

Mathematical Analysis of Thresholding Schemes

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Ginzburg-Landau Type Models and Dynamics

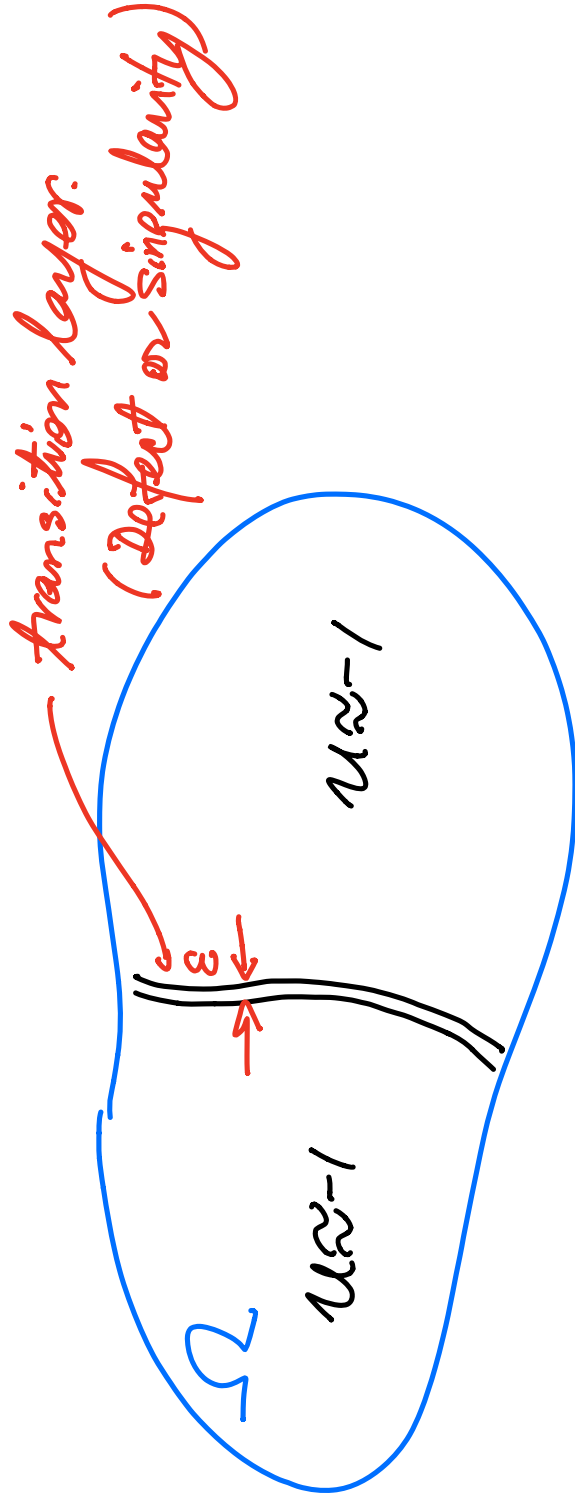
- $J_\varepsilon(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 + \frac{1}{\varepsilon^2} W(u) dx, \quad \Omega \subseteq \mathbb{R}^n$
- $u_t = - \frac{\delta J_\varepsilon(u)}{\delta u} = \Delta u - \frac{1}{\varepsilon^2} W'(u)$
↙ L^2 -gradient flow of J_ε
- $u: \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$
(eg. $n=2,3, \quad m=1,2$)
- $W: \mathbb{R}^m \longrightarrow \mathbb{R}_+, \text{ multi-well structure}$

Typical Examples

(1) $m=1$, i.e. u is scalar valued.

$$w(u) = (1-u^2)^2$$

$$(w(u) \geq 0, = 0 \text{ iff } u = 1, -1)$$

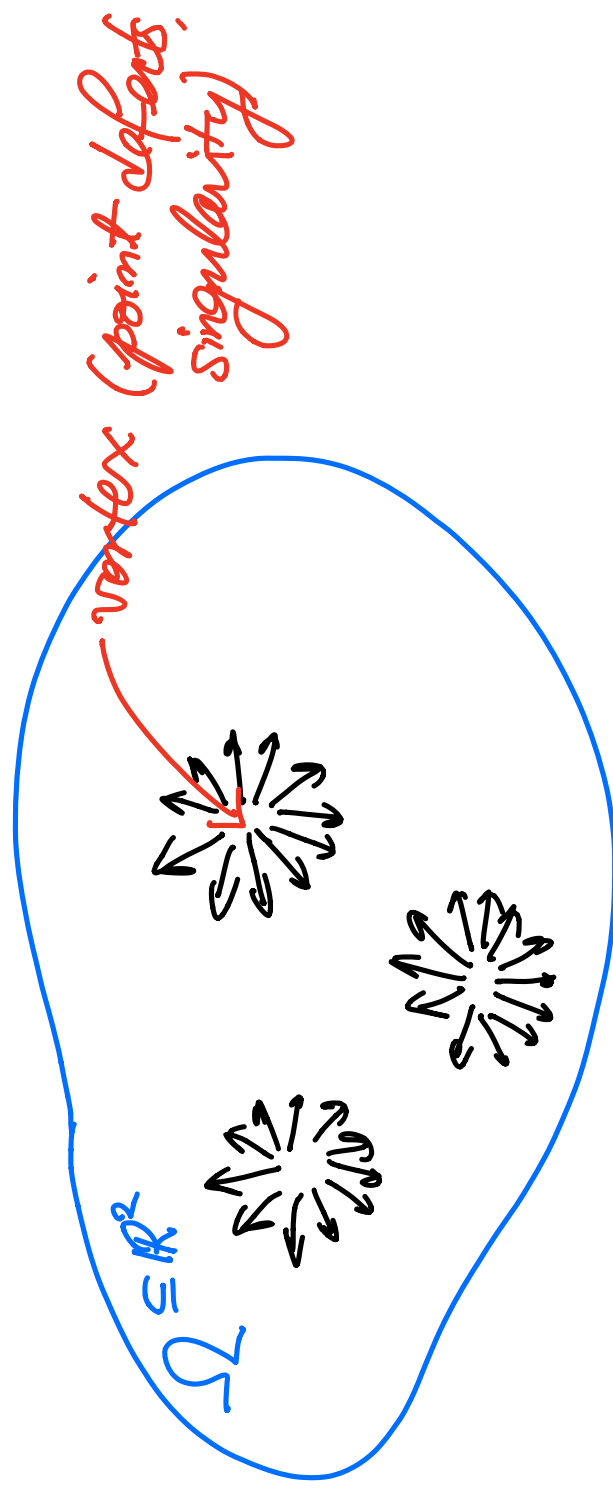


$$(2) \quad n=2, \quad m=2, \quad u: \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \cong \mathbb{C}$$

complex valued

$$W(u) = (1 - |u|^2)^2$$

$$(W(u) \geq 0, \quad = 0 \quad \text{iff} \quad |u| = 1, \quad \text{i.e. } u \in S^1)$$

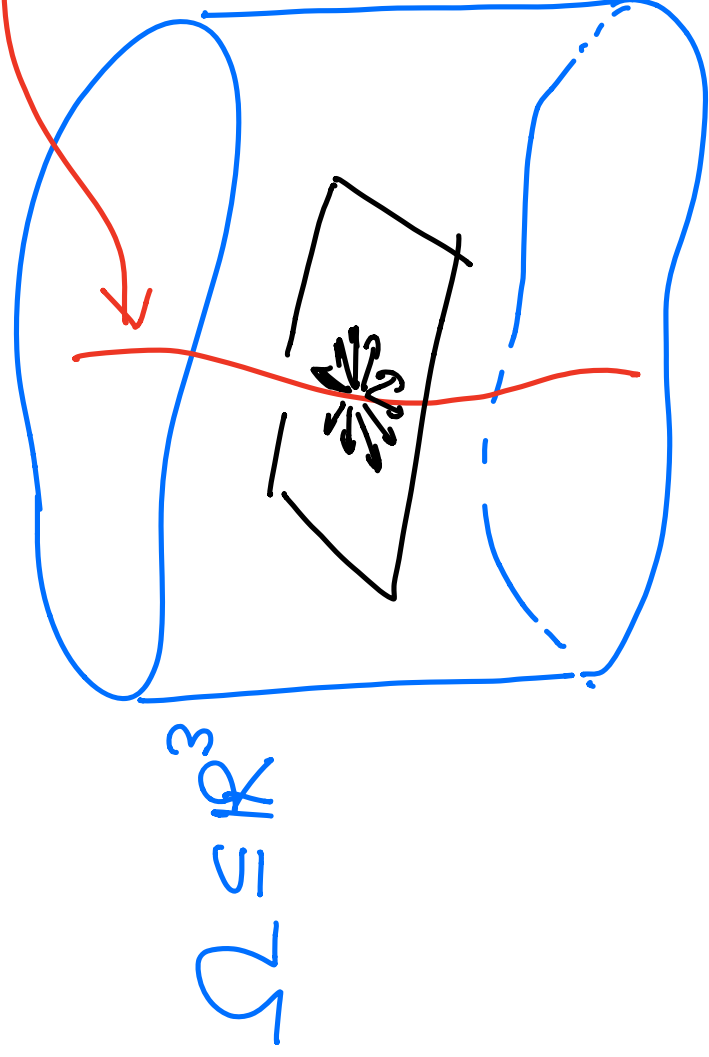


$$(3) \quad n=3, m=2, \quad u: \Omega \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \simeq \mathbb{C}$$

$$W(u) = (1 - |u|^2)^2$$

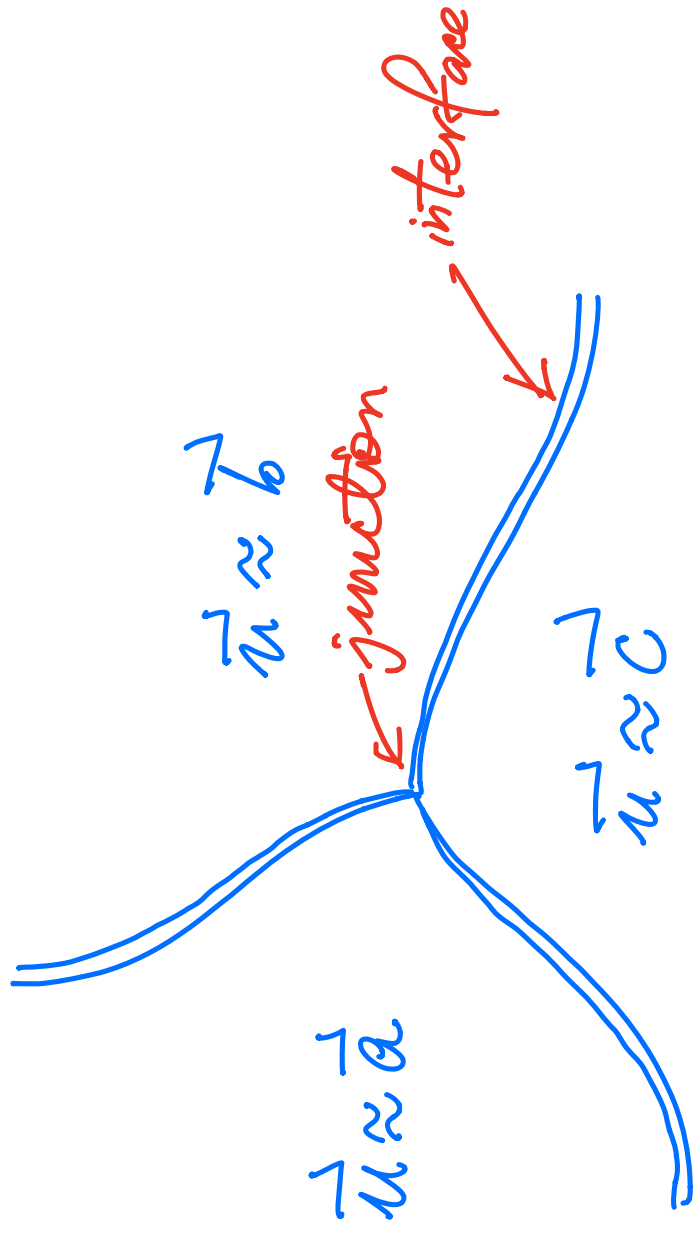
$$(W(u) = 0 \iff u \in S')$$

line defect, singularity
(filament)



(4) Multi-phase (e.g. $m=3$) Bronsard-Leitich

$$(W(u)=0 \Leftrightarrow u=a, b, c)$$



(5) Harmonic Maps

$$u: \Omega \longrightarrow \mathbb{R}^m$$

$$W(\vec{u}) \geq 0, = 0 \text{ iff } \vec{u} \in \bigcup_i N_i$$



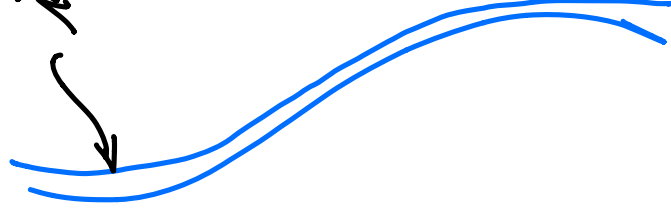
Riemannian manifolds

interface

$$\vec{u} \in N_i$$

$$\vec{u} \in N_j$$

(Harmonic maps)



Asymptotic Limit ($\varepsilon \rightarrow 0$) of Structures and Dynamics of Defects or Singularities

$$u_t = \Delta u - \frac{1}{\varepsilon^2} V(u)$$

A very simple numerical algorithm was first proposed by Merriman-Bence-Oster (MBO) also called the thresholding scheme.

It is essentially a time splitting scheme:

(1) Let $\mathcal{S} = \{u: W(u) = 0\}$

(2) Let $u_0: \Omega \rightarrow \mathcal{S}$

(3) Iteration steps: $\Delta t > 0$, given $u_n, n \geq 0$

(i) Linear Diffusion: $u_n^* = v(\Delta t)$ where

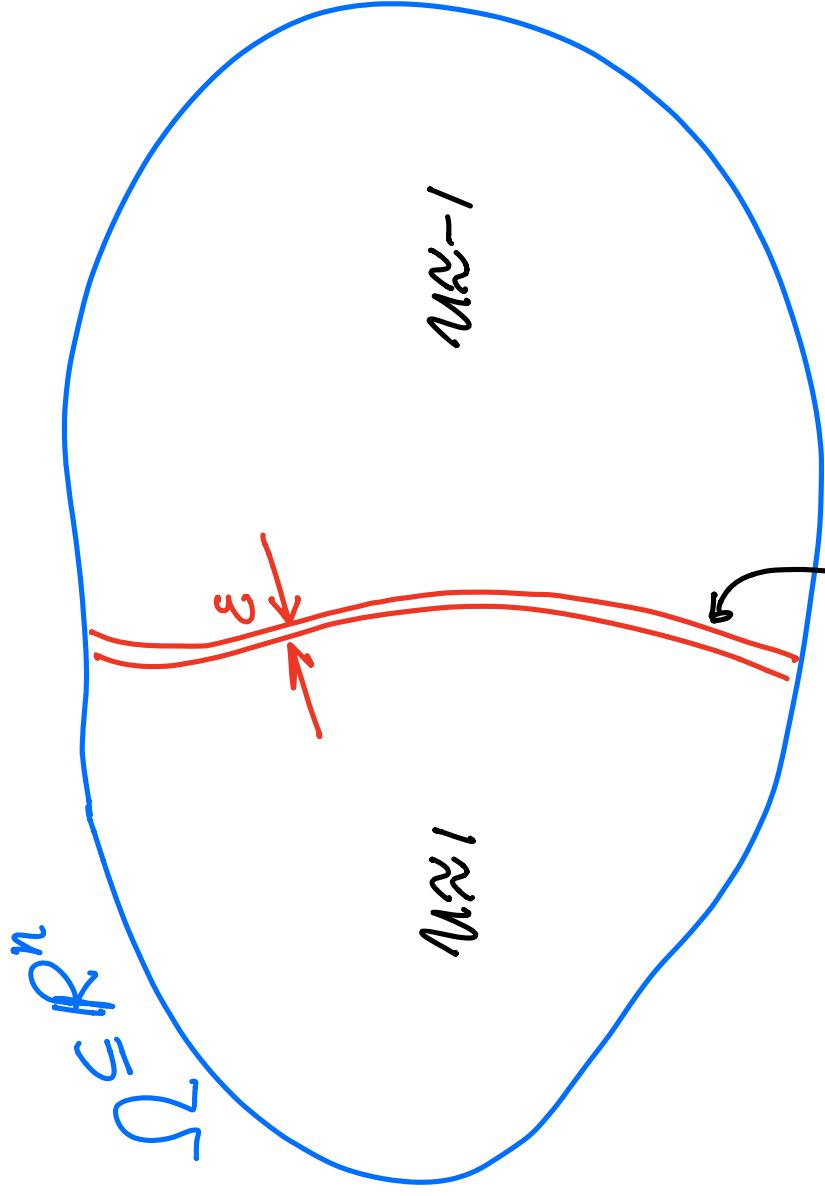
$(\uparrow) \quad v(\Delta t) \text{ solves: } \quad v_t = \Delta v, \quad v(0) = u_n$

(ii) Projection onto $\mathcal{S}$: $u_{n+1} = \text{Proj}_{\mathcal{S}} u_n^*$

(4) A sequence $\{u_n\}_{n \geq 0}$ is formed. Its convergence as $\Delta t \rightarrow 0$ is then analysed.

Motion by Mean Curvature ($m=1$)

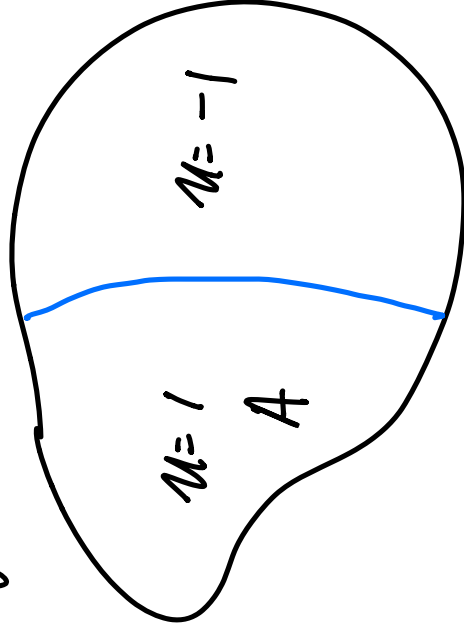
Allen-Cahn Functional and Equation



as $\varepsilon \rightarrow 0$, the interface moves by its
mean curvature

(1) Gamma convergence of Allen-Cahn functional:

$$J_\varepsilon(u) = \int \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \longrightarrow C_0 \int |\nabla u| = C_0 \text{Per}(\partial A)$$



$$(u: \Omega \rightarrow \{-1, 1\})$$

$$A = \{x: u=1\}$$

(2) Mean curvature

= First Variation of Area Functional

MO for Allen-Cahn Equation

$$u_t = \Delta u - \frac{1}{\varepsilon^2} u(1-u^2)$$

(1) Given an initial set $\Omega_0 \subseteq \mathbb{R}^n$, define

$$u_0(x) = 2\chi_{\Omega_0}(x) - 1 = \begin{cases} 1 & \text{if } x \in \Omega_0 \\ -1 & \text{if } x \notin \Omega_0 \end{cases}$$



(2) Given $u_n: \mathbb{R}^n \rightarrow [-1, 1]$

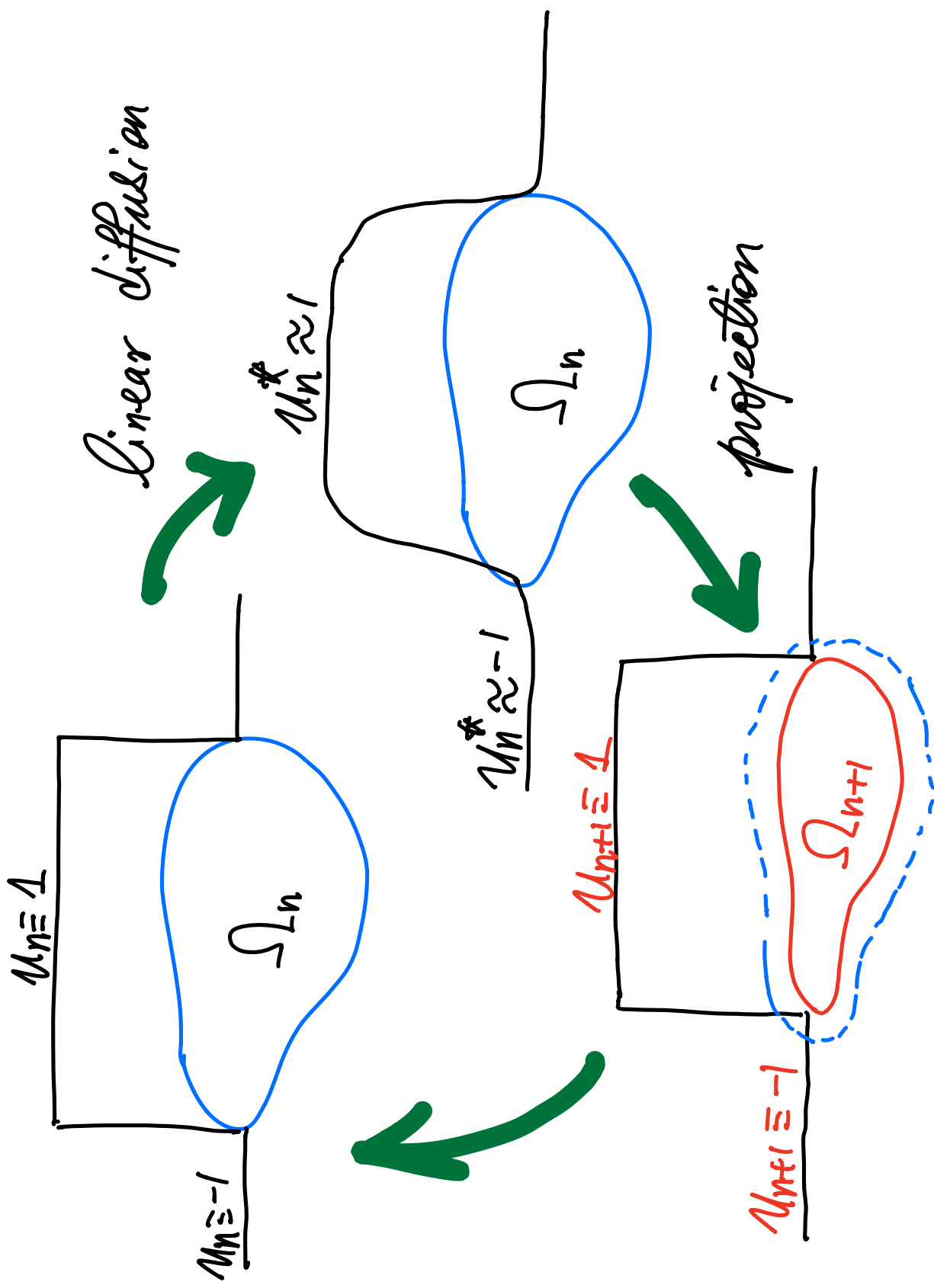
Define:

(1) $u_n^* := v(\Delta t)$ where $v(\cdot)$ solves

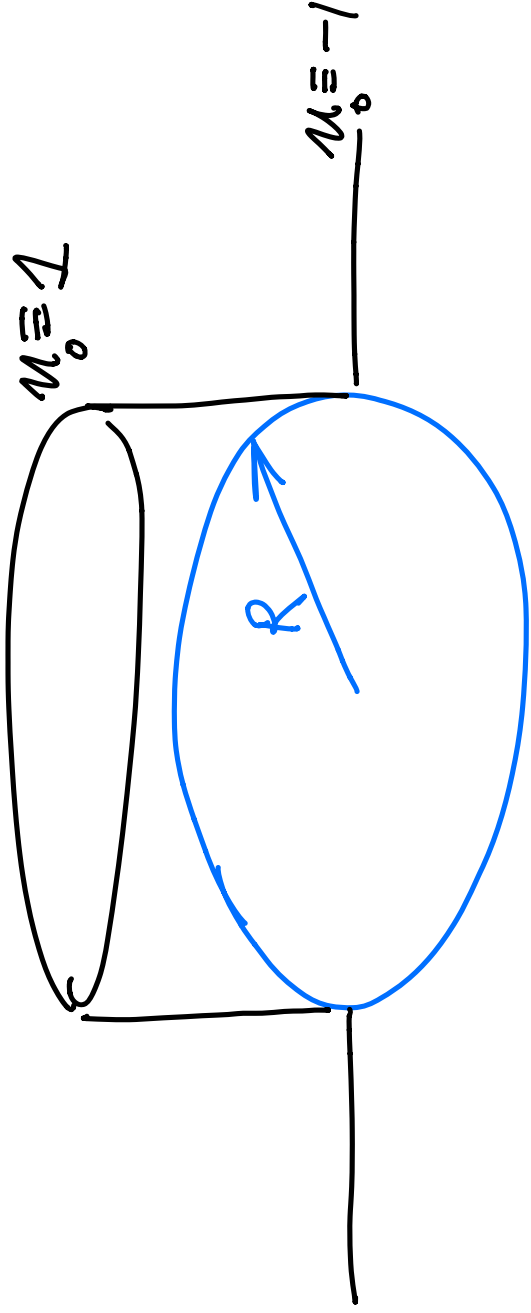
$$u_t = \Delta v, \quad 0 < t < \Delta t, \quad v_0 = u_n$$

$$(2) \quad u_{n+1}^{(*)} := \begin{cases} 1 & \text{if } u_n^*(x) > 0 \\ -1 & \text{if } u_n^*(x) < 0 \end{cases}$$

(3) Define $\Omega_n = \{x: u_n(x) > 0\}$, $\Gamma_n = \partial\Omega_n$
 $\{\Omega_n\}_{n \geq 0}$ or $\{\Gamma_n\}_{n \geq 0}$ can be shown to converge to
motion by mean curvature



Heuristics (Radially Symmetric case)



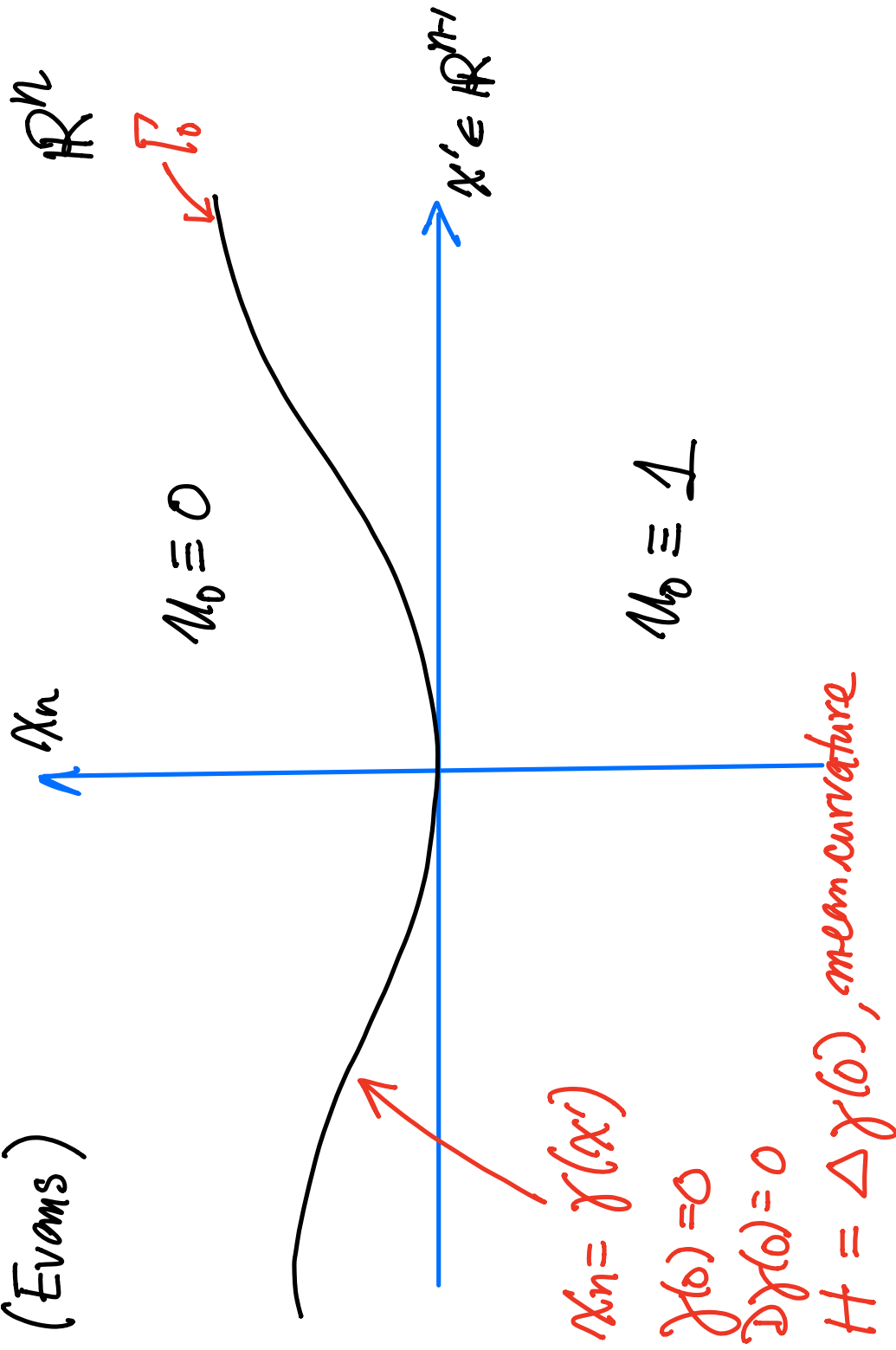
Diffusion step: $u(x, t) = u(r, t)$

$$u_t = \Delta u$$

$$u_t = u_{rr} + \frac{n-1}{r} u_r$$

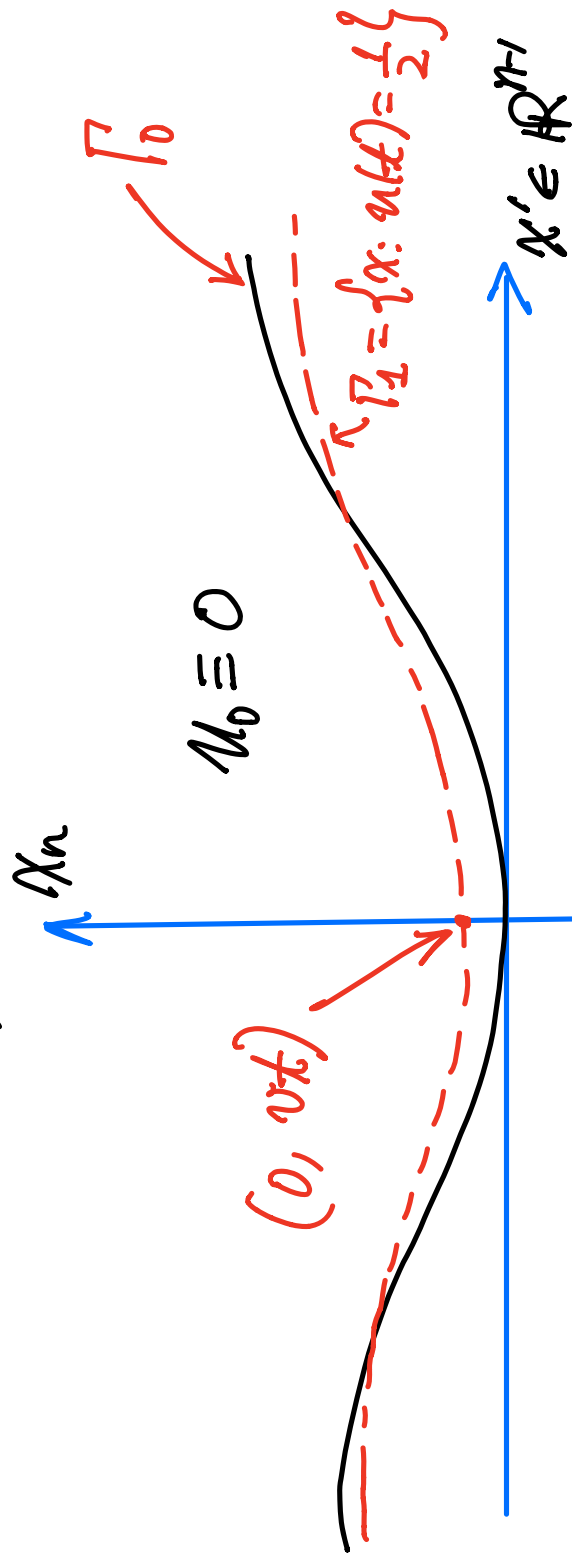
$\frac{n-1}{r} u_r$ $\swarrow$ drift term =
curvature

Heuristics (Evans) (In the case of a graph)



Let $u(x', x_n, t)$ be the solution of

$$u_t = \Delta u, \quad u(0) = u_0$$



Find vt such that

$$u(0, vt, t) = \frac{1}{2}$$

$$x = (x', x_n) \in \mathbb{R}^n$$

$$u_0 \equiv 1$$

$$\frac{1}{2} = u(0, vt, t)$$

$$= \frac{1}{(4\pi t)^{n/2}} \int_{\Omega_0} e^{-\frac{|x_1|^2 + |x_n - vt|^2}{4t}} dx_n dx'$$

$$= \frac{1}{(4\pi t)^{n/2}} \int_{[-1,1]^{n-1}} \int_{-1}^1 e^{-\frac{|x_1|^2}{4t}} \int_{-\frac{|x_n - vt|^2}{4t}}^{\delta(x_1)} dx_n dx' + O(e^{-\frac{c}{t}})$$

$$(y = \frac{x}{vt}) = \frac{1}{(4\pi)^{n/2}} \int_{\frac{1}{vt}[-1,1]^{n-1}} \int_{-\frac{1}{vt}}^{\frac{\delta(\sqrt{t}y')}{\sqrt{t}}} e^{-\frac{|y_1|^2}{4}} - \frac{|y_n - \sqrt{t}\frac{1}{2}|^2}{4} dy_n dy'$$

Hence

$$\int_{\mathbb{R}^{n-1}} \int_0^{\infty} \frac{1}{\sqrt{t}} (\gamma(\sqrt{t} z') - t\nu) e^{-\frac{|z'|^2}{4t}} dz' dt = \text{e.s.t.}$$

(Taylor expansion)

$$\frac{1}{\sqrt{t}} (\gamma(\sqrt{t} z') - t\nu) = \sqrt{t} \left[\frac{1}{2} \gamma_{x_i x_j}(0) z_i z_j - \nu \right] + O(t^{3/2})$$

$$\int_{\mathbb{R}^{n-1}} e^{-\frac{|z'|^2}{4}} \left(\frac{1}{2} \gamma_{x_i x_j}(0) z_i z_j - \nu \right) dz' = O(\sqrt{t})$$

$$\nu = \Delta \gamma(0) + o(\sqrt{t}) = H + o(\sqrt{t})$$

Convergence Proof : (Barles - Georgelin)

Level Set Formulation of Viscosity Solution

$$u(x, t) < 0$$

$$u(x, t) > 0$$

$$\Gamma = \{x : u(x, t) = 0\}$$

$$u_t = |\nabla u| \operatorname{div} \frac{\nabla u}{|\nabla u|}$$

$$= \Delta u - \frac{\langle \tilde{D}^2 u, Du \rangle}{|Du|^2} = t \left[\left(I - \frac{Du \otimes Du}{|Du|^2} \right) \tilde{D}^2 u \right]$$

$$= -F(\tilde{D}^2 u, Du) \quad (*)$$

① $u: \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is called a viscosity sub-soln of (*) if $\forall \varphi \in C^2(\mathbb{R}^n \times \mathbb{R}_+)$ and $u - \varphi$ has a local max. pt. at (x, t) , then

$$\frac{\partial \varphi}{\partial t} - \left(\Delta \varphi - \frac{(\nabla^2 \varphi \nabla \varphi, \nabla \varphi)}{|\nabla \varphi|^2} \right) \Big|_{(x, t)} \leq 0,$$

if $\nabla \varphi(x, t) \neq 0$

$\nabla \frac{\partial \varphi}{\partial t} \leq 0$, if $\nabla \varphi(x, t) = 0$, $\nabla^2 \varphi(x, t) = 0$

(2)

$$\lim_{*} \underline{\lim} u_n(x, t) := \lim_{\substack{y \rightarrow x \\ n \Delta t \rightarrow t}} u_n(y, n \Delta t)$$

$$\lim_{*} \overline{\lim} u_n(x, t) := \lim_{\substack{y \rightarrow x \\ n \Delta t \rightarrow t}} u_n(y, n \Delta t)$$

Then $\lim_{*} \underline{\lim} u_n$ & $\lim_{*} \overline{\lim} u_n$ are

viscosity super- and sub-solutions of
(*)

A General Convergence Method (Barles-Jouanolis)

(1) Let $G_t: BUC(\mathbb{R}^n) \rightarrow BUC(\mathbb{R}^n)$, $t > 0$

$$\left. \begin{array}{l} \text{satisfies} \\ G_t(u+c) = G_t(u) + c \\ u \leq v \Rightarrow G_t u \leq G_t v \end{array} \right\}$$

(2) There exists a continuous function

$$F: \underline{\mathcal{S}}^m \times (\mathbb{R}^n \setminus \{0\}) \rightarrow \mathbb{R}$$

which is degenerate elliptic such that
($\mathcal{S}^n =$ space of symmetric matrices)

for all smooth function φ and $x \in \mathbb{R}^n$,

$$\lim_{t \rightarrow 0^+} \frac{1}{t} (\hat{G}_t \varphi - \varphi) \leq -F_*(D^2 \varphi(x), D\varphi(x))$$

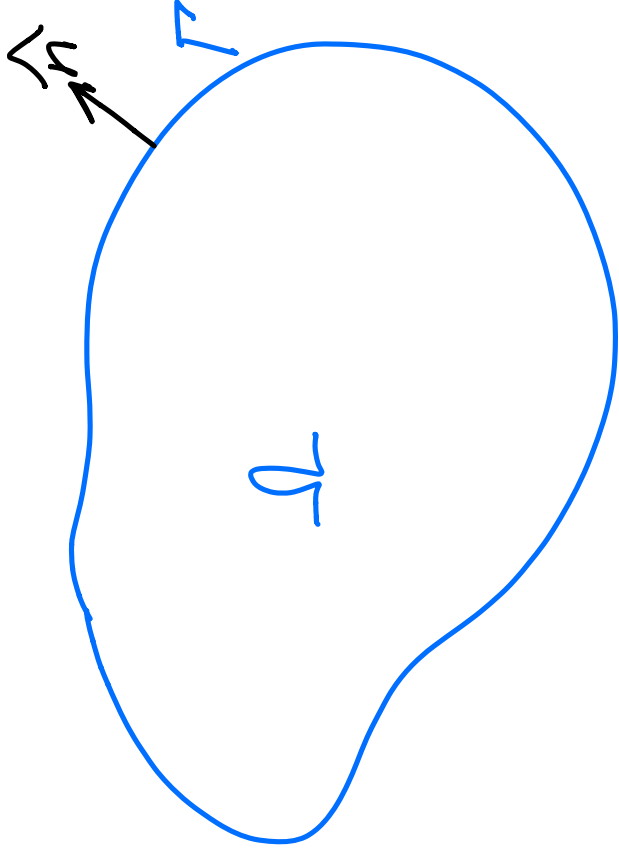
$$\lim_{t \rightarrow 0^+} \frac{1}{t} (\hat{G}_t \varphi - \varphi) \geq -F_*(D^2 \varphi(x), D\varphi(x))$$

(3) Define: $u_{\Delta t}(\cdot, t) = \hat{G}_{t-t_n}(u_{\Delta t}(\cdot, (n-1)\Delta t))$

(4) Then $u_{\Delta t}(\cdot, t)$ converges to the (unique) viscosity solution of

$$u_t + F(D^2 u, Du) = 0$$

Anisotropic Curvature Motion



$$E(\Gamma) = \int_{\Sigma} \alpha(\hat{n}) d\mathcal{H}^{n-1}, \quad \sigma: S^{n-1} \longrightarrow \mathbb{R}^+$$

surface energy integrand

Anisotropic MBO (Ishii-Peres-Souganidis)

(1) Kernel (Given)

- $f(x) \geq 0$, $f(x) = f(-x)$, $\int_{\mathbb{R}^n} f(x) dx = 1$
- $\int_{\mathbb{R}^n} |x|^{2tp} f(x) dx < \infty$, $\underbrace{tp \in S^{n-1}}_0 < \int_{\mathbb{R}^n} (1+|x|^2) f(x) dx^{n-1} < \infty$
(existence of 2nd moment)
- $p \in S^{n-1} \rightarrow \int_{\mathbb{R}^n} f(x) dx^{n-1}$, $\int_{\mathbb{R}^n} x_i x_j f(x) dx^{n-1}$ are cont.

(2) Thresholding Algorithm:

Given $\Omega_n \subseteq \mathbb{R}^n$

$$\bullet \quad G_{\Delta t}^n(I_{\Omega_n}) = \int_{\mathbb{R}^n} \frac{1}{\Delta t^{n/2}} f\left(\frac{x-y}{\Delta t^{1/2}}\right) 1_{\Omega_n}(y) dy$$

$$\bullet \quad \Omega_{n+1} = \left\{ x : G_{\Delta t}^n(I_{\Omega_n}) \geq \frac{1}{2} - c\sqrt{\Delta t} \right\}$$

$$(3) \quad \Omega_{\Delta t}(t) = \Omega_n, \quad \text{for } n\Delta t \leq t < (n+1)\Delta t$$

(4) Formulas for Velocity:

$$\underline{v(X, p)} = \left(\int_{p^\perp} f(x) dH^{n-1} \right)^{-1} \left(-\frac{1}{2} \int_{p^\perp} \langle X_{xx} \rangle f(x) dH^{n-1} + c \right)$$

$$(X \in \mathcal{S}^n, \quad p \in S^{n-1})$$

$$\underline{F(X, p)} = -|p| v \left(-|p|^{-1} (I - \bar{p} \otimes \bar{p}) X (I - \bar{p} \otimes \bar{p}), -\bar{p} \right)$$

(F is degenerate elliptic: $X \leq Y \Rightarrow F(X, p) \geq F(Y, p)$)

F is geometric: $F(\lambda X + \mu p \otimes p, \lambda p) = F(X, p).$

(5) $\{1_{\Omega^{\Delta t}(t)}\}_{t \geq 0}$ converges as $\Delta t \rightarrow 0$
to a viscosity solution of

$$u_t + F(\tilde{D}u, Dn) = 0$$

Ishii-Peres-Souganidis's work gives a formula
for velocity, given kernel:

$$\text{Kernel}(\varphi) \xrightarrow{\quad} \text{Velocity}(v)$$


Design of Kernel (Bonnetier-Bretin-Chambolle)

- (1) Given an anisotropy $\sigma: S^{n-1} \rightarrow \mathbb{R}_+$,
 $(\sigma: \mathbb{R}^n \rightarrow \mathbb{R}_+, \sigma(p) = |p| \sigma(\frac{p}{|p|})$ is convex)

- (2) Velocity law:

$$V_n = \kappa_\sigma \hat{n}_\sigma$$

where $\hat{n}_\sigma = (\nabla_p \sigma)(\hat{n})$ (Cahn-Hoffman vector)
 $\kappa_\sigma = \operatorname{div}(\hat{n}_\sigma)$ (σ -Curvature)

(3) Formula of Kernel : (Inverse Fourier Transform)

$$K_{\sigma,t}(x) = \mathcal{F}^{-1} \left[e^{-4\pi^2 t \sigma(\xi)^2} \right] (x)$$

- The Kernel above is positive in Fourier domain but might not be positive in physical domain
- It might not have 2nd order moment
- Consistency of the scheme is checked.

(See Elsey-Esedoglu for more recent algorithm for kernel construction.

Other Relevant Works

(1) Swartz-Y.

A "new proof" of convergence of MBO that can be extended to handle kernels with some negative parts

(2) Esedoglu-Ruuth-Tsai

MBO scheme for higher order flows (such as surface diffusion and Willmore flows)

(3) Caffarelli-Sengulidis :

MBO scheme for non-local mean curv. flow

Variational Interpretation of MBO

(Esedoglu-Otto)

The MBO scheme has a gradient flow or minimizing movement structure.

MBO:

$$u_n = 2\mathbb{1}_{\Omega_n} - 1 \rightarrow G_{\text{sf}} * u_n \rightarrow u_{n+1} = \frac{G_{\text{sf}} * u_n}{|G_{\text{sf}} * u_n|}$$

$\Leftrightarrow$ minimize $-v(G_{\text{sf}} * u_n)(x)$, $|v| \leq 1$.
(point-wise)

$$\begin{aligned}
& \min_{|v| \leq 1} - \int v G^* u_n \\
&= \int v G^* (v - u_n - v) \\
&= \int -v G^* v + \int v G^* (v - u_n) \\
&= \int -v G^* v + \int (v - u_n) G^* (v - u_n) \\
&\quad + \int u_n G^* (v - u_n) \\
&\quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
&\quad \quad \quad \int u_n G^* v - \int u_n G^* u_n
\end{aligned}$$

$$\Rightarrow -\int v G^* u_n$$

$$= -\frac{1}{2} \int v G^* v + \frac{1}{2} \int (v - u_n) G^* (v - u_n)$$

(Since $v \in \mathcal{H}_1$, write $v = 21_\Sigma - 1$)

$$= -\frac{1}{2} \int (21_\Sigma - 1) G^* (21_\Sigma - 1)$$

$$= -\frac{1}{2} \int 4 1_\Sigma G^* 1_\Sigma - 2 \cdot G^* 1_\Sigma - 2 1_\Sigma \cdot G^* 1$$

$$- \frac{1}{2} \int 1 G^* 1$$

$$= -2 \int 1_\Sigma \cdot G^* 1_\Sigma - 1 \cdot G^* 1_\Sigma$$

$$= 2 \int (1 - 1_\Sigma) \tilde{G}^* 1_\Sigma$$

Hence

$$\min - \int v \tilde{G}^* u_n \quad (|v| \leq 1)$$

is equivalent to

$$\min \int (1 - 1_\Sigma) \tilde{G}^* 1_\Sigma + \int (1_\Sigma - 1_{\Sigma_n}) \tilde{G}^* (1_\Sigma - 1_{\Sigma_n})$$

$$(v = 21_\Sigma - 1 \in \{-1, 1\}, \quad u_n = 21_{\Sigma_n} - 1 \in \{-1, 1\})$$

$$\min \underbrace{\frac{1}{\sqrt{h}} \int (1 - \Sigma) \zeta_h^* \Sigma + \frac{1}{\sqrt{h}} \int (\Sigma - \Sigma_n) \zeta_h^* (\Sigma - \Sigma_n)}_{\text{Dissipation or motion delimiter.}}$$

Approximation of Area
Functional

Dissipation or
motion delimiter.

$$\approx \boxed{E_h(\Sigma) + \frac{1}{h} \|\Sigma - \Sigma_n\|^2}_{\text{distance}}$$

Resemble minimizing movement algorithm
of Almgren-Taylor-Wang & Luckhaus-
Sturzenhecker

Consistency of Algorithm (Σ, Σ_n -smooth sets)

$$(1) E_h(\Sigma) = \frac{1}{\sqrt{h}} \int (1 - 1_\Sigma) \zeta_h^* 1_\Sigma$$

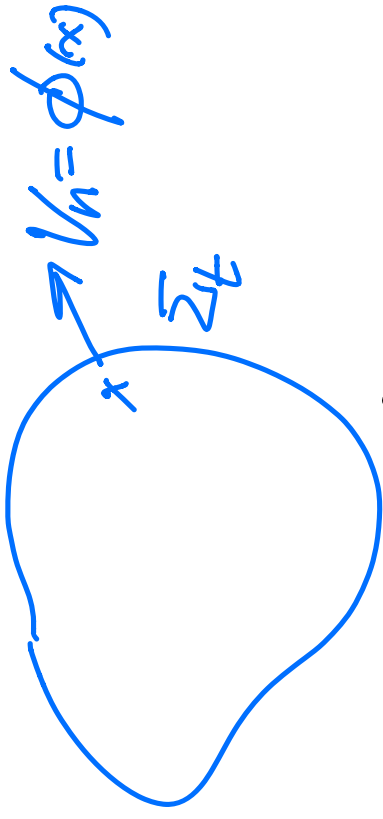
$$\xrightarrow{h \rightarrow 0} \int_{\partial \Sigma} \sigma(\hat{n}(x)) d\mathcal{H}^{n-1}(x) \quad (\text{surface energy})$$

where $\sigma: S^{n-1} \rightarrow \mathbb{R}_+$

$$\sigma(n) = \frac{1}{2} \int_{\mathbb{R}^n} |n \cdot x| \phi(x) dx$$

(Surface energy integrand)

(2) Let Σ_t evolve by normal velocity ϕ



Then the limit of the dissipation is given by:

$$\begin{aligned} \lim_{h \rightarrow 0} \lim_{t \rightarrow \infty} \frac{\sqrt{h}}{t^2} \int (\mathbb{1}_{\Sigma_t} - \mathbb{1}_{\Sigma_0}) G_h^* (\mathbb{1}_{\Sigma_t} - \mathbb{1}_{\Sigma_0}) d^n x \\ = \int_{\Sigma_0} \frac{1}{\mu(n\omega)} \phi^2(x) dA^{n-1}(x) \leftarrow \text{mobility} \end{aligned}$$

Mobility function:

$$\mu(n) = \frac{1}{\int_{\{x: x \cdot n = 0\}} G(x) dH^{n-1}(x)}$$

(3) The MBO scheme with Kernel G is consistent with the velocity function

$$V_n = \mu(n) \left[\left(\partial_{S_1}^2 \sigma(n) + \sigma(n) \right) \tau_1 + \left(\partial_{S_2}^2 \sigma(n) + \sigma(n) \right) \tau_2 \right]$$

differentiation along
principal direction

principal curvatures

Another representation of V_n :

$$V_n = \frac{\mu(n)}{2} \int_{\mathcal{H}^\perp} \left[K_1 \langle e_1, y \rangle^2 + K_2 \langle e_2, y \rangle^2 \right] \hat{G}(x) d\mathcal{H}^{n-1}$$

(principal direction)

(See Elsery-Esedoglu for more explicit relationship between σ of K , in particular when $\hat{K}(\xi) \geq 0$ or $K(x) \geq 0$)

Extension to Multi-phase Evolution

(Esedoglu-Otto, Lauritzen-Otto)

Energetic Information (Approximation of Surface Energy)

$$E_h(\chi) = \sum_{i,j} \sigma_{ij} \frac{1}{V_h} \int \chi_i \tilde{G}_h * \chi_j \, d^n x$$

where $\chi = \{\chi_i\}$, $\chi_i = 1_{\Omega_i} \in \{0,1\}$, i th phase

$$\sum_i \chi_i = 1$$

$$\underbrace{E_h(\gamma)}_{\text{approx. perimeter}} = \sum_{i,j} \sigma_{ij} \frac{1}{\sqrt{h}} \int \gamma_i \otimes \gamma_j$$

$\downarrow \Gamma \text{ conv.}$

$$\underbrace{E(\gamma)}_{\text{perimeter}} = \infty \sum_{i,j} \sigma_{ij} \left[\int |\nabla \gamma_i| + |\nabla \gamma_j| - |\nabla(\gamma_i + \gamma_j)| \right]$$

$$= \infty \sum_{i,j} \sigma_{ij} \text{Area}(\partial \Omega_i \cap \partial \Omega_j)$$

Multi-phase MBO Algorithm

(1) Given $\{x_i^n = 1\Omega_i^n, \sum_i x_i^n = 1\}$,

Define: $\phi_i = G_h * \left(\sum_i \sigma_{ij} x_j^n \right)$

(2) Thresholding:

$$\Omega_i^{n+1} = \begin{cases} x_i^n & \phi_i(x) < \phi_i(x) \\ 1\Omega_i^n & \forall i \neq i \end{cases}$$

($x_i^{n+1} = 1\Omega_i^{n+1}$)

Variational Reformulation of (1) & (2)

$$\chi^{n+1} = \{\chi_i^{n+1}\}$$

$$\in \arg\min \{E_h(\chi) - E_h(\chi - \chi^n)\}$$

surface energy



dissipation $\approx \frac{1}{h} \|\chi - \chi^n\|^2$



Condition on $\sigma = \{\sigma_{ij}\}$:

$$\left\{ \begin{array}{l} \sigma_{ij} = \sigma_{ji} \geq \sigma_{\min} > 0, \quad i \neq j, \quad \sigma_{ii} = 0 \\ \sigma_{ij} < \sigma_{ik} + \sigma_{kj} \quad i \neq j \neq k \\ \vec{\sigma} \leq -cI \quad \text{on} \quad (1, \dots, 1)^\perp \end{array} \right.$$

Definition of Motion by Mean Curvature (Tolks)

$$(i) \quad \chi = \alpha(\chi_i, t) : \mathbb{R}^n \times [0, T] \rightarrow \{0, 1\}$$

(ii) Uniform energy bound:

$$\sup_{t \in [0, T]} E(\chi(t)) < \infty$$

(iii) Existence of Velocities:

$$\exists \quad v_i(t) \quad \text{s.t.} \quad \sum_i \int_0^T \int v_i^2 |\nabla \chi_i| \, dx \, dt < \infty$$

s.t. for all $\xi, \int$

$$\textcircled{1} \sum_{i,j} \tilde{Q}_{ij} \int_0^T \int (\nabla \cdot \xi - n_i \nabla \cdot \xi \cdot n_j - 2 \xi \cdot n_i \nabla_i) \times (k(x+y) \Delta / -1/2 \Delta) \times (\nabla \chi_i / +1/2 \Delta) dx dt = 0$$

$$(V_{ij} = \frac{1}{2} H_{ij} \text{ on } \Sigma_{ij} = \partial \Omega_i \cap \partial \Omega_j)$$

$$\textcircled{2} \int_0^T \int \partial_t \chi_i dx dt + \int \chi_i(0) dx = - \int_0^T \int \chi_i \nabla \chi_i / dx dt$$

$$(V_i = \frac{\partial \chi_i}{\partial t})$$

Outline of Proof of Convergence (Laur-Otto)

Under the assumption of convergence of energy:

$$\int_0^T E_h(\chi_h^n(t)) dt \rightarrow \int_0^T E(\chi(t)) dt$$

(1) A priori Energy Estimates $\Rightarrow$ Space-Time Compactness

$$E_h(\chi^{n+1}) + \frac{1}{h} \|\chi^{n+1} - \chi^n\|_V^2 \leq E_h(\chi^n)$$

$$\sup_T E_h(\chi_h^n) + \int_0^T \|\partial_t \chi_h^n\|_V^2 dt < \infty$$

(2) Euler-Lagrange Equation (for x^n)

Since $x^n \in \operatorname{argmin}_{\mathcal{X}} E_h(x) - E_h(x - x^{n-1})$

Hence

$$\delta E_h(x^n, \xi) - \delta E_h(x^n - x^{n-1}, \xi) = 0 \quad \neq \xi$$

$$\delta E_h(x^n, \xi) = \frac{2}{\sqrt{n}} \sum_{i,j} \sigma_{ij} \int x_i^n \xi_h^y (-\nabla x_i^n \cdot \xi) dx \quad (1) \quad (2)$$

$$- \delta E_h(x^n - x^{n-1}, \xi) = \frac{2}{\sqrt{n}} \sum_{i,j} \sigma_{ij} \int (x_i^n - x_i^{n-1}) \xi_h^y (\nabla x_i \cdot \xi) dx$$

$$\textcircled{1} = \delta E_h(\chi^n, \xi) = \frac{2}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \chi_i^n \delta_h * u_g (-\nabla \chi_j^n, \xi) dx$$

$$= \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \chi_i^n \nabla \xi : (\delta_h I_d - h \nabla^2 \delta_h) * \chi_j^n dx$$

(+err)

↓ $h \rightarrow 0$, with energy convergence assumption

$$\sum_{i,j} \sigma_{ij} \int_0^T \int (\nabla \xi) \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|) dx dt$$

$$- \sum_{i,j} \sigma_{ij} \int_0^T \int (n_i \cdot \nabla \xi : n_j) \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|) dx dt$$

[Consistency Statements:

$$\begin{aligned}
 (i) \quad & \lim_{h \rightarrow 0} \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{i,j} \int \chi_i \delta_h * \chi_j dx \\
 &= \sum_{i,j} \sigma_{i,j} \frac{1}{2} \int \int (\nabla \chi_i + \nabla \chi_j) \cdot (\nabla \chi_i + \nabla \chi_j) dx \\
 (ii) \quad & \lim_{h \rightarrow 0} \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{i,j} \int \chi_i M : \nabla^2 \delta_h * \chi_j dx \\
 &= \sum_{i,j} \sigma_{i,j} \int (\nu M_{ij}) \frac{1}{2} (\nabla \chi_i + \nabla \chi_j) \cdot (\nabla \chi_i + \nabla \chi_j) dx
 \end{aligned}$$

$$\textcircled{2} = -\sum_{ij} E_{ij} (\chi^n - \chi^{n-1}) \zeta$$

$$= \frac{2}{\sqrt{n}} \sum_{ij} \sum_{i_1, \dots, i_n} \sigma_{ij} \int (\chi^n - \chi^{n-1})$$

(2-phase case)

$$= -2 \int (\chi^n - \chi^{n-1}) \zeta \left(\frac{\chi}{\sqrt{n}} \right) \int dx$$

$$\underbrace{\hspace{10em}}_{n \approx}$$

$$\underbrace{\hspace{10em}}_{(\pm \chi/\sqrt{n}) \approx}$$

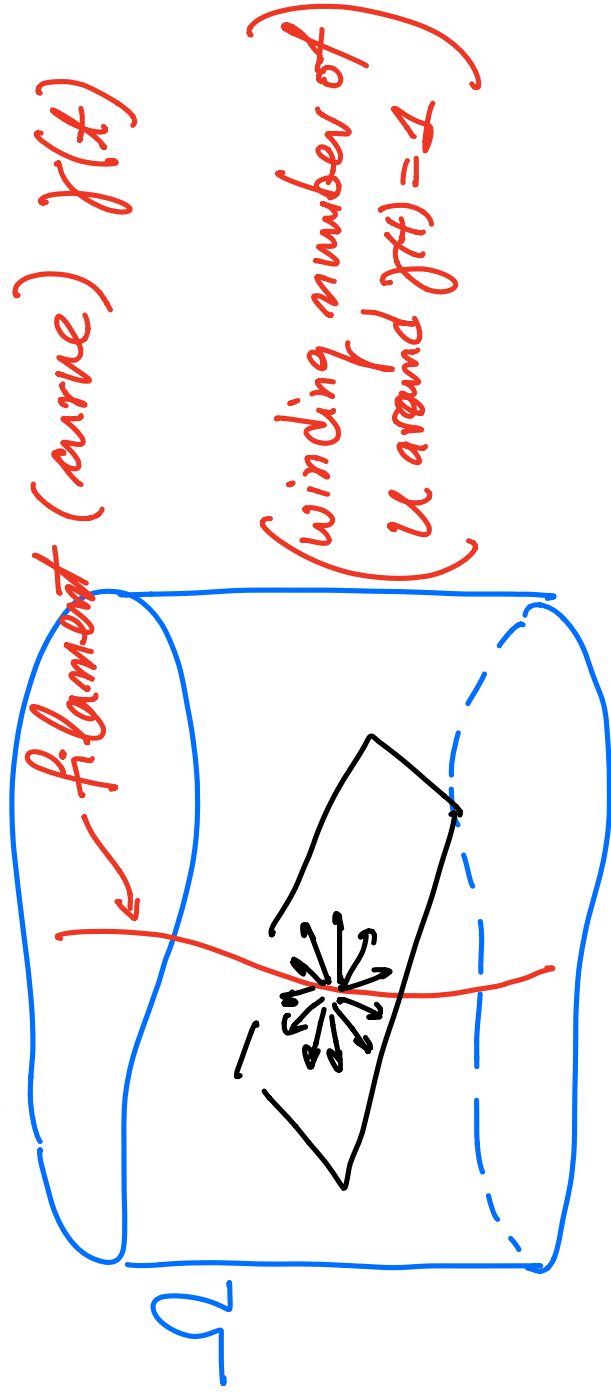
$\rightarrow \chi \rightarrow 0$

$$= -2 \sum_{ij} \sigma_{ij} \int_0^1 \zeta \cdot n_i \left(\frac{\chi}{\sqrt{n}} \right) \times \left(\frac{\chi}{\sqrt{n}} \right)^{\frac{1}{2}} \Delta - \left(\frac{\chi}{\sqrt{n}} \right)^{\frac{1}{2}} \Delta - \left(\frac{\chi}{\sqrt{n}} \right)^{\frac{1}{2}} \Delta$$

Curvature Motion of Filament in $\mathbb{R}^3$

$$u: \Omega \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \simeq \mathbb{C}$$

$$F_\varepsilon(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2 dx$$



Heat Flow of u :

$$u_t = \Delta u + \frac{1}{\varepsilon^2} u(1-u)^2 \quad (*)$$

(Rubinstein)

MBO for u :

$$u^n \rightarrow G_{\Delta t} * u^n \rightarrow u^{n+1} = \frac{G_{\Delta t} * u^n}{|G_{\Delta t} * u^n|}$$

(Ruth-Merriman-Xin-Osher)

Analyze the motion of the filament.

Asymptotics of (*) (Rubinstein)

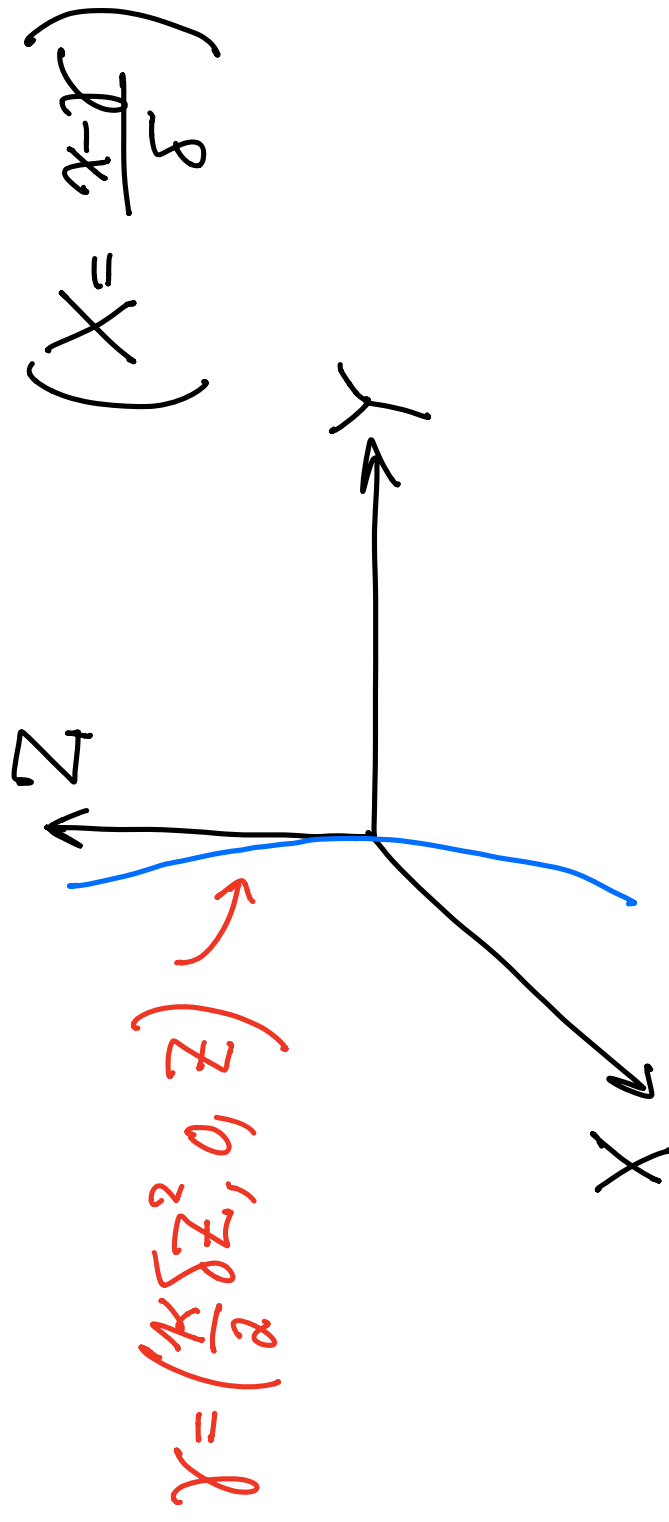
Outer Region ($\text{dist}(x, \gamma^*) \geq O(1)$)

$$u^\varepsilon(x, t) = u^0 + O(\varepsilon^2)$$

$$u^0 = e^{iS}, \quad S_t = \Delta S$$

Intermediate Region ($\text{dist}(x, \gamma^*) \sim O(\delta), \quad \delta \gg \varepsilon$)

$$u^\varepsilon = u^0 + \left(\frac{\varepsilon}{\delta}\right)^2 u', \quad u^i = u^i\left(\frac{x - \gamma^*}{\delta}\right)$$



$$u^0(x) = e^{iS}$$

$$\Delta_{xy} S - \delta x \delta y + \delta \nabla_{xy} S \cdot \dot{y} = O(\delta^2)$$

$$S = R \int_0^{\bar{\Phi}} \left(\frac{\partial H}{\partial R} - \delta(\kappa - \dot{\gamma}) \delta(\cos \bar{\Phi}) + \delta \dot{\gamma}_2 H(\sin \bar{\Phi}) \right) d\bar{\Phi} + O(\delta^2)$$

$$H = -\exp\left(\frac{\delta}{2}(\kappa - \dot{\gamma})X - \frac{\delta}{2}\dot{\gamma}_2 Y\right) K_0\left(\frac{\delta}{2}\sqrt{(\kappa - \dot{\gamma})^2 + \dot{\gamma}_2^2} R\right)$$

$$R = \sqrt{X^2 + Y^2}, \quad \bar{\Phi} = \tan^{-1} \frac{Y}{X}$$

K_0 - Bessel function

Inner Region ($\text{dist}(x, \gamma_H) = O(\varepsilon)$)

$$u^\varepsilon = u^0 + \varepsilon u^1, \quad X = \frac{x - \gamma_H}{\varepsilon}$$

$$\tilde{\Delta} u^0 + u^0(1 - |u^0|^2) = 0$$

$$L[u^1] = -\tilde{\nabla} u^0 \cdot \dot{\gamma} + \kappa u^0$$

$$L\varphi = \tilde{\Delta}\varphi + (1 - |u^0|^2)\varphi - \dot{\gamma} \cdot (u^0)^2 \dot{\gamma} - |u^0|^2 \varphi$$

Solvability Condition for u' , for $v \in \text{Null}(L)$

$$\text{Re} \left[-\dot{\gamma} \int_{BA} \tilde{\nabla} u^0 \bar{v} + \kappa \int_{BA} u_x^0 \bar{v} \right] = \text{Re} \left[\int_{BA} \left[-v \frac{\partial u'}{\partial r} - u' \frac{\partial \bar{v}}{\partial r} \right] \right]$$

$$u^0 = e^{i\phi} h(r), \quad \frac{1}{r} \frac{d}{dr}(r h) - \frac{1}{r^2} h + h - h^3 = 0$$

$$h(0) = 0, \quad h(\infty) = 1$$

$$u' \sim -ie^{i\phi} \frac{r}{2} \left[\ln(\epsilon b r) + \alpha \right] \left[(\sin \phi)(x - \dot{y}_1) + (\cos \phi - 1) \dot{y}_2 \right]$$

$\hookrightarrow b = [(x - \dot{y}_1)^2 + \dot{y}_2^2]^{\frac{1}{2}}$

$$\boxed{\text{Choose } v = \frac{\partial u^0}{\partial x} \Rightarrow}$$

$$-\pi \dot{\gamma}_2 \ln A - \pi D_A \dot{\gamma}_2 = -\pi \dot{\gamma}_2 \ln(\varepsilon b A) - \pi \ln\left(\alpha + \frac{1}{2}\right) \dot{\gamma}_2 + o(1)$$

$(1 \gg 3)$
 $(\varepsilon \ll 1)$
 $\dot{\gamma}_2 = 0$

$$\boxed{\text{Choose } v = \frac{\partial u^0}{\partial y} \Rightarrow}$$

$$\pi(\kappa - \dot{\gamma}_1)(\ln A + D_A) = \pi(\kappa - \dot{\gamma}_1) \left[\ln(\varepsilon x A) + \left(\alpha + \frac{1}{2}\right) \right] + o(A)$$

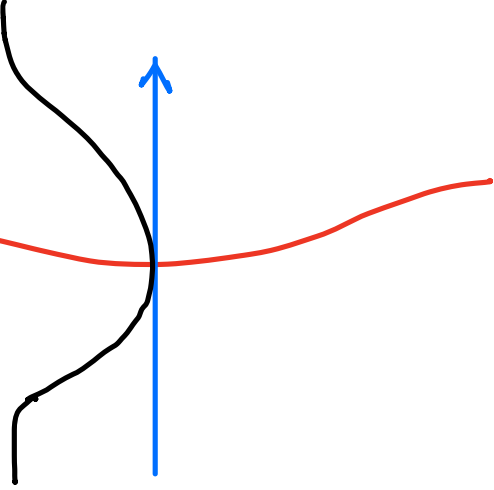
$\dot{\gamma}_1 = \kappa \leftarrow \text{motion by mean cur. along normal.}$

Analysis using cut-off function (Lin)

χ_z $\leftarrow$ filament
(energy concentrates)

$$\phi(r) = \begin{cases} r^2 & r < a \\ 4a^2 & r \geq 2a \end{cases}$$

$$r(x, t) = \text{dist}(x, \chi_z)$$



Consider the localized Ginzburg-Landau energy

$$\int \phi(r) \underbrace{\left[\frac{1}{2} |\nabla u|^2 + \frac{1}{4\epsilon^2} (1 - |u|^2)^2 \right]}_{e(u)} dx$$

Goal: $\frac{d}{dt} \int \phi(r) e(u) \leq c \int \phi(r) e(u)$

$\Downarrow$ Gronwall

$$\int \phi(r) e(u) dx \leq C \exp \left(\int \phi(r) e(u) dx \right)_{t=0} \leq C$$

$$\begin{aligned}
& \frac{d}{dt} \int \phi e(u) \\
&= \int \phi_t e(u) + \int \phi \left[\langle \nabla u, \nabla u_t \rangle - \frac{u(1-|u|^2)}{\varepsilon^2} u_t \right] \\
&= \int \phi_t e(u) + \int \phi \left[-\Delta u \cdot u_t - \frac{u(1-|u|^2)}{\varepsilon^2} u_t \right] \\
&\quad - \int \nabla \phi \cdot \nabla u u_t \\
&= - \int \phi u_t^2 + \int \phi_t e(u) - \underbrace{\int \nabla \phi \cdot \nabla u \cdot u_t}_{\downarrow}
\end{aligned}$$

$$- \int \nabla \phi \cdot \nabla u \, u \, \tau$$

$$= - \int \phi_{x_i} u_{x_i} \left[u_{x_j x_j} + \frac{u(1-u)^2}{\varepsilon^2} \right]$$

$$= \int (\phi_{x_i} u_{x_i})_{x_j} u_{x_j} - \phi_{x_i} u_{x_i} \frac{u(1-u)^2}{\varepsilon^2}$$

$$= \int \phi_{x_i x_j} u_{x_j} + \phi_{x_i} u_{x_i x_j} u_{x_j} + \phi_{x_i} u_{x_i} \left(\frac{1-u}{4\varepsilon^2} \right)_{x_j} x_j$$

$$= \int \phi_{x_i x_j} u_{x_i} u_{x_j} + \phi_{x_i} \left(\frac{1}{2} u_{x_j}^2 \right)_{x_i} + \phi_{x_i} \left(\frac{(1-u)^2}{4\varepsilon^2} \right)_{x_i}$$

$$= \int \phi_{x_i x_j} u_{x_i} u_{x_j} + \phi_{x_i} \left(\frac{1}{2} u_{x_j}^2 + \frac{(1-u)^2}{4\varepsilon^2} \right)_{x_i}$$

$$= \int \phi_{x_i x_j} u_{x_i} u_{x_j} - (\Delta \phi) e(u)$$

$$\frac{d}{dt} \int \phi(r) e(u) dx$$

$$= - \int \phi u_t^2 + \int (\phi - \Delta \phi) e(u) dx + \int \phi_{x_i} x_j u_{x_i} u_{x_j}$$

Note:

$$\bullet (\phi_{i|x_j}) \leq 2I \Rightarrow \phi_{i|x_j} u_i u_j \leq 2|\nabla u|^2$$

$$\bullet \text{ On } \gamma_z: \phi_z = 0, \Delta \phi = 4$$

$$\frac{d}{dz} \nabla \phi - \Delta \nabla \phi = 0$$

$$\bullet (\phi_z - \Delta \nabla)(r) = (\phi_z - \Delta \phi)(0) + O(r^2)$$

$$\leq -4 + C\phi$$

Hence

$$\frac{d}{dt} \int \phi e(u)$$

$$\leq \int 2|\nabla u|^2 dx + \int (-4 + c\phi) e(u) dx$$

$$= \int (2|\nabla u|^2 - 4e(u)) dx + c \int \phi e(u) dx$$

$$= \int (2|\nabla u|^2 - 2|\nabla u|^2 - \frac{(1-|u|^2)^2}{\varepsilon^2}) dx + c \int \phi e(u) dx$$

$$\leq c \int \phi(r) e(u)$$

Proof of Convergence of MBO for Filament

(Laur-Y.)

MBO energy and Gradient Flow Structure

$$E_h(u) = \frac{1}{h_c} \int (1 - u \hat{G}_h * u) dx$$

$u^{n-1} \longrightarrow \frac{\hat{G}_h * u^{n-1}}{|\hat{G}_h * u^{n-1}|}$ is equivalent to

$$\min E_h(u) + \frac{1}{h_c} \int (u - u^{n-1}) \hat{G}_h * (u - u^{n-1})$$

Localized Version

$$E_h(u, \phi) = \frac{1}{h} \int \phi(1-u \cdot \hat{G}_h^* u) \, dx$$

$u^{n-1} \longrightarrow \frac{\hat{G}_h^* u^{n-1}}{|\hat{G}_h^* u^{n-1}|}$ is then equivalent to

$$\min E_h(u, \phi) + \frac{1}{h} \int \phi(u - u^{n-1}) \cdot \hat{G}_h^* (u - u^{n-1}) \\ + \frac{1}{h} \int |\hat{G}_h^* u^{n-1}| \cdot [\hat{G}_h^*, \phi] u^{n-1} \, dx$$

Some open problems (MBO in phase-field) ($\varepsilon \rightarrow 0$)

- (1) General anisotropic filament motion
- (2) Harmonic maps coupled with minimal surf.
- (3) Schrodinger flow:
$$i u_t = \Delta u - \frac{1}{\varepsilon^2} W'(u)$$

(filament: $V_n = \kappa \hat{b}$.)
- (4) Higher order flows, e.g. Willmore or surf. diff.