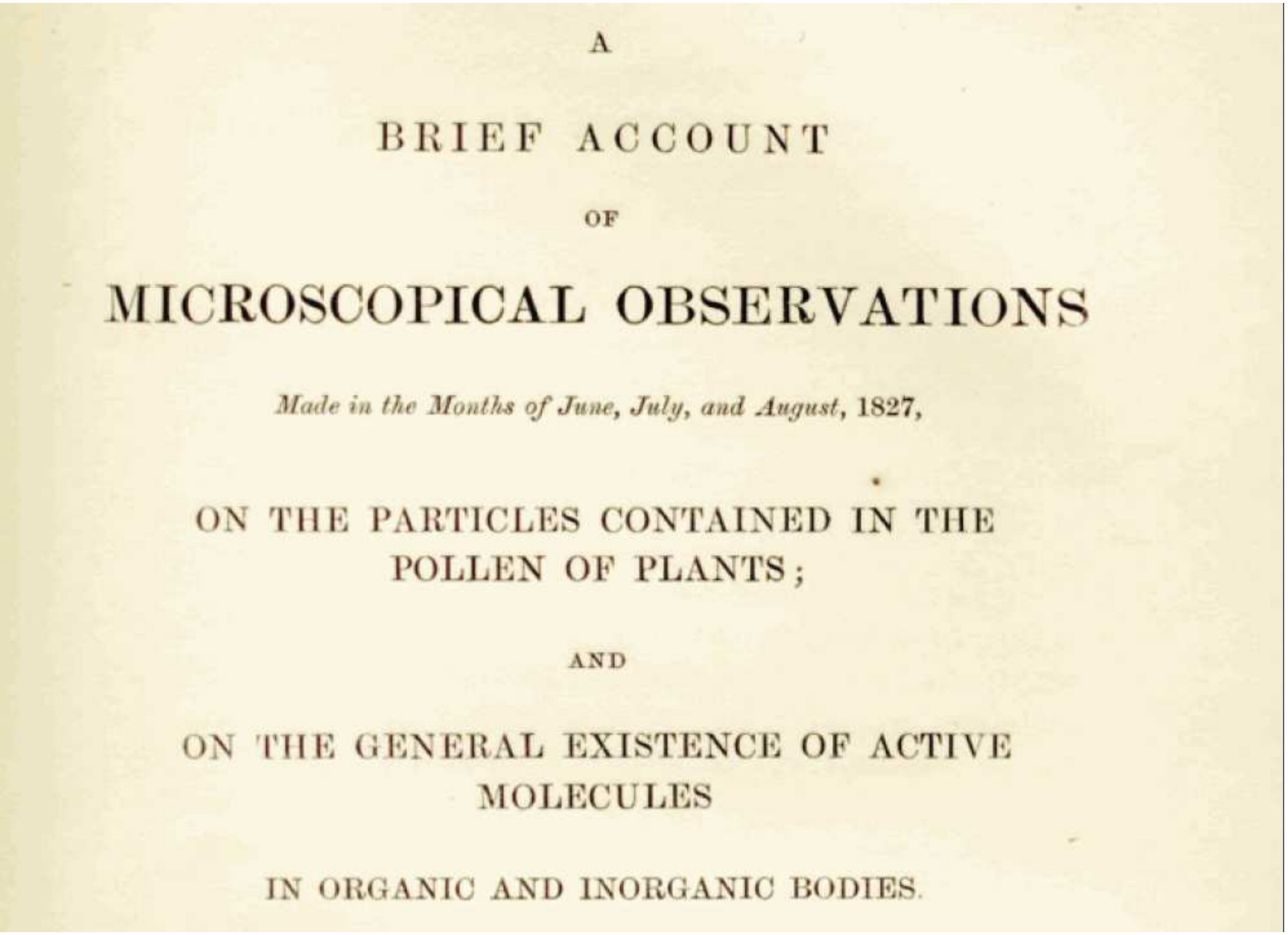
Random Walk and Brownian Motion – 2





Brownian Motion – Some Historical Notes

Robert Brown observed the random motion of **pollens in water**.



A

BRIEF ACCOUNT

OF

MICROSCOPICAL OBSERVATIONS

Made in the Months of June, July, and August, 1827,

ON THE PARTICLES CONTAINED IN THE
POLLEN OF PLANTS;

AND

ON THE GENERAL EXISTENCE OF ACTIVE
MOLECULES

IN ORGANIC AND INORGANIC BODIES.

Brownian Motion – Some Historical Notes – 2

Albert Einstein made use of the concept of Brownian motion to estimate the **size** of solute molecules in a solvent.

INVESTIGATIONS ON
THE THEORY OF THE
BROWNIAN MOVEMENT
BY
ALBERT EINSTEIN, PH.D.

one of the **five** fundamental papers Einstein wrote in 1905:

Einstein's Miraculous Year:
Five Papers that Changed the Face of Physics.

Princeton University Press, 2005, John Stachel (ed.)

Brownian Motion – Some Historical Notes – 3

- The 1926 Nobel Prize in physics was received by **J. Perrin** for his precise determination of the **Avogadro's number**. He compared the experimentally measured diffusivity of spherical granules in water with the **Stokes-Einstein** formula for the **diffusion coefficient**:

$$D = \frac{k_B T}{6\pi\eta R}, \quad \text{and} \quad D = \lim_{t\to\infty} \frac{\langle X(t)^2 \rangle}{2t}$$

- Other contributors to the understanding of Brownian Motion: **Smoluchowski; Langevin; Ornstein-Ohlenback; Wiener; Levy; ....**

Gambler Ruin Problem

Given initial capital C, between 0 and A, you keep on playing until your capital becomes 0 or A.

If $p = \frac{1}{2}$ (i.e. the game is fair), then

$$\textbf{Prob(Win)} = \frac{C}{A}, \quad \textbf{Prob(Lose)} = 1 - \frac{C}{A}$$

The **expected duration** of the game, either lose or win is:

$$C(A - C)$$

If $p \neq \frac{1}{2}$, the above formulas become ($z = \frac{q}{p}$)

$$\textbf{Prob(Win)} = \frac{z^C - 1}{z^A - 1}, \quad \textbf{Prob(Lose)} = \frac{z^A - z^C}{z^A - 1}$$

$$\textbf{Expected duration} = \left(\frac{C}{q-p}\right) - \left(\frac{A}{q-p}\right)\left(\frac{1-z^C}{1-z^A}\right)$$

Winning by Losing Strategies

Parrondo's Paradox. G. P. Harmer, D. Abbott, *Nature, 1999*

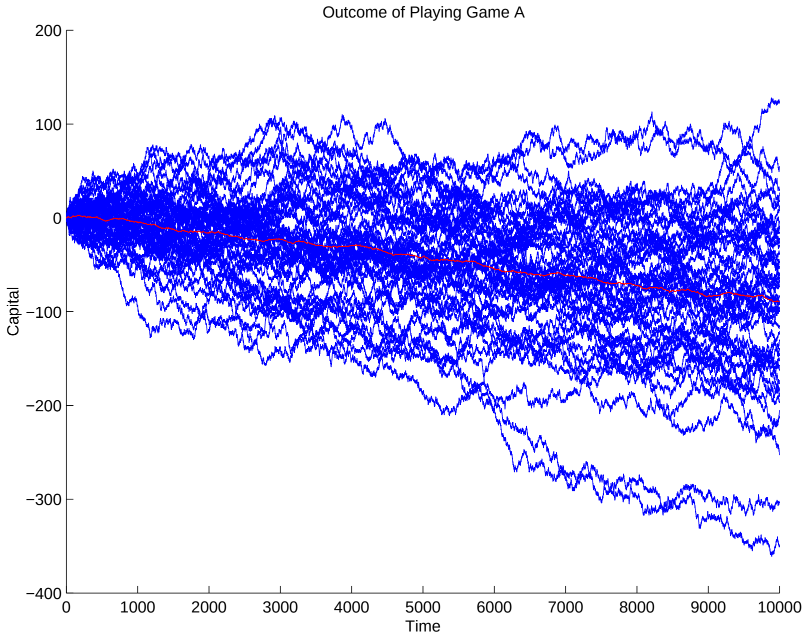
Now you can choose two different games to play:

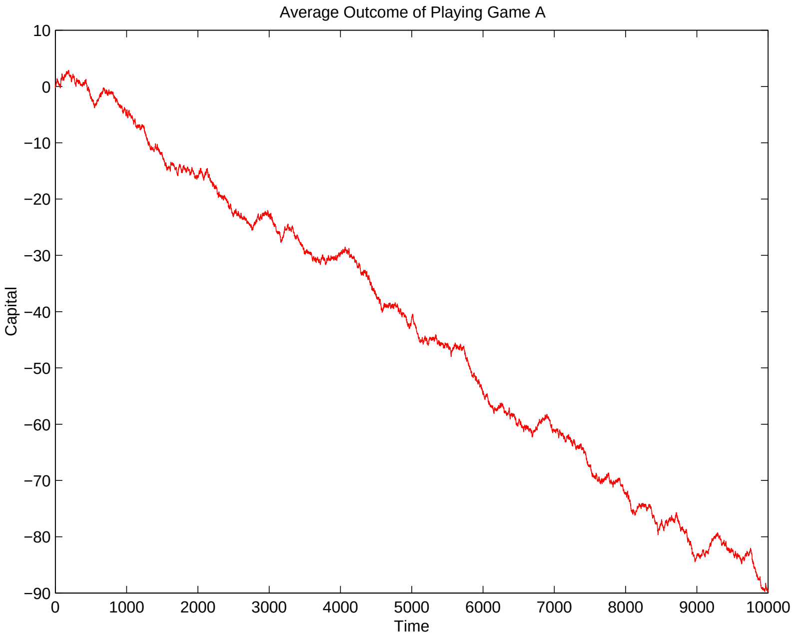
- **Game A.** One slightly biased coin is used:

$$\textbf{Prob(Head)} = 0.495$$

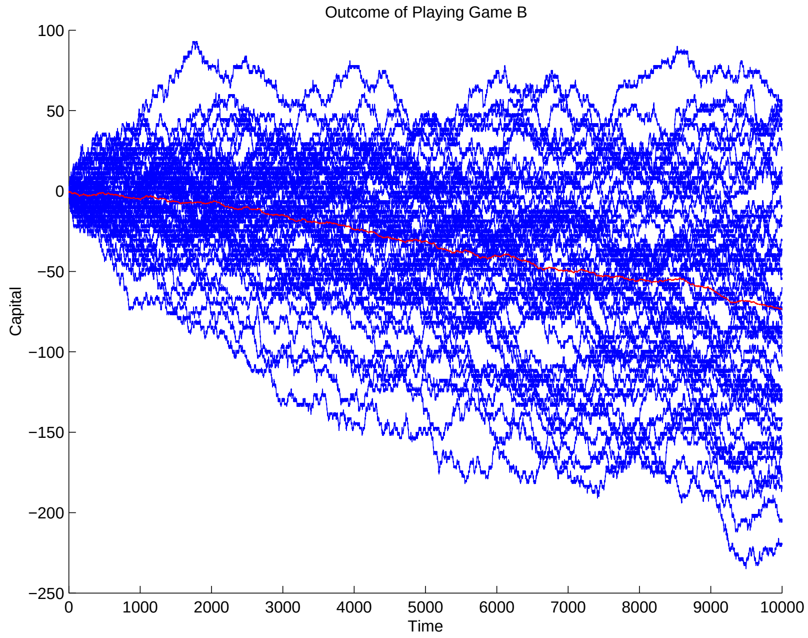
- **Game B.** **Two different** coins are used, depending on your current sum (capital).
    1. If your current sum is divisible by three i.e. $0, 3, 6, 9 \ldots$, you toss the coin with **Prob(Head)**$) = 0.095$;
    2. otherwise, you toss the coin with **Prob(Head)** $= 0.745$.

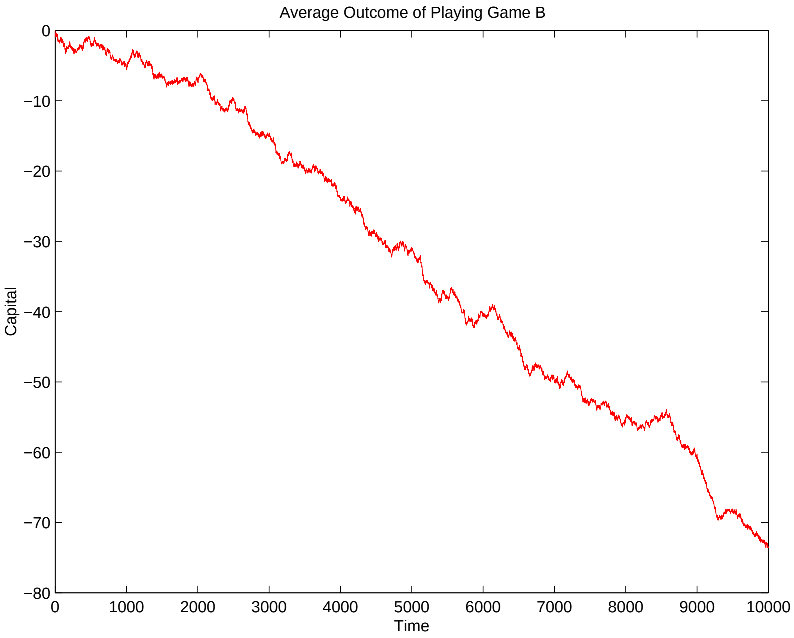
Playing Game A Only





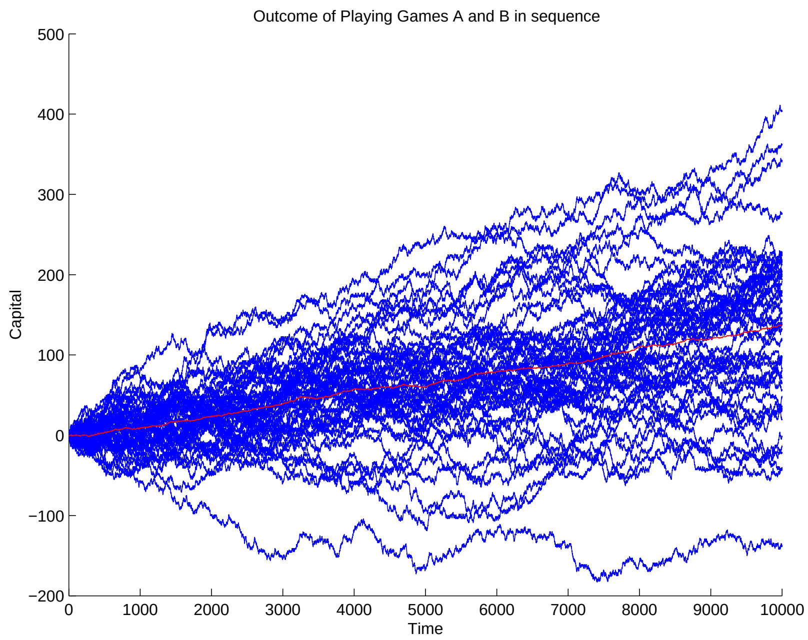
Playing Game B only

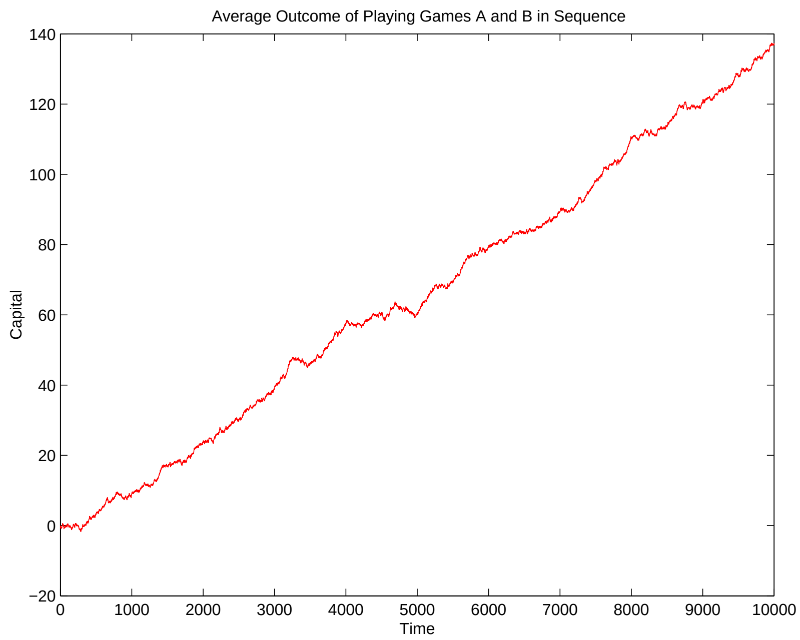




Playing Games of A and B in sequence:

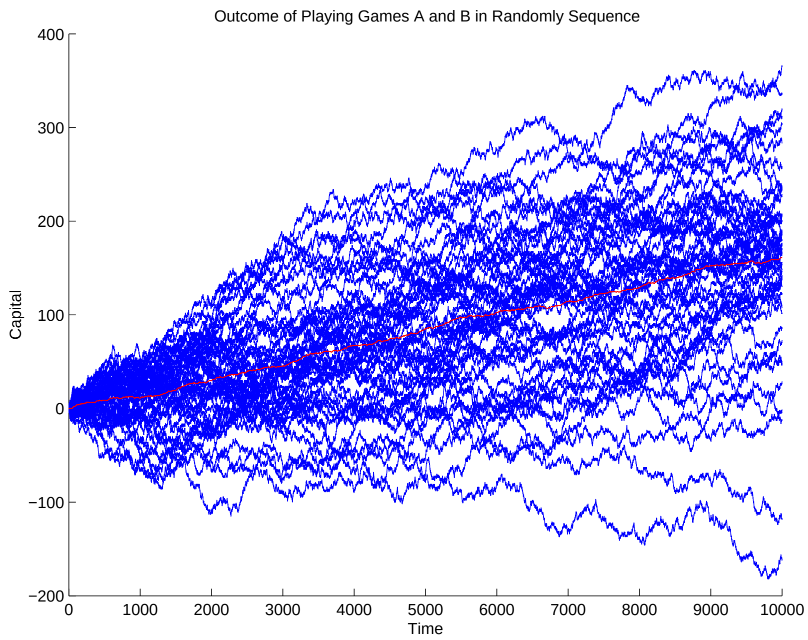
AABBAABBAABB....

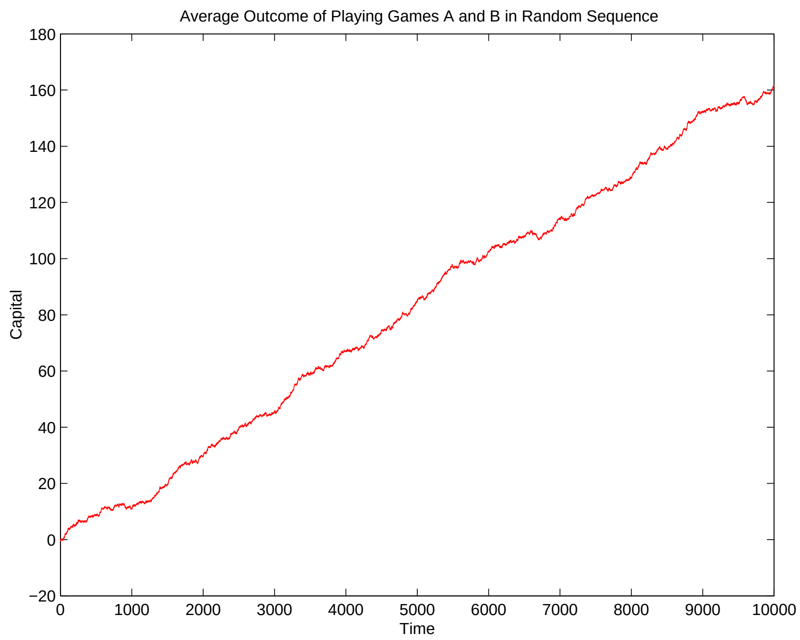




Playing Games A and B at Random:

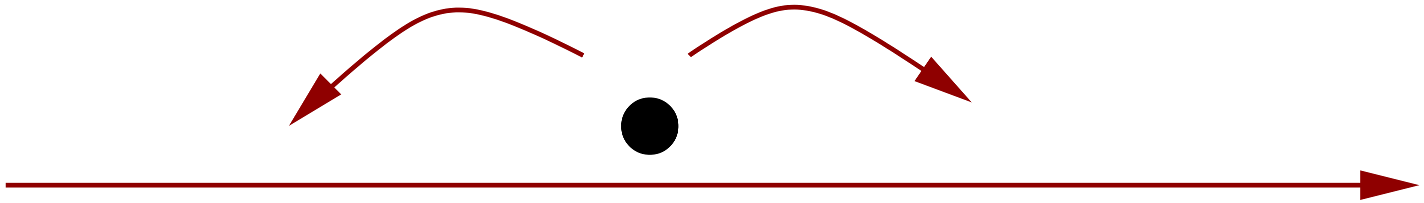
AABABAABBBBABB...



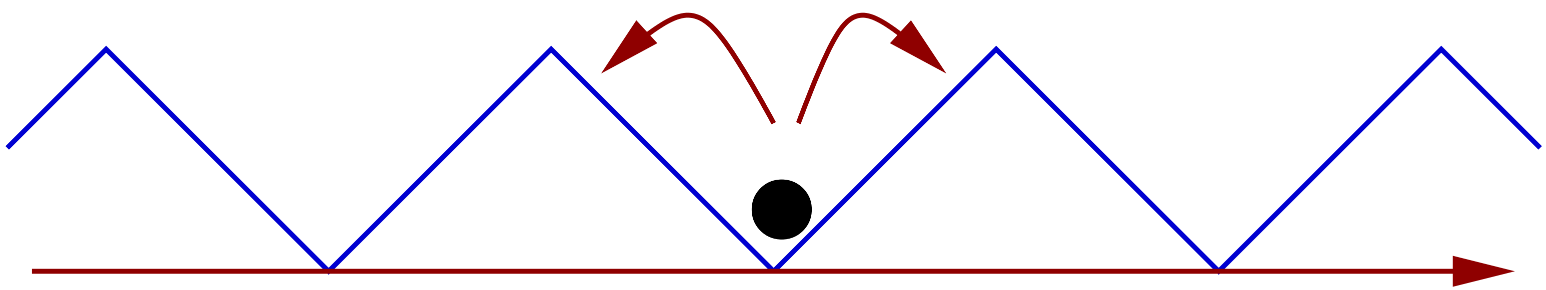


Random Walk in Periodic Environment

- **Homogeneous Environment.**
  (No global drift)

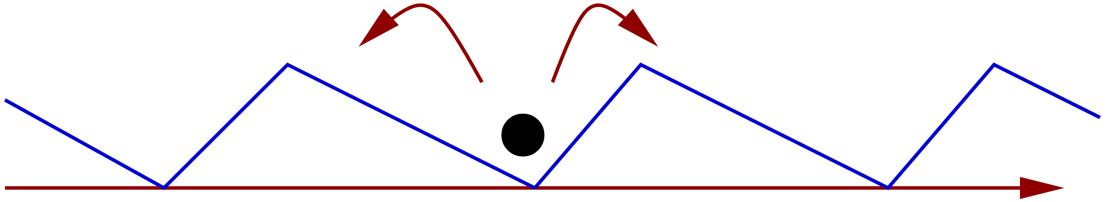


- **Inhomogeneous (Symmetric) Environment.**
  (No global drift)

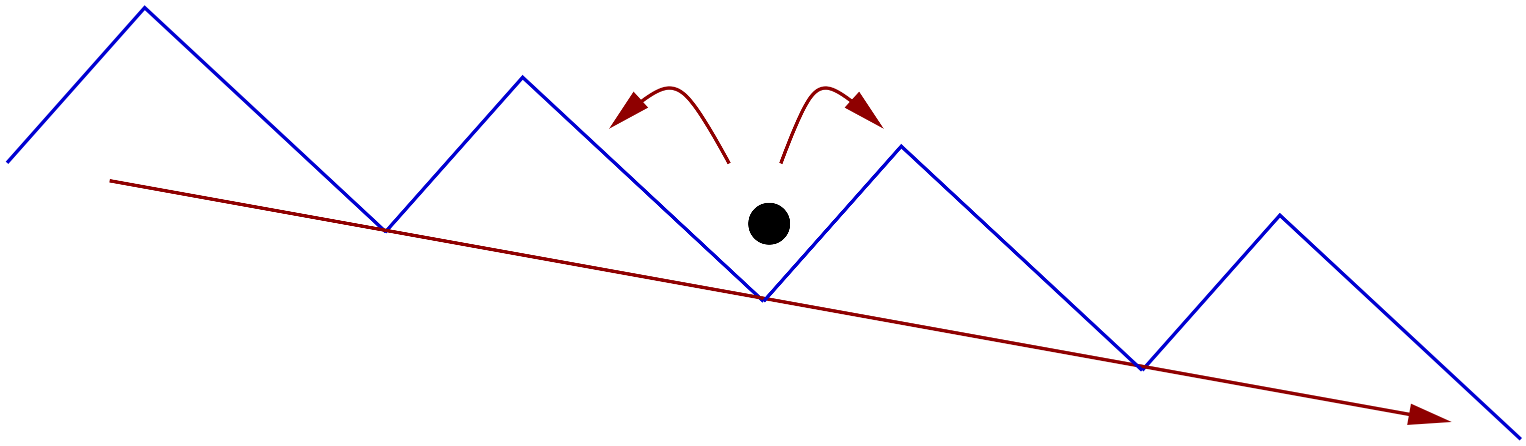


Random Walk in Periodic Environment – 2

- **Inhomogeneous (Asymmetric, Unbiased) Environment. (Global drift?)**

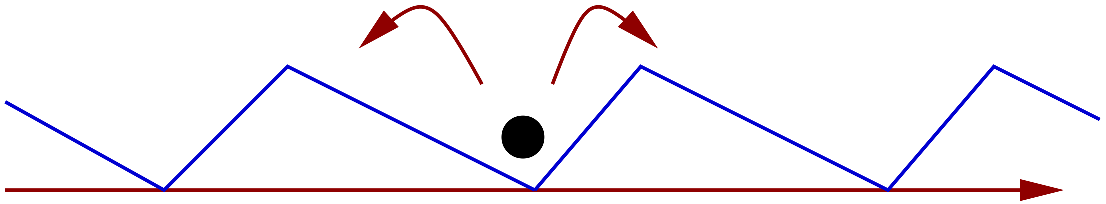


- **Inhomogeneous (Biased) Environment. (Global drift?)**

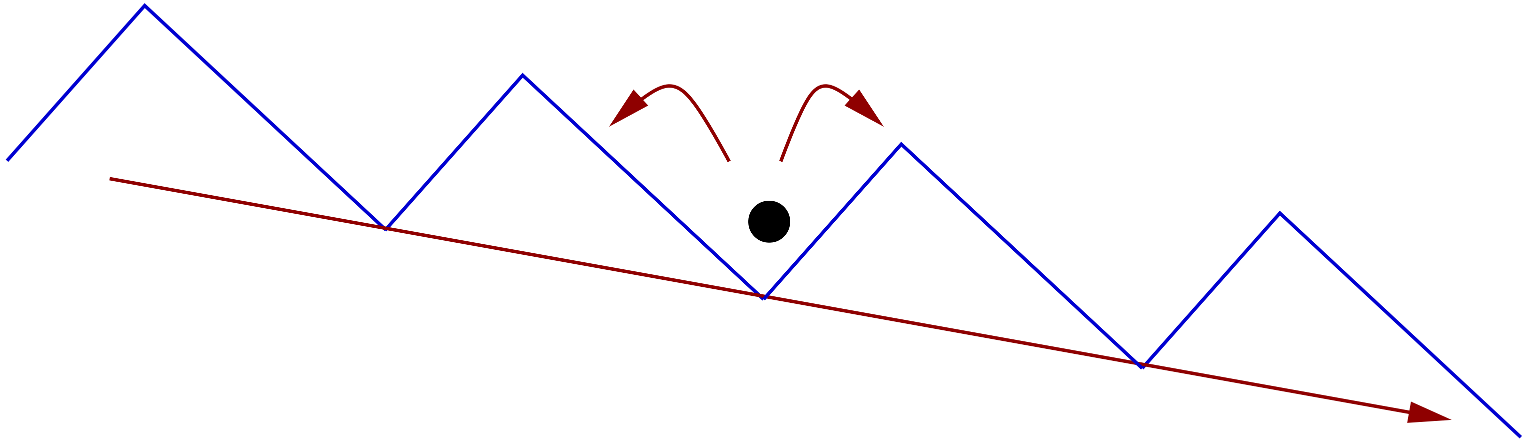


Random Walk in Periodic Environment – 2

- **Inhomogeneous (Asymmetric, Unbiased) Environment. (Still no global drift)**



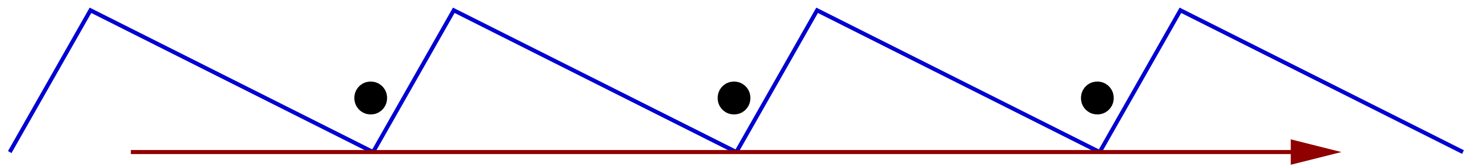
- **Inhomogeneous (Biased) Environment. (Global drift)**



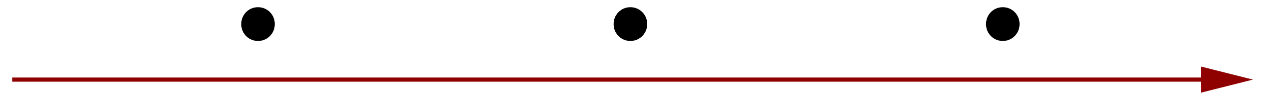
Extraction of useful work

from **random noise** and **un-biased but non-equilibrium fluctuations**

- **(I) Potential $U(x)$.**



- **(II) Potential $\equiv 0$.**

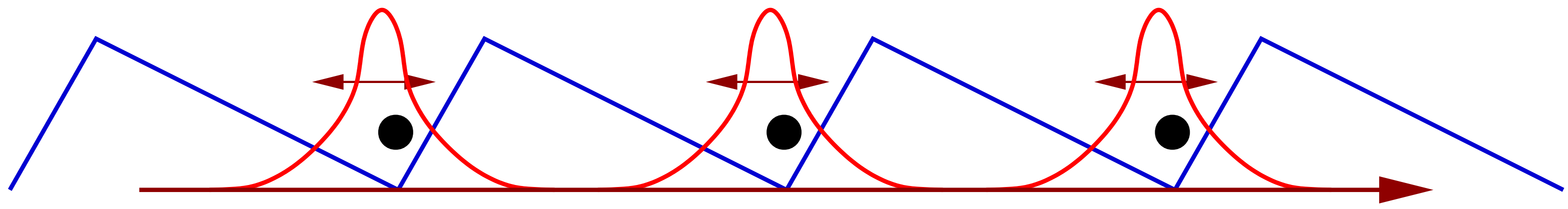


(I) and (II) **individually** does not give **global drift**.

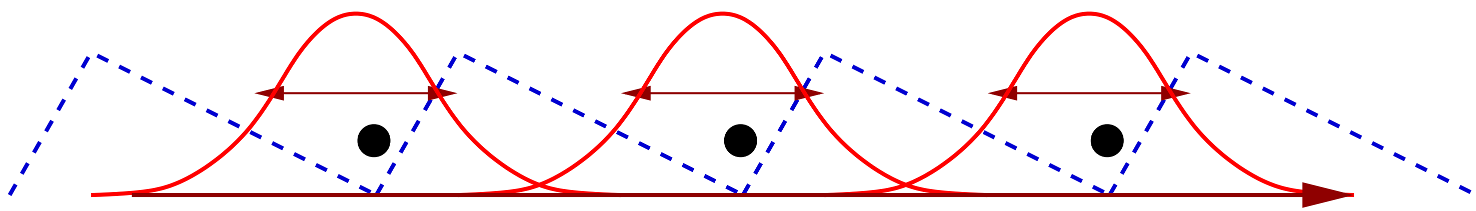
Extraction of useful work – 2

now **alternate (I) and (II)** with certain **frequency**.

- **(I) Potential $U(x)$ is ON.**

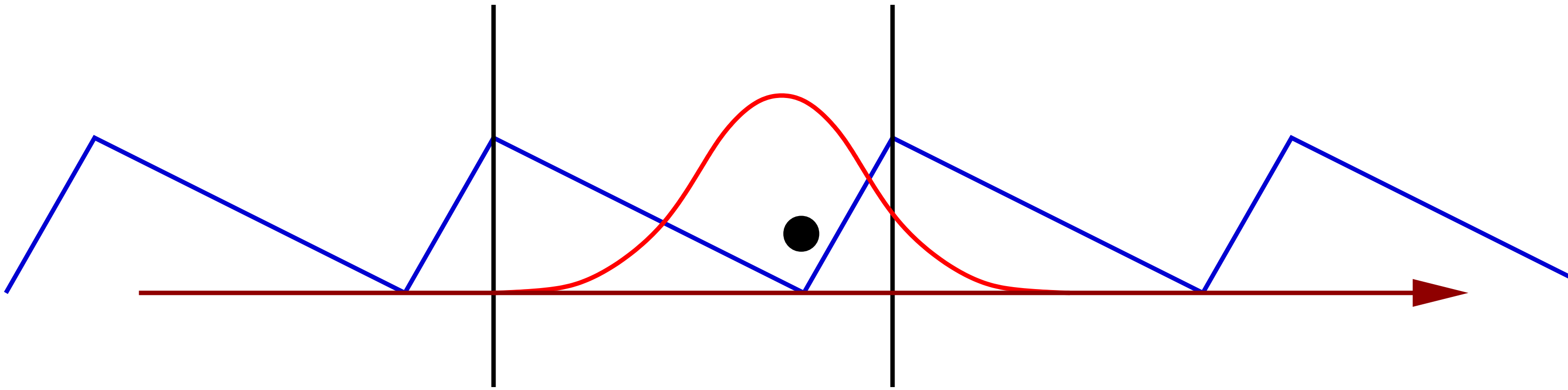


- **(II) Potential $U(x)$ is OFF.**

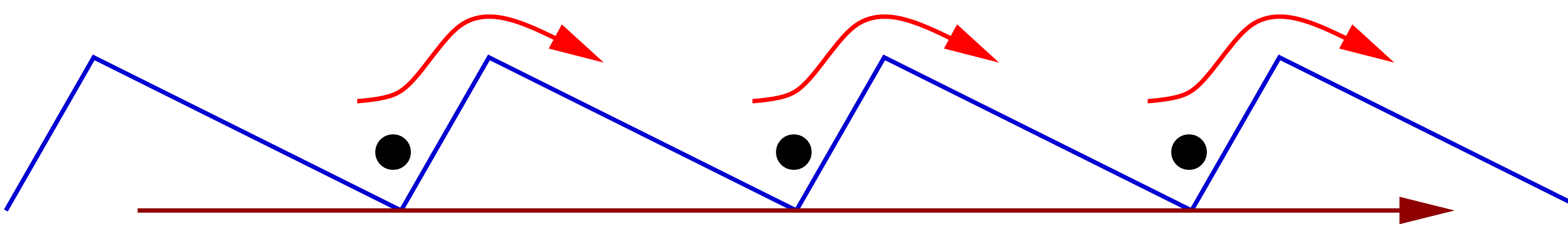


Extraction of useful work – 3

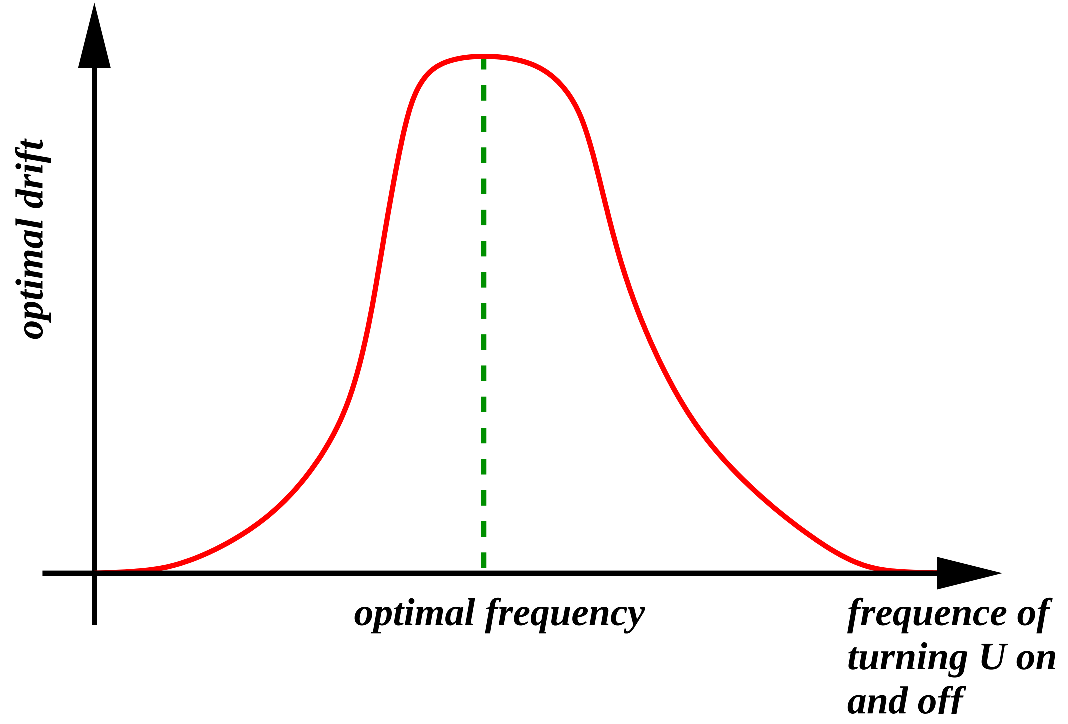
- **(I) Potential $U(x)$ is ON again.**



- So **effectively**, there is a **global drift**.



Optimal Frequency – Stochastic Resonance

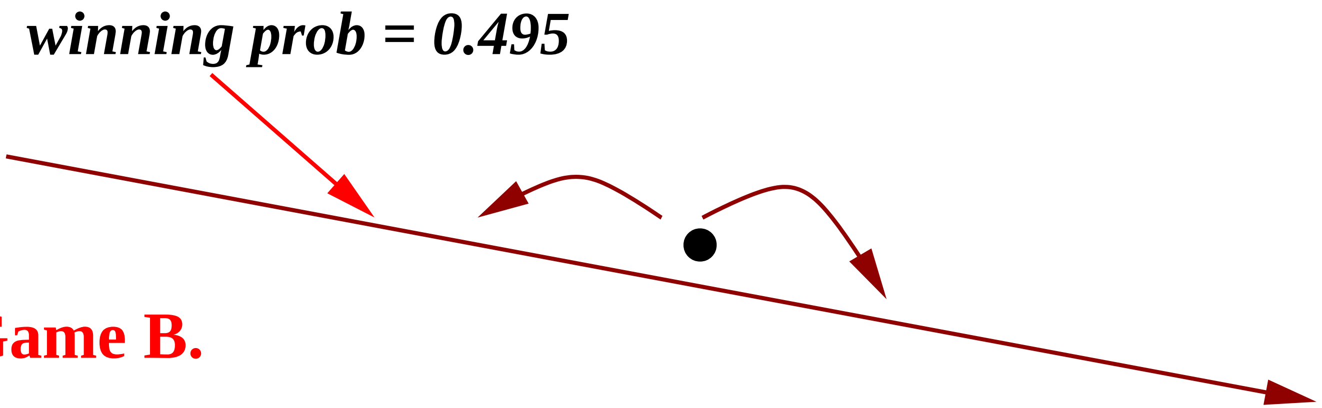


Key ingredients:

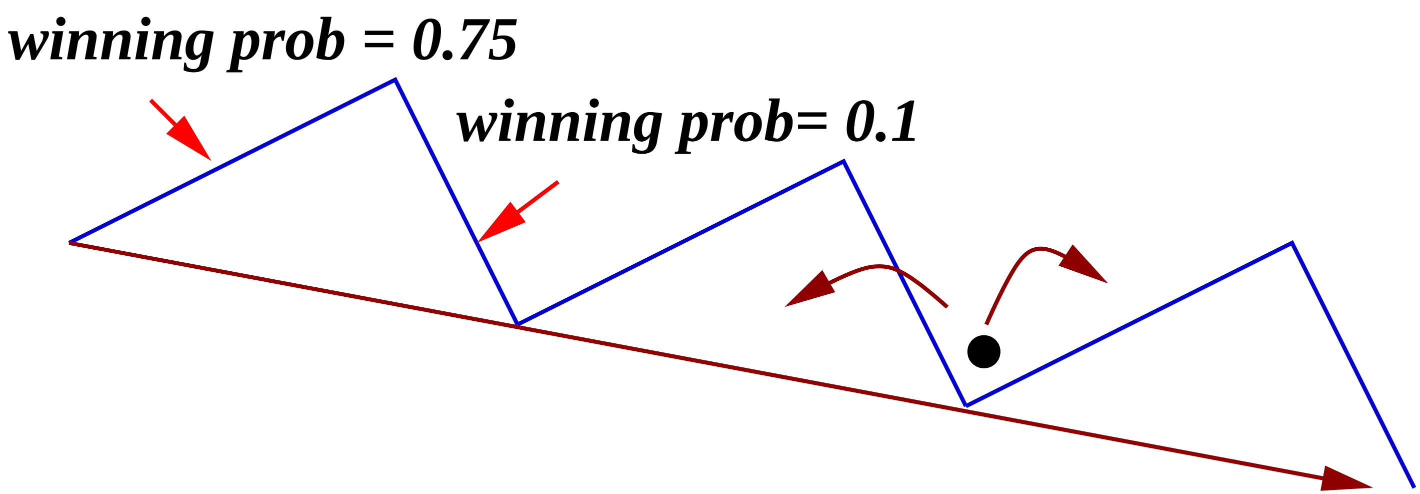
- **Spatially asymmetric environment;**
- **Non-equilibrium fluctuations;**
- **Noise!**

Back to Parrondo's Paradox

Playing Game A.



Playing Game B.



So **effectively**, there is a **winning tendency**.

Applications of Stochastic Resonance

Astumian, Science, Vol 276, 1997

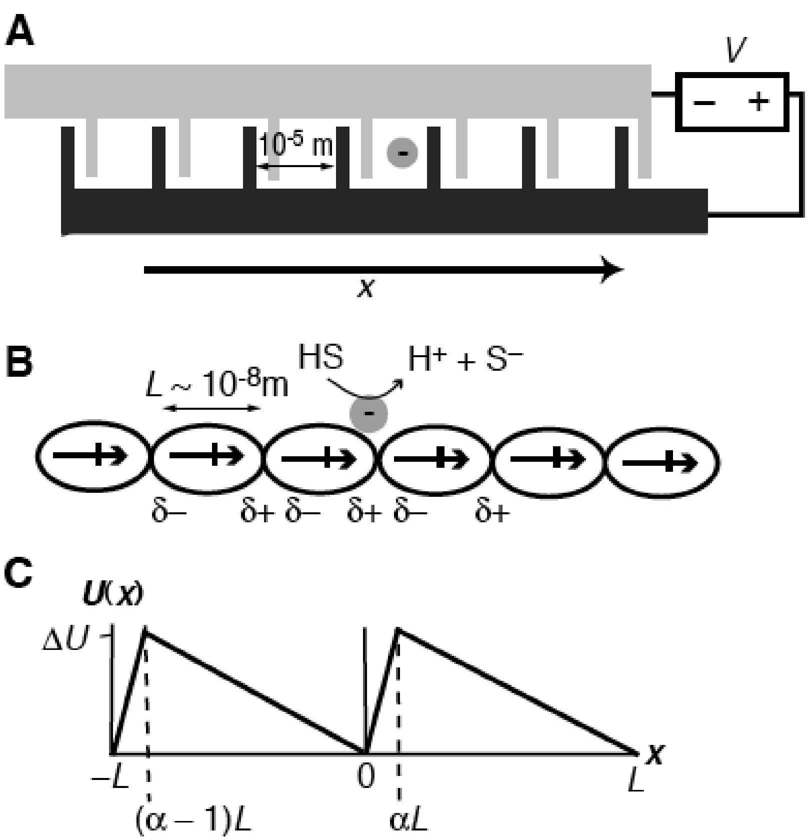
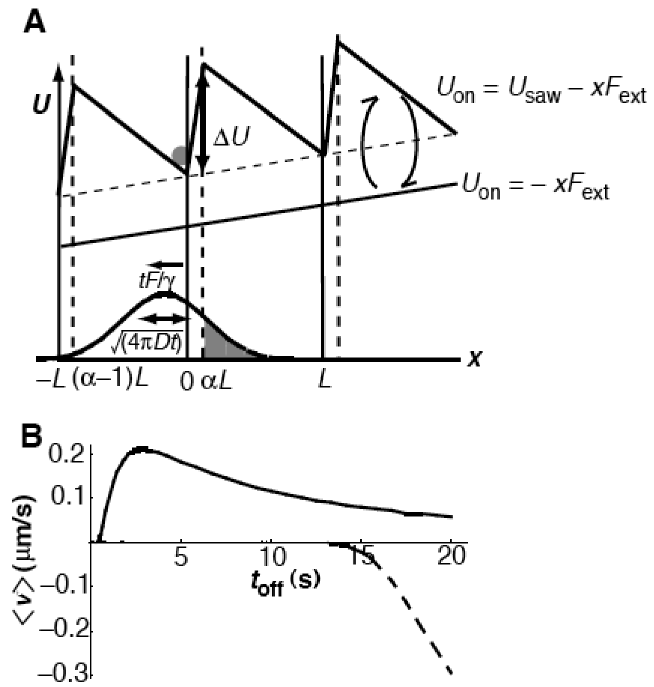
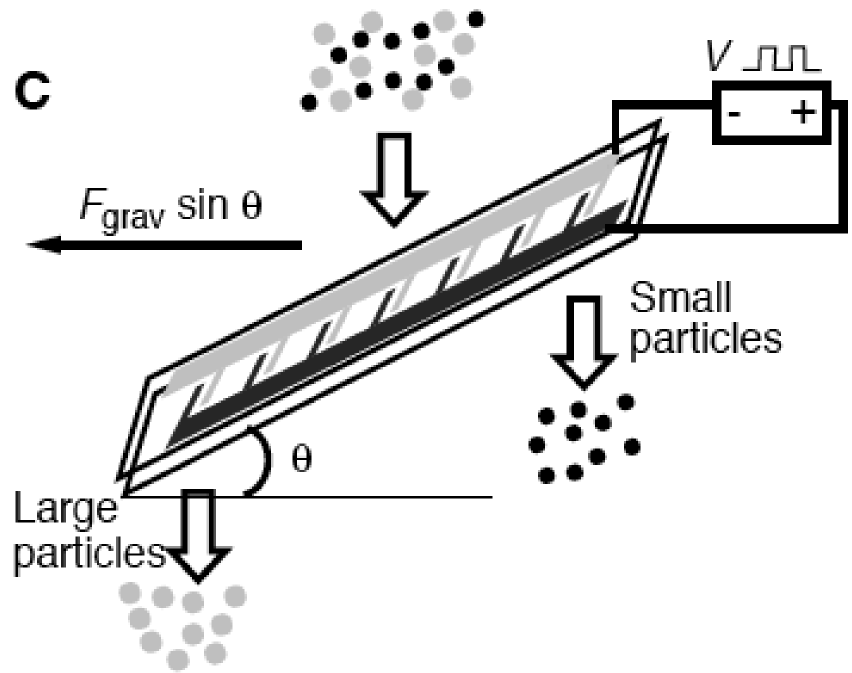


Fig. 1. Two specific models by which an anisotropic periodic potential can arise. (**A**) When a voltage difference *V* is applied to interdigitated electrodes deposited on a glass slide with anisotropically positioned "teeth," the resulting potential is spatially periodic but anisotropic. With photolithography it is possible to achieve a spatial period *L* for such a structure as small as 10^{-5} m or even somewhat less. (**B**) A linear array of dipoles aligned head to tail on which a charged Brownian particle (possibly a protein) moves. The individual dipoles could be macromolecular monomers that aggregate to form an extended linear polymer. The length of the individual monomer is $L \sim 10^{-8}$ m, a reasonable value for many aggregating proteins. If the particle catalyzes the reaction $HS \rightarrow H^+ + S^-$, the charge on the particle, and hence its electrostatic potential energy of interaction with the dipole array $U(x, t)$, fluctuates depending on its chemical state. (**C**) The potential $U(x)$ for both (A) and (B) is approximately an anisotropic sawtooth function U_{saw}, drawn for simplicity as a piece-wise linear function. The amplitude is ΔU, and the wells (potential minima) are spaced periodically at positions iL. The anisotropy is parameterized by α. The amplitude ΔU of the sawtooth potential can be modulated in (A) by using an external switching device to control the applied voltage.

Applications of Stochastic Resonance –2

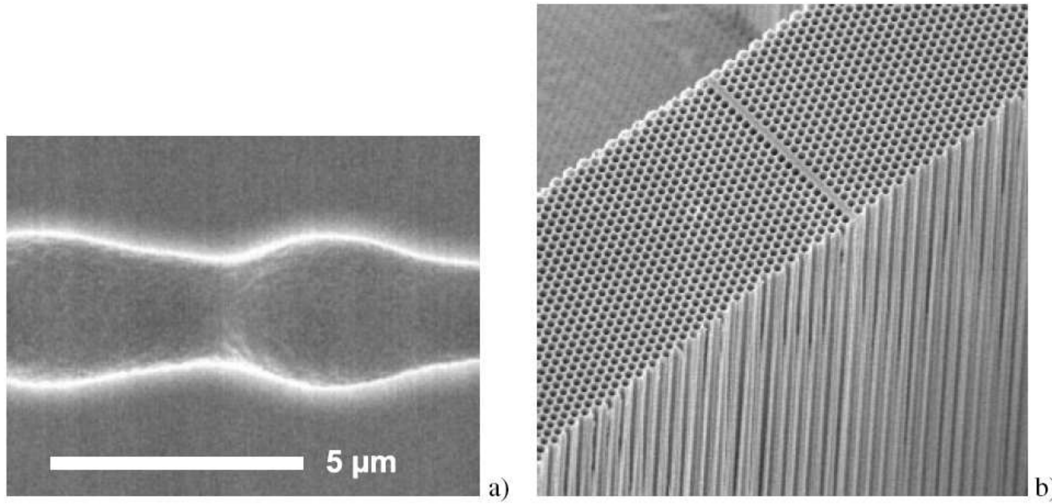
Astumian, Science, Vol 276, 1997

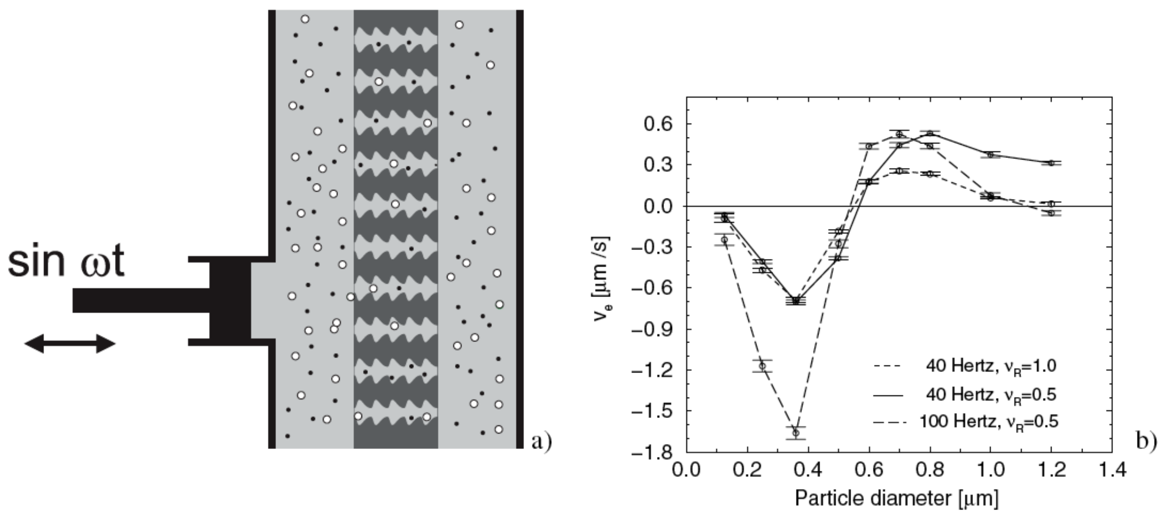




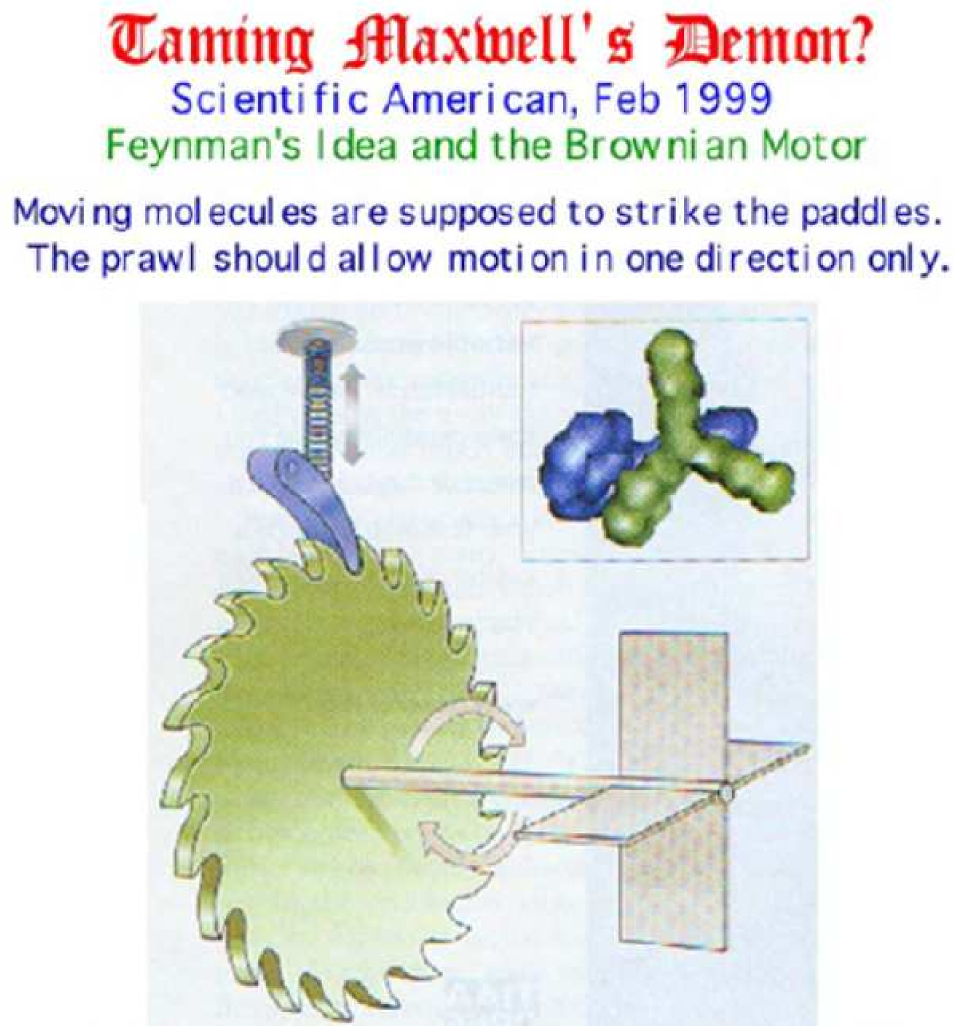
Applications of Stochastic Resonance –3

Hanggi, et. al., Ann. Phys., Vol 14, 2005





Applications of Stochastic Resonance?



Feynman's Lecture Notes in Physics

THANK YOU