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Convergence Estimates of Multilevel Additive and Multiplicative Algorithms for Non-symmetric and Indefinite Problems

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New uniform estimates for multigrid algorithms are established for certain non-symmetric indefinite problems. In particular, we are concerned with the simple additive algorithm and multigrid ($V(1, 0)$-cycle) algorithms given in [5]. We prove, without full elliptic regularity assumption, that these algorithms have uniform reduction per iteration, independent of the finest mesh size and number of refinement levels, provided that the coarsest mesh size is sufficiently small.

KEY WORDS elliptic equations; multilevel methods; finite element

1. Introduction

In recent years, multilevel additive and multiplicative methods have been investigated to effectively solve the non-symmetric discrete equations which arise in numerical approximation of partial differential equations ([3], [6], [12], [17], [18], and [20]). So far, convergence rate estimates for these methods are weakly dependent on the number of refinement levels J that could deteriorate when J is becoming large. In contrast, the theory developed in this paper gives uniform convergence rates for both the additive and multiplicative algorithms introduced in Section 3. We analyze non-symmetric additive preconditioners

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and multiplicative iteration operators here, in contrast to the analyses of the multilevel methods as presented in [20] and [17], but our analysis can also be applied to those which are symmetric. We prove that both algorithms have uniform convergence rates, without full elliptic regularity assumption, with respect to the number of refinement levels J and finest mesh parameter h_J provided that the coarsest mesh parameter h_0 is sufficiently small. That is, the finite element approximation on the coarsest grid is accurate enough. Such an assumption is common for the convergence theory of multilevel methods (e.g., [3], [6], [12], [17], and [18]) as well as the theory of finite element discretization of indefinite problems (e.g., [1], [15] and [20]).

We consider a class of second order, non-symmetric, and indefinite elliptic partial differential equations on a two-dimensional polygonal or three-dimensional polyhedral domain Ω. The problems are solved numerically by using a nested sequence of linear finite element spaces generated by quasi uniform refinement of the coarsest mesh. Using these vector spaces we build multilevel additive and multiplicative algorithms that are similar to those considered in [5] but for solving certain symmetric positive-definite counterparts.

The crucial parts in the theoretical development here are that the underlying non-symmetric discrete operators are 'small' when restricted to coarser grids and that the non-symmetric part of the original bilinear form is uniformly bounded by the energy norm induced from the symmetric second order terms. The former property, which is similar to the strengthened Cauchy–Schwarz inequality, plays an important role in obtaining the uniform estimates for both additive and multiplicative cases. Similar ideas were also used in [22] and [5] for solving symmetric positive-definite problems. To obtain the latter property, we use the fundamental approximation result given in [16] that is not based on any regularity assumption for the original non-symmetric problems.

The outline of the remainder of the paper is as follows. We give a general formulation for some non-symmetric and indefinite boundary value problems in Section 2. We then introduce the projection operators and the algorithms in Section 3. Finally, we present an abstract convergence analysis for the additive algorithm as well as the multiplicative algorithm in Section 4.1 and Section 4.2 respectively.

2. The model problem

For simplicity, throughout this paper, let Ω be a two-dimensional polygon. Extensions of the results in subsequent sections to higher dimensions are straightforward and omitted here. We consider the solution of the elliptic boundary value problem

$$\begin{cases} -\nabla \cdot (A\nabla u) + \vec{b} \cdot \nabla u + cu = f, \text{ in } \Omega \\ \qquad\qquad\qquad\qquad\quad u = 0, \text{ on } \partial\Omega \end{cases} \tag{2.1}$$

We shall impose some weak assumptions on the coefficients above. Assume that $A \in R^{2\times 2}$ is a given symmetric matrix function, which is uniformly positive definite for almost all $x \in \overline{\Omega}$; and that a_{ij} is in $W^{\gamma,p}(\Omega)$, the Sobolev space of order γ defined in terms of the L^p-norm (cf. [11]) for some positive parameters $\gamma \in (0, 0.5)$ and $p > 2/\gamma$. Note that functions a_{ij}, that are piecewisely smooth with respect to subregions with Lipschitz continuous boundaries, are in such a space. Assume also that $\vec{b} \equiv (b_1, b_2)^t$ is a continuously differentiable vector function on $\overline{\Omega}$, and that c is a bounded scalar function. Moreover, we assume that (2.1) has a unique solution for each f in $H^{-1}(\Omega)$, the dual space of the Sobolev space $H_0^1(\Omega) \equiv W_0^{1,2}(\Omega)$, with the norm $\|\cdot\|_{-1}$.

Multiplying (2.1) by a smooth test function v that vanishes on $\partial\Omega$ and integrating by parts gives variational form of (2.1): Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall\, v \in H_0^1(\Omega) \tag{2.2}$$

Here, $(\cdot, \cdot)$ denotes the L^2 inner product with induced norm $\|\cdot\|$ and for any $u, v \in H_0^1(\Omega)$ bilinear form $a(u, v)$ is defined as

$$a(u, v) = a^s(u, v) + b(u, v)$$

where

$$\begin{cases} a^s(u, v) = \int_\Omega (A\nabla u)^t(\nabla v)\,dx, \text{ and} \\ b(u, v) \;\; = \int_\Omega (\vec{b}\cdot\nabla u + cu)v\,dx \end{cases} \tag{2.3}$$

Under the above assumptions, it is straightforward to verify that $a^s(u, v)$ is bounded and uniformly elliptic (coercive) in $H_0^1(\Omega)$. As a consequence the energy norm

$$|v|_a \equiv \sqrt{a^s(v, v)}, \quad \forall\, v \in H_0^1(\Omega)$$

is equivalent to the H^1 norm, $\|v\|_1$. By the assumptions on the coefficients in problem (2.1), the symmetric bilinear form $a^s(\cdot, \cdot)$ is uniformly equivalent to the form corresponding to the constant coefficient operator $-\Delta$. Hence, we assume that there is an α in $(0, 1]$ such that solutions u of (2.1) with $A = I$, $\vec{b} = \vec{0}$, and $c = 0$ satisfy the following regularity estimate:

$$\|u\|_{1+\alpha} \;\leq\; C\|f\|_{-1+\alpha} \tag{2.4}$$

Here, $\|f\|_{-1+\alpha}$ is the interpolated norm between $L^2(\Omega)$ and $H^{-1}(\Omega)$. This regularity assumption is weak (for more discussions, see [5]).

Here and henceforth, we may drop the subscript for the finest mesh parameter h_J and use C with or without script to denote a generic positive constant independent of the number of levels J and the finest mesh parameter h_J. We note the following inequalities regarding the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ in terms of the L^2 and energy norms

$$|b(u, v)| \leq \begin{cases} C|u|_a\|v\| \\ C\|u\||v|_a \end{cases} \tag{2.5}$$

$$|a(u, v)| \leq C|u|_a|v|_a, \quad \forall\, u, v \in H_0^1(\Omega)$$

In addition, we have the Gårding's inequality:

$$|v|_a^2 - C\|v\|^2 \leq a(v, v), \quad \forall\, v \in H_0^1(\Omega)$$

We consider a finite element approximation to problem (2.2) in terms of a collection of nested finite dimensional subspaces of $H_0^1(\Omega)$ characterized by the triangulation parameters h_j, $j = 0, 1, \ldots, J$. To define these subspaces, we start with an intentionally coarse triangulation $\mathcal{T}_0$ of $\overline{\Omega}$ with the properties that the boundary $\partial\Omega$ is composed of edges of some triangles T in $\mathcal{T}_0$ and that every triangle of $\mathcal{T}_0$ is shape regular. Each triangle T of $\mathcal{T}_0$ is regularly refined several times, giving a family of nested triangulation $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_J \equiv \mathcal{T}$ such that triangle of $\mathcal{T}_{k+1}$ is generated by subdividing a triangle of $\mathcal{T}_k$ into congruent triangles (cf.[9]). As a result the ratio

$$\gamma_j = \frac{h_j}{h_{j+1}}$$

is an integer bigger than or equal to one and bounded above by a fixed constant for $j = 0, 1, \cdots, J-1$. Throughout this paper, for simplicity of the presentation, we let

$$\gamma_j = 2, \quad \forall\, j \in \{0, 1, \ldots, J-1\} \tag{2.6}$$

the case corresponds to the uniform refinement that each triangle of $\mathcal{T}_{k+1}$ is generated by subdividing a triangle of $\mathcal{T}_k$ into four congruent ones.

For each j=0, 1, ..., J, we associate the triangulation $\mathcal{T}_j$ with the piecewise linear finite element space V_j (cf. [2] and [9]). It is easy to verify that the family of spaces $\{V_j\}$ is nested, i.e.,

$$V_0 \subset V_1 \subset \ldots \subset V_J \equiv V_h \equiv V_{h_J} \tag{2.7}$$

The finite element approximation to the solution of problem (2.2) is to seek $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h \tag{2.8}$$

We use nodal bases functions $\{\phi_i(x)\}$ as a basis for V_h. After substituting the expression of u_h and v_h in terms of the basis into (2.8), we obtain a system of linear equations

$$A_J y = g$$

where $A_J = \big(a(\phi_j, \phi_i)\big)$ is the stiffness matrix; $g_i = f(\phi_i)$ is the i-th component of the vector g; and y_i is the i-th component of solution vector y satisfying

$$u(x) = \sum_i y_i \phi_i(x)$$

Note that the matrix A_J is non-symmetric and indefinite with condition number $\kappa(A_J) = O(h^{-2})$ and that existence and uniqueness of the finite element solution u_h to problem (2.8) does not immediately follow in general. However, it does hold if h is sufficiently small. In subsequent sections, we assume existence of solution to the above non-symmetric and indefinite system of linear equations.

3. Additive and multiplicative algorithms

In this section, we present construction of multilevel additive and multiplicative algorithms that are used to solve the system of linear equations resulting in (2.8)

$$A_J u = f \tag{3.1}$$

where A_J is a non-symmetric and indefinite operator on a finite dimensional vector space V_J. We start by introducing a linear iterative process for solving (3.1) and definition of operators in Subsection 3.1. Then, in Subsection 3.2 and Subsection 3.3, we describe the additive and multiplicative algorithms.

Given an approximate solution u^{old} of (3.1), we produce a new approximation u^{new} by the following three steps:

1. Compute the residual $r^{\text{old}} = f - A_J u^{\text{old}}$. If $r^{\text{old}} = 0$ or very small, then stop. Otherwise, perform the following two steps.
2. Solve the residual equation $A_J e = r^{\text{old}}$ approximately: compute $\tilde{e} = Br^{\text{old}}$ where B is an approximate inverse of A_J.
3. Update $u^{\text{new}} = u^{\text{old}} + \tilde{e}$, then go to Step 1.

It is well known that choice of B in the above iterative process plays an important role for effectively solving both symmetric and non-symmetric systems of linear equations. One way to choose B is by solving certain subspace problems like the additive algorithm introduced in Subsection 3.2.

3.1. Definition of operators

Assume that we are given a nested sequence of finite dimensional vector spaces as (2.7). We introduce the following operators:

1. the projection $Q_j : V_J \to V_j$ is defined for $u \in V_J$ by
$$(Q_j u, v) = (u, v), \quad \forall v \in V_j \quad (j = 0, 1, \ldots, J)$$
2. the projection $P_j^s : V_J \to V_j$ is defined for $u \in V_J$ by
$$a^s(P_j^s u, v) = a(u, v), \quad \forall v \in V_j \quad (j = 1, \ldots, J)$$
3. the projection $P_0 : V_J \to V_0$ is defined for $u \in V_J$ by
$$a(P_0 u, v) = a(u, v), \quad \forall v \in V_0$$
4. the operator $A_j^s : V_j \to V_j$ is defined for $u \in V_j$ by
$$(A_j^s u, v) = a^s(u, v), \quad \forall v \in V_j \quad (j = 1, 2, \ldots, J)$$
5. the operator $A_0 : V_0 \to V_0$ is defined for $u \in V_J$ by
$$(A_0 u, v) = a(u, v), \quad \forall v \in V_0$$
6. the operator $A_J : V_J \to V_J$ is defined for $u \in V_J$ by
$$(A_J u, v) = a(u, v), \quad \forall v \in V_J$$

Moreover, we note that any $v \in V_J$ can be written as

$$v = \sum_{j=0}^{J} (Q_j - Q_{j-1}) v \quad (Q_{-1} = 0,\ Q_J = I) \tag{3.2}$$

where 0 and I are the respective null and identity operators. Also, it is straightforward to verify that

$$Q_j A_J = A_j^s P_j^s \quad (j = 1, 2, \cdots, J)$$

3.2. Additive algorithm

Given u^{old}, compute a new approximation u^{new} by the following two steps.

1. Instead of solving the whole residual equation, we seek a correction $\tilde{e}_j$ by following equation in each subspace V_j
$$\tilde{e}_j = R_j Q_j r^{\text{old}}$$
where R_j is a certain 'smoother' or subspace solver in V_j.
2. An update of the approximation of u is then obtained by summing over all corrections $\{\tilde{e}_j\}$ as
$$u^{\text{new}} = u^{\text{old}} + \sum_{j=0}^{J} \tilde{e}_j$$

In this algorithm, it is clear that u^{new} can be written as a linear iterative process of the form
$$u^{\text{new}} = u^{\text{old}} + B^a(f - A_J u^{\text{old}})$$
Hence, the multilevel additive preconditioner B^a is
$$B^a = \sum_{j=0}^{J} R_j Q_j$$

It is straightforward to verify that the preconditioned operator $B^a A_J$ satisfies the relations
$$B^a A_J = \sum_{j=0}^{J} R_j Q_j A_J = \sum_{j=0}^{J} R_j A_j^s P_j^s \equiv \sum_{j=0}^{J} T_j$$

We use B^a as a preconditioner in co-operation with GMRES type iterative methods (cf. [10] and [14]) to solve the non-symmetric problem (2.2). It is well known that, unlike the conjugate gradient method for the symmetric positive-definite problems, the GMRES method for solving non-symmetric problems may not converge without proper preconditioning. A preconditioner for the GMRES method is not only to speed up convergence rate but also to guarantee the convergence of the method. Besides, Eisenstat, Elman and Schultz [10] proved that the rate of convergence of the GMRES method can be approximated in terms of the minimal eigenvalue of the symmetric part of the preconditioned operator $B^a A_J$, which is defined as
$$\alpha_0 = \inf_{v \in V_J} \frac{a^s(B^a A_J v, v)}{|v|_a^2}$$
together with the energy norm α_1 of the operator $B^a A_J$. The asymptotic convergence rate with respect to the energy norm for GMRES method is $1 - (\alpha_0^2/\alpha_1^2)$ (cf. [10]). In Section 4.1, we prove that both α_0 and α_1 are uniformly bounded below and above, respectively, with respect to the number of refinement levels J and the finest mesh parameter h_J providing the coarsest mesh parameter h_0 is sufficiently small.

In subsequent sections, for simplicity, we let the smoothing operators R_0 be A_0^{-1} and R_j be $(1/\lambda_j)I$ for $j = 1, 2, \ldots, J$. Here, λ_j is the spectral radius of the operator A_j^s, and it is straightforward to show that $\lambda_j = O(h_j^{-2})$ for our model problem. With these notations, we have the following identities

$$T_0 = R_0 Q_0 A_J = R_0 A_0 P_0 = P_0, \quad \text{and}$$
$$T_j = R_j Q_j A_J = \frac{1}{\lambda_j} Q_j A_J = \frac{1}{\lambda_j} A_j^s P_j^s \quad (j = 1, 2, \dots, J)$$

Also, we note that one coarsest problem is solved per preconditioning step due to the choice of the smoother R_0 on the coarsest mesh.

3.3. Multiplicative algorithm

We consider the following multiplicative algorithm (V(1, 0)-cycle). Given $u^\ell \in V_J$, an approximation to the solution of (3.1), we define the next approximation $u^{\ell+1}$ as follows:

1. Set $y_J = u^\ell$.
2. Compute an update from each successively coarser level according to

$$y_{J-j-1} = y_{J-j} + R_{J-j} Q_{J-j}(f - A_J y_{J-j}) \quad \text{for} \quad j = 0, 1, \cdots, J$$

3. Set $u^{\ell+1} = y_{-1}$.

It is straightforward to show that

$$u - u^{\ell+1} = (I - T_0)(I - T_1) \cdots (I - T_J)(u - u^\ell)$$

The convergence of the multiplicative algorithm followed from norm estimate of the following error reduction operator

$$E_J = (I - T_0)(I - T_1) \cdots (I - T_J)$$

In Section 4.2, we show that the energy norm of the error reduction operator E_J is uniformly bounded by a positive constant that is strictly less than one, independent of the number of refinement levels J and the finest mesh parameter h_J, providing the coarsest mesh parameter h_0 is sufficiently small.

4. Convergence analysis

In this section, we present and prove uniform convergence theorems for the additive and multiplicative algorithms. We first prove that the minimal eigenvalue of the symmetric part of the preconditioned operator $B^a A_J$ and its energy norm are uniformly bounded below and above respectively, independent of the number of refinement levels J and the finest mesh parameter h_J, providing that the coarsest mesh parameter h_0 is sufficiently small. The former is presented in Theorem 4.1 and the latter in Theorem 4.2. We then present the uniform convergence theorem for multiplicative case in Theorem 4.3.

We first note the following well-known approximation and boundedness properties for the operators $\{Q_j\}$ in terms of L^2 and energy norms: for any $v \in H_0^1(\Omega)$ we have the following inequalities

$$\|(Q_j - Q_{j-1})v\|^2 \le C\lambda_j^{-1} a^s(v, v), \quad j = 1, 2, \dots, J \tag{4.1}$$

and

$$a^s(Q_j v, Q_j v) \leq C a^s(v, v), \quad j = 0, 1, \ldots, J \tag{4.2}$$

where $\lambda_j^{-1} \leq Ch_j^2$ for our application. By the construction of the triangulation it is simple to verify that the sum of $\left\{\lambda_j^{-\frac{1}{2}}\right\}$ satisfying

$$\sum_{j=1}^{J} \lambda_j^{-\frac{1}{2}} \leq Ch_0 \sum_{j=1}^{J} 2^{-j} \leq Ch_0 \tag{4.3}$$

We next cite a lemma regarding the sum of $\|(Q_j - Q_{j-1})v\|^2$ for $j = 1, 2, \cdots, J$. For symmetric positive-definite problems, this lemma leads to a much weaker assumption that replaces the full regularity assumptions for multigrid type algorithms (see [5] for a detailed proof).

Lemma 4.1. *For every v in V_J there exists a positive constant C independent of J and h_J satisfying*

$$|Q_0 v|_a^2 + C \sum_{j=1}^{J} \lambda_j \|(Q_j - Q_{j-1})v\|^2 \leq C|v|_a^2$$

In [16], Schatz and Wang proved the following fundamental result for the non-symmetric indefinite elliptic problems introduced in Section 2 without regularity assumption. This lemma is used in estimating $\|b(\cdot, \cdot)\|$.

Lemma 4.2. *For any fixed $\varepsilon > 0$ and for every v in V_J, these exists $H > 0$ such that P_0, the operator obtained by restricting the original problem to the coarsest grid, has the following relations: $\forall v \in V_J$,*

$$\|P_0 v - v\| \leq \varepsilon |P_0 v - v|_a \leq C\varepsilon |v|_a, \qquad \forall \ \ 0 < h_0 \leq H$$

4.1. *Additive version*

We now prove that the sum of the operators $\{T_j^* T_j\}$ is positive definite with respect to the energy norm in Lemma 4.3, and that $\{T_j\}$ are non-negative up to a small perturbation in Lemma 4.5. These lemmas play a fundamental role in the development of the uniform convergence estimates for both the additive and multiplicative preconditioners.

Lemma 4.3. *For sufficiently small h_0, there exists a positive constant C_0 independent of J and h_J such that*

$$C_0 a^s(v, v) \leq \sum_{j=0}^{J} a^s(T_j v, T_j v), \quad \forall v \in V_J \tag{4.4}$$

Proof Substituting (3.2) for v we obtain the following equation

$$a^s(v, v) = a^s(v, Q_0 v) + \sum_{j=1}^{J} a^s(v, (Q_j - Q_{j-1})v)$$

By the definition of the bilinear form $a(\cdot, \cdot)$, the operators, and Lemma 4.2, we establish an upper bound for the first term of the right hand side in the previous equality as follows

$$\begin{aligned} a^s(v, Q_0 v) &= a^s(T_0 v, Q_0 v) + b((P_0 - I)v, Q_0 v) \\ &\leq C|T_0 v|_a |Q_0 v|_a + C\|(P_0 - I)v\| |Q_0 v|_a \\ &\leq C|T_0 v|_a |Q_0 v|_a + C\varepsilon |v|_a |Q_0 v|_a \end{aligned} \tag{4.5}$$

To obtain an upper bound for the summation term, a little manipulation is needed. It follows from the definition of the operators $\{T_j\}$, (2.5), and (4.1) that

$$\begin{aligned} \sum_{j=1}^{J} a^s(v, (Q_j - Q_{j-1})v) &\leq \sum_{j=1}^{J} \lambda_j (T_j v, (Q_j - Q_{j-1})v) + C \sum_{j=1}^{J} |v|_a \|(Q_j - Q_{j-1})v\| \\ &\leq C \sum_{j=1}^{J} \lambda_j |T_j v|_a \|(Q_j - Q_{j-1})v\|_{-1} + C \sum_{j=1}^{J} h_j |v|_a^2 \end{aligned} \tag{4.6}$$

Since

$$\begin{aligned} \|(Q_j - Q_{j-1})v\|_{-1} &= \sup_{\phi \in H_0^1(\Omega)} \frac{((Q_j - Q_{j-1})v, \phi)}{\|\phi\|_1} \\ &= \sup_{\phi \in H_0^1(\Omega)} \frac{((Q_j - Q_{j-1})v, (Q_j - Q_{j-1})\phi)}{\|\phi\|_1} \\ &\leq C\lambda_j^{-\frac{1}{2}} \|(Q_j - Q_{j-1})v\| \end{aligned}$$

by (4.5), (4.6), and the Cauchy–Schwarz inequality we have that

$$\begin{aligned} a^s(v, v) \leq & \left(\sum_{j=0}^{J} |T_j v|_a^2 \right)^{\frac{1}{2}} \left(|Q_0 v|_a^2 + \sum_{j=1}^{J} \lambda_j \|(Q_j - Q_{j-1})v\|^2 \right)^{\frac{1}{2}} \\ & + C\varepsilon |v|_a |Q_0 v|_a + C|v|_a^2 \sum_{j=1}^{J} h_j \end{aligned}$$

Lemma 4.1 and (4.2) then yield

$$a^s(v, v) \leq C_1 \left(\sum_{j=0}^{J} |T_j v|_a^2 \right)^{\frac{1}{2}} |v|_a + C_2(\varepsilon + h_0)|v|_a^2$$

Choosing a sufficiently small h_0 such that $1 - C_2(\varepsilon + h_0) > 0$, (4.4) now follows with constant

$$C_0 = \left(\frac{1 - C_2(\varepsilon + h_0)}{C_1} \right)^2$$

This completes the proof of the lemma. ∎

From the definition of bilinear form $a^s(\cdot, \cdot)$ and the fact that coefficients $\{a_{ij}\}$ in (2.1) belong to $W^{\gamma,p}(\Omega)$ with $\gamma \in (0, 0.5)$, we have the following property (see [5] for a detailed proof): for $i \le j$, there exists a constant $\tilde{C}$ not depending on the mesh parameter satisfying

$$|a^s(w, \phi)| \le \tilde{C} h_j^{(-1+\gamma)} h_i^{-\gamma} |w|_a \|\phi\|, \quad \forall\, w \in V_i, \quad \phi \in V_j$$

This property can be easily extended to non-symmetric bilinear form $a(\cdot, \cdot)$, i.e.,

$$|a(w, \phi)| \le C\left(h_j^{(-1+\gamma)} h_i^{-\gamma} + 1\right)|w|_a \|\phi\|, \quad \forall\, w \in V_i, \quad \phi \in V_j \tag{4.7}$$

Lemma 4.4. *For any $v \in V_J$, we have the following estimates*

$$\|T_j v\| \;=\; \frac{1}{\lambda_j}\|A_j^s P_j^s v\| \;\le\; C h_j |v|_a \tag{4.8}$$

Proof (4.8) follows from the definition of the operators and (4.7) with $i = j$ that

$$\begin{aligned} \|T_j v\| &= \frac{1}{\lambda_j}\|A_j^s P_j^s v\| = \frac{1}{\lambda_j} \sup_{\phi \in V_j} \frac{(A_j^s P_j^s v, \phi)}{\|\phi\|} \\ &= \frac{1}{\lambda_j} \sup_{\phi \in V_j} \frac{a^s(P_j^s v, \phi)}{\|\phi\|} = \frac{1}{\lambda_j} \sup_{\phi \in V_j} \frac{a(v, \phi)}{\|\phi\|} \\ &\le C h_j |v|_a \end{aligned}$$

■

Lemma 4.5. *Given $\varepsilon > 0$ and sufficiently small h_0, the energy norm of $T_j v$ ($j = 0, 1, \cdots, J$) satisfying*

$$a^s(T_0 v, T_0 v) \le a^s(T_0 v, v) + C\varepsilon a^s(v, v), \ \ \forall v \in V_J \tag{4.9}$$

and

$$a^s(T_j v, T_j v) \le a^s(T_j v, v) + C h_j a^s(v, v), \ \ \forall v \in V_J \quad \forall j = 1, 2, \ldots, J \tag{4.10}$$

Proof We will prove (4.10) only since the proof of (4.9) is similar. From the definition of the operators, the fact that $(A_j^s - \lambda_j I)$ is non-positive, and Lemma 4.4, we obtain that

$$\begin{aligned} a^s(T_j v, T_j v) - a^s(v, T_j v) &= a^s(T_j v, T_j v) - a(v, T_j v) + b(v, T_j v) \\ &= a^s(T_j v, T_j v) - a^s(P_j^s v, T_j v) + b(v, T_j v) \\ &= ((A_j^s - \lambda_j I) T_j v, T_j v) + b(v, T_j v) \\ &\le C |v|_a \|T_j v\| \\ &\le C h_j |v|_a^2 \end{aligned}$$

(4.10) now follows by rearranging the sums on both sides. ■

The next remark is an immediate consequence of Lemma 4.5 by summing over $j = 0, 1, 2, \cdots, J$ in (4.9) and (4.10).

Remark 1. For any fixed $\varepsilon > 0$, we obtain that

$$\sum_{j=0}^{J} a^s(T_j v, T_j v) \le C \sum_{j=0}^{J} a^s(v, T_j v) + C(h_0 + \varepsilon) a^s(v, v), \quad \forall v \in V_J$$

We are now ready to prove that the smallest eigenvalue of the symmetric part of operator $B^s A_J$ is bounded below uniformly by a constant, providing that the coarsest mesh parameter h_0 is sufficiently small.

Theorem 4.1. *For any $v \in V_J$, there exists $H > 0$ such that for any $0 < h_0 \le H$, we have*

$$a^s(v, v) \le C a^s(v, B^a A_J v), \quad \forall v \in V_J \tag{4.11}$$

Proof (4.11) follows from Lemma 4.3 and Remark 4.1. ■

Next, we show that the preconditioned operator $B^a A_J$ is uniformly bounded above in the energy norm. To this end, note the well-known results (cf. [4],[5], [7],[13], [19] and [21]): for every v in V_J,

$$\sum_{j=1}^{J} \lambda_j^{-1} \| Q_j A_J^s v \|^2 \le C a^s(v, v) \tag{4.12}$$

Since non-symmetric operator A_J is a small perturbation to symmetric operator A_J^s, it is intuitively clear that (4.12) also holds for A_J.

Lemma 4.6. *For every v in V_J, we have that*

$$\sum_{j=1}^{J} \lambda_j^{-1} \| Q_j A_J v \|^2 \le C a^s(v, v), \quad \textit{for all } v \in V_J \tag{4.13}$$

Proof By the definition of the operators and the bilinear form, it is clear to verify the identity

$$(Q_j A_J v, Q_j A_J v) = (Q_j A_J^s v, Q_j A_J^s v) + b(v, Q_j A_J v) + b(v, Q_j A_J^s v)$$

To bound the perturbation term $b(v, Q_j A_J^s v)$ above, we apply (2.6), the Cauchy–Schwarz inequality, and (4.12) to get that

$$\begin{aligned}
\sum_{j=1}^{J} \lambda_j^{-1} b(v, Q_j A_J^s v) &\le C \sum_{j=1}^{J} \lambda_j^{-1} |v|_a \| Q_j A_J^s v \| \\
&\le C |v|_a \left(\sum_{j=1}^{J} \lambda_j^{-1} \right)^{\frac{1}{2}} \left(\sum_{j=1}^{J} \lambda_j^{-1} \| Q_j A_J^s v \|^2 \right)^{\frac{1}{2}} \\
&\le C h_0 |v|_a^2
\end{aligned} \tag{4.14}$$

Using (2.3) and Lemma 4.4 in the second perturbation term yield

$$\begin{aligned}\sum_{j=1}^{J} \lambda_j^{-1} b(v, Q_j A_J v) &\le C \sum_{j=1}^{J} \lambda_j^{-1} |v|_a \|A_j^s P_j^s v\| \\ &\le C|v|_a^2 \sum_{j=1}^{J} h_j \\ &\le C h_0 |v|_a^2\end{aligned} \tag{4.15}$$

(4.13) now follows from (4.12), (4.14), and (4.15). ∎

Theorem 4.2. *There exists a positive constant H such that for any $0 < h_0 \le H$, we have that*

$$a^s(B^a A_J v, B^a A_J v) \le C a^s(v, v), \quad \forall v \in V_J \tag{4.16}$$

Proof From the triangle inequality, we have

$$a^s(B^a A_J v, B^a A_J v) \le 2\left(a^s(T_0 v, T_0 v) + a^s\left(\sum_{j=1}^{J} T_j v, \sum_{j=1}^{J} T_j v\right)\right)$$

It thus suffices to show that T_0 and $\sum_{j=1}^{J} T_j$ are bounded in the energy norm. Boundedness of T_0 follows from the Cauchy–Schwarz inequality and Lemma 4.2 shows that

$$\begin{aligned}a^s(T_0 v, T_0 v) = a^s(P_0 v, T_0 v) &\le |P_0 v|_a |T_0 v|_a \\ &\le \left(|v|_a + |(I - P_0)v|_a\right)|T_0 v|_a \\ &\le (1 + C)|v|_a |T_0 v|_a\end{aligned}$$

To show boundedness of $\sum_{j=1}^{J} T_j$ in the energy norm, let $w = \sum_{j=1}^{J} T_j v$. By the definition of the operators $\{T_j\}$, the Cauchy–Schwarz inequality, (4.12), and Lemma 4.6, we have that

$$\begin{aligned}|w|_a^2 &= \sum_{j=1}^{J} a^s(T_j v, w) \\ &= \sum_{j=1}^{J} \lambda_j^{-1} (Q_j A_J v, Q_j A_J^s w) \\ &\le \left(\sum_{j=1}^{J} \lambda_j^{-1} \|Q_j A_J v\|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{J} \lambda_j^{-1} \|Q_j A_J^s w\|^2\right)^{\frac{1}{2}} \\ &\le C|v|_a |w|_a\end{aligned}$$

which implies that

$$a^s\left(\sum_{j=1}^{J} T_j v, \sum_{j=1}^{J} T_j v\right) \le C a^s(v, v)$$

Hence, this completes the proof of theorem. ■

4.2. *Multiplicative version*

Our goal is to prove that the energy norm of the error reduction operator E_J is uniformly bounded above by a constant strictly less than one, independent of the number of refinement levels J and the finest mesh parameter h_J, providing that the coarsest mesh parameter h_0 is sufficiently small.

We shall outline lemmas that play a crucial role in the development of the uniform convergence estimate for the multiplicative algorithm. For a detailed analysis, readers should consult the framework in [8]. We first give and verify the following important observation: for our application, the non-symmetric operator T_j is 'small' when applying functions in V_i for i smaller than or equal to j.

Lemma 4.7. *For sufficiently small coarsest triangulation parameter h_0, there is a positive number $0 < \delta < 1$ and a positive constant C, independent of mesh parameters J and h_J, satisfying*

$$a^s(v, T_j v) \le C\delta^{2(j-i)} a^s(v, v), \quad \forall v \in V_i \quad \textit{and} \quad i \le j \tag{4.17}$$

Proof By the definitions of the operators and the L^2 norm, it follows from (2.5) and (4.7) that

$$\begin{aligned}
a^s(v, T_j v) &= \frac{1}{\lambda_j} a(v, A_j^s P_j^s v) - \frac{1}{\lambda_j} b(v, A_j^s P_j^s v) \\
&\le \frac{C}{\lambda_j}\left(\sup_{\phi \in V_j} \frac{(A_j^s P_j^s v, \phi)}{\|\phi\|}\right)^2 + \frac{C}{\lambda_j}|v|_a\left(\sup_{\phi \in V_j} \frac{(A_j^s P_j^s v, \phi)}{\|\phi\|}\right) \\
&\le \frac{C}{\lambda_j}\left(\sup_{\phi \in V_j} \frac{a(v, \phi)}{\|\phi\|}\right)^2 + \frac{C}{\lambda_j}|v|_a\left(\sup_{\phi \in V_j} \frac{a(v, \phi)}{\|\phi\|}\right) \\
&\le \frac{C}{\lambda_j}\left(h_j^{(-1+\gamma)} h_i^{-\gamma} + 1\right)^2 |v|_a^2 + \frac{C}{\lambda_j}\left(h_j^{(-1+\gamma)} h_i^{-\gamma} + 1\right)|v|_a^2
\end{aligned}$$

Using the fact that $\lambda_j^{-1} \le C h_j^2$ and (2.6), we obtain

$$\begin{aligned}
a^s(T_j v, v) &\le C\left(\left(\frac{h_j}{h_i}\right)^{2\gamma} + h_j^2 + h_j\left(\frac{h_j}{h_i}\right)^{\gamma}\right)|v|_a^2 \\
&\le C\delta^{2(j-i)}|v|_a^2
\end{aligned}$$

where

$$\delta = \begin{cases} \left(\frac{1}{2}\right)^{\gamma}, & \text{if } \left(\frac{h_j}{h_i}\right)^{2\gamma} \geq h_j^2 \\ \left(\frac{h_0^{1/j}}{2}\right), & \text{otherwise} \end{cases}$$

is a positive constant less than one for sufficiently small h_0 (e.g., h_0 is less than or equal to one). This completes the proof of lemma. ■

Since the bilinear form $a(\cdot,\cdot)$ is equal to $a^s(\cdot,\cdot)$ up to a small perturbation, it is straightforward to obtain the following inequality analogous to (4.17).

Lemma 4.8. *For every $v \in V_i$ and $i \leq j$ we have that*

$$a(v, T_j v) \leq C\hat{\delta}^{2(j-i)} a^s(v, v) \tag{4.18}$$

Here,

$$\hat{\delta} = \Delta^{\frac{1}{2(j-i)}} \left(\Delta = \max\left\{\delta^{2(j-i)}, h_0 2^{-j}\right\}\right) \tag{4.19}$$

is a positive constant less than one for sufficiently small h_0.

Proof Using (4.17), the definition of bilinear form $a(\cdot,\cdot)$ and the operator T_j, and Lemma 4.4, we obtain

$$\begin{aligned} a(v, T_j v) &\leq C\delta^{2(j-i)} a^s(v, v) + C\|T_j v\| |v|_a \\ &\leq C\left(\delta^{2(j-i)} + h_j\right)|v|_a^2 \\ &= C\left(\delta^{2(j-i)} + h_0 2^{-j}\right)|v|_a^2 \end{aligned}$$

Now, (4.18) is valid with (4.19). ■

By making use of Lemma 4.8, it is straightforward to prove the following estimates (for more details, see [8]).

Lemma 4.9. *For any v in V_J, there exists $H > 0$ such that for any $0 < h_0 \leq H$, we have*

$$\sum_{j=0}^{J} a^s(T_j v, T_j v) \leq C_1 \sum_{j=0}^{J} a^s(T_j E_{j-1} v, T_j E_{j-1} v)$$

and

$$\sum_{j=0}^{J} a^s(T_j E_{j-1} v, T_j E_{j-1} v) \leq C_2 a^s(v, v) - C_3 a^s(E_J v, E_J v)$$

with

$$C_2 = C_3 + \frac{O(h_0 + \epsilon)}{1 - O(h_0)} \quad \text{and} \quad C_3 = \frac{\tilde{C}}{1 - O(h_0)}$$

Here, C_1 and $\tilde{C}$ are constants, independent of the finest mesh parameter h_J and the number of levels J.

The following uniform convergence theorem for the multiplicative algorithm follows from Lemma 4.3 and Lemma 4.9.

Theorem 4.3. *For any v in V_J, there exists $H > 0$ such that for any $0 < h_0 \leq H$, we have*

$$a^s(E_J v, E_J v) \leq \hat{C} a^s(v, v)$$

Here, the $\hat{C} \in (0, 1)$ is a constant, independent of the finest mesh parameter h_J and the number of levels J.

5. Conclusion

In this paper, we present and prove uniform convergence results for some multilevel additive and multilevel multiplicative algorithms for certain non-symmetric and indefinite problems. We note that the elliptic regularity assumption used in our proof is weak, that the theory for multiplicative case (multigrid $V(1, 0)$-cycle) can be extended to the multigrid algorithm of arbitrary cycles, and that our results can be extended to the case involving a locally refined mesh (cf. [5]).

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