3.12.3 *The fourth order accuracy of P^2 FEM as a finite difference scheme*

The standard finite element error estimate for schemes in this section is third order in L^2-norm. But it can be proven that (3.36) is actually fourth order accurate in the 2-norm over grid points.

First of all, we can check that the finite difference approximation to the second order derivative in (3.36) is only second order accurate, even for the one in (3.36b). Second, if we use this second order approximation to solve a second order PDE such as $-u''(x) = f$, we get a fourth order accurate scheme (in ℓ^2-norm or 2-norm)! As a matter of fact, it can be rigorously proven that this scheme is fourth order accurate for commonly used linear second order PDEs for

- Elliptic equation $-\Delta u = f$.
- Parabolic equation $u_t = \Delta u$.
- Wave equation $u_{tt} = \Delta u$.
- Schrödinger equation $\mathbb{i} u_t = \Delta u$.
- Helmholtz equation $-\Delta u - k^2 u = f$.
- Variable coefficient version of the equations above.

Remark 3.17. *In general, by the standard superconvergence theory of P^k ($k \geq 2$) finite element method* (3.41) *(even for a variable coefficient problem in multiple dimensions), function values of $u_h(x)$ are $(k+2)$-th order accurate at Gauss-Labotto points for each small interval in 2-norm, as opposed to $(k+1)$-th order in the L^2-norm error estimate, and derivatives of $u_h(x)$ are $(k+1)$-th order accurate at Gauss points, as opposed to k-th order in the H^1-norm error estimate. For a discrete Laplacian scheme like* (3.36)*, i.e., the finite difference implementation of* (3.41)*, the proof of the $(k+2)$-th order accuracy in 2-norm in 2D and 3D can be found in [Li and Zhang (2020c)] for elliptic equations, in [Li* et al. *(2022)] for parabolic, wave and Schrödinger equations , and in [Zhang (2025)] for the Helmholtz equation.*

All error estimates here are *a priori* error estimates, which means that the order holds if the exact solution $u(x)$ is smooth enough. For instance, the fourth order accuracy of (3.37) can be proven only if assuming $u \in H^4(\Omega)$. In practice, we often use high order accurate schemes for nonsmooth solutions, for which high order *a priori* error estimates can no longer hold. So a natural question is whether it still makes sense to use a high order accurate scheme like (3.36) on uniform meshes, which is nonetheless often used in applications. Figure 3.7 shows a comparison of between the second order finite difference (3.28) and the fourth order finite difference (3.36) for solving the following generalized Allen-Cahn equation

$$\phi_t + u\phi_x + v\phi_y = \mu\Delta\phi - \frac{F'(\phi)}{\varepsilon}, \quad (x, y) \in \Omega, \tag{3.38}$$

where u, v are given incompressible velocity field, and $F'(\cdot)$ is some fixed energy potential term. With the first order accurate implicit explicit (IMEX) time discretization, it becomes

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} + u^{n+1}\phi_x^{n+1} + v^{n+1}\phi_y^{n+1} = \mu\Delta\phi^{n+1} - \frac{F'(\phi^n)}{\varepsilon}. \tag{3.39}$$

For the differential operators in (3.39), we can used two finite difference schemes derived from P^1 and P^2 finite element method with quadrature. For the second order derivative, they are (3.28) and (3.36). In Figure 3.7, we can see that the solution $\phi(x, y)$ has a sharp interface with large gradient, thus the smoothness assumption we need for proving that (3.36) is a fourth order scheme may no longer hold. Nonetheless, the fourth order spatial discretization is still superior because the second order spatial discretization gives a wrong solution on the same coarse 239×239 grid. Higher order time accuracy here does not help the second order spatial discretization on the same coarse 239×239 grid. This is somehow intuitive since usually time evolution is a lot smoother, thus spatial error is dominant in these problems. In other words, when spatial error dominates, higher order accurate spatial discretization often performs better.

3.13 Efficient inversion of Laplacian in three dimensions

For solving the Poisson equation on a rectangular domain, the same simple and efficient implementation in Section 2.9 can also be extended to arbitrary high order finite element method with Q^k polynomial basis on rectangular meshes. In two dimensions, the Q^k polynomial space has dimension $(k+1)^2$, spanned by $x^i y^j$ for $i, j = 0, 1, \cdots, k$. In three dimensions, Q^k polynomial space is similarly defined with dimension $(k + 1)^3$, while P^k polynomial space has dimension $\frac{1}{6}(k+1)(k+2)(k+3)$. The Q^k basis is used for finite element methods on rectangular meshes, while P^k is used for finite element methods on simplicial meshes consisting of triangles and pyramids.

Now consider solving $-u_{xx} - u_{yy} - u_{zz} = f$ on a cube $\Omega = [0, 1]^3$ with homogeneous Dirichlet boundary conditions. Let us use Q^2 element as an example, and consider a 3D rectangular mesh shown in Figure 3.8 (a).