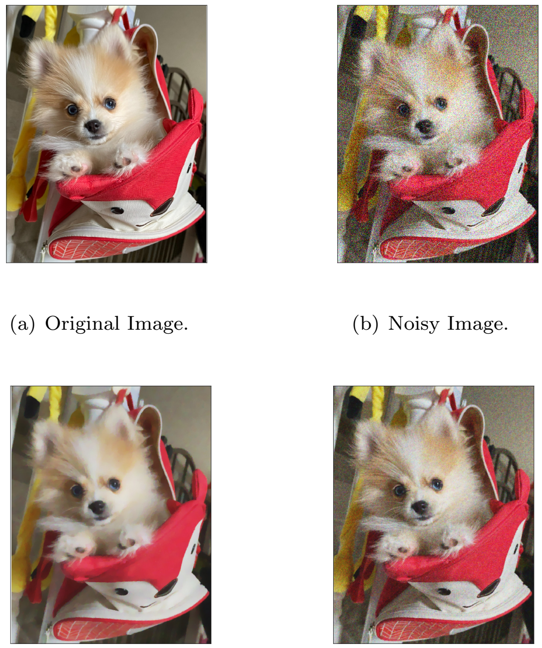

(a) Original Image. (b) Noisy Image.

(c) $C = 4$. (d) $C = 8$.

Fig. 4.7 The minimizer of ROF model (4.13) with different $\lambda = \frac{C}{h}$.

4.2.3 *Primal, dual and primal-dual forms*

Using all notation above, the discrete ROF model can be written as a convex minimization problem

$$\min_{U \in \mathbb{R}^{n \times n}} f(\mathcal{K}U) + g(U), \tag{4.14a}$$

where $\mathcal{K} = \nabla_h : \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}^{2(n \times n)}$ is a linear mapping

$$\mathcal{K}(U) = \nabla_h U = (UD^T, DU), \tag{4.14b}$$

and

$$f(P, Q) = \sum_{i,j} h^2 \sqrt{P^2(i,j) + Q^2(i,j)}, \quad g(U) = \lambda \sum_{i,j} h^2 |U(i,j) - a(i,j)|^2. \tag{4.14c}$$

It is straightforward to verify that the adjoint operator of $\mathcal{K}$ is given by

$$\mathcal{K}^* = -\nabla_h \cdot : \mathbb{R}^{2(n\times n)} \longrightarrow \mathbb{R}^{n\times n}$$
$$(P, Q) \longmapsto (PD + D^T Q)$$

The convex minimization (4.14a) is in *primal* form, and it has an equivalent *dual* form:

$$-\min_{\mathbf{P}\in\mathbb{R}^{2(n\times n)}} f^*(\mathbf{P}) + g^*(-\mathcal{K}^*\mathbf{P}), \tag{4.15}$$

where f^* and g^* are *convex conjuate* (a.k.a., Legendre transform) of the convex function f and g respectively. See Appendix C.3 for the definition and properties of the convex conjugate functions. Both (4.14a) and (4.15) also have an equivalent *primal-dual* form:

$$\min_{U\in\mathbb{R}^{n\times n}} \max_{\mathbf{P}\in\mathbb{R}^{2(n\times n)}} \langle \mathcal{K}U, \mathbf{P}\rangle - f^*(\mathbf{P}) + g(U), \tag{4.16}$$

Remark 4.2. *The minimizer U^* to* (4.14a) *and the minimizer $\mathbf{P}^*$ to* (4.15) *are related via the first order optimality condition:*

$$0 = \mathcal{K}^*\mathbf{P} + \nabla g(U),$$

which gives $0 = \mathcal{K}^*\mathbf{P} + \lambda(U - A) \Leftrightarrow U = A - \frac{1}{\lambda}\mathcal{K}^*\mathbf{P}$.

To be more explicit, for the TV-norm denoising problem of a 2D image $A \in \mathbb{R}^{n\times n}$, the primal problem can be written as

$$\min_{U\in\mathbb{R}^{n\times n}} \|\mathcal{K}U\|_1 + \frac{\lambda}{h}\|U - A\|_F^2,$$

where $\|\cdot\|_F$ is the Frobenius norm for a matrix $\|U - A\|_F = \sqrt{\sum_{i,j}|U(i,j) - a(i,j)|^2}$ and the 1-norm for a pair of matrices $\mathbf{V} = (P, Q)$ is

$$F(\mathbf{V}) = \|(P, Q)\|_1 = \sum_{i,j} \sqrt{P(i,j)^2 + Q(i,j)^2}.$$

Up to a constant shift, the dual problem can be written as

$$-\min F^*(\mathbf{V}) + \frac{h}{2\lambda}\|\mathcal{K}^*\mathbf{V} - \frac{\lambda}{h}A\|_F^2,$$

where the convex conjugate or Legendre transform of $F(\mathbf{V})$ is

$$F^*(\mathbf{V}) = F^*(P, Q) = \sum_{i,j} \iota_{\{x^2+y^2\le 1\}}(P(i,j), Q(i,j)),$$

with a convex closed proper (extended) function $\iota_{\{x^2+y^2\le 1\}}$ called *indicator function of a convex closed set (see Appendix C.1)*:

$$\iota_{\{x^2+y^2\le 1\}}(x, y) = \begin{cases} 0, & x^2 + y^2 \le 1 \\ +\infty, & x^2 + y^2 > 1 \end{cases}.$$