4.2.4 *Poisson solver in ADMM for the primal problem*

The alternating direction method of multipliers (ADMM) introduced in [Glowinski and Marroco (1975)] is a widely used popular algorithm for solving convex minimization problems like (4.14a). See Appendix C for some background knowledge about ADMM. For solving (4.14a), one particular version of ADMM with only one algorithm parameter $\sigma > 0$ is written as

$$\text{(ADMM)}: \quad \begin{aligned} \mathbf{P}_{k+1} &= \operatorname{argmin}_{\mathbf{P}} f(\mathbf{P}) + \langle \mathbf{Q}_k, \mathbf{P} - \mathcal{K}U_k \rangle + \frac{\sigma}{2}\|\mathbf{P} - \mathcal{K}U_k\|^2 \\ U_{k+1} &= \operatorname{argmin}_U g(U) + \langle \mathbf{Q}_k, \mathbf{P}_{k+1} - \mathcal{K}U \rangle + \frac{\sigma}{2}\|\mathbf{P}_{k+1} - \mathcal{K}U\|^2 \\ \mathbf{Q}_{k+1} &= \mathbf{Q}_k + \sigma(\mathbf{P}_{k+1} - \mathcal{K}U_{k+1}). \end{aligned}$$

This particular version of ADMM is equivalent to the another popular and powerful algorithm, called *Douglas-Rachford splitting* (DRS) introduced in [Lions and Mercier (1979)]. ADMM and DRS are equivalent in the following sense (see Appendix C):

$$\text{ADMM on primal} \Leftrightarrow \text{Douglas-Rachford on dual},$$

$$\text{ADMM on dual} \Leftrightarrow \text{Douglas-Rachford on primal}.$$

In particular, it was proven in [Lions and Mercier (1979)] that DRS with any positive step size converges to a minimizer for a convex problem, which implies the particular version of ADMM in this section converges with any positive parameter $\sigma > 0$.

We will see how to implement ADMM later. For now, we focus on why we need a Poisson solver, which is related to the second line in ADMM. By ignoring terms that are not dependent of U, the second line is written as

$$U_{k+1} = \operatorname{argmin}_U g(U) - \langle \mathbf{Q}_k, \mathcal{K}U \rangle + \frac{\sigma}{2}\|\mathcal{K}U - \mathbf{P}_{k+1}\|^2.$$

Notice that $g(U)$ is a simple quadratic function, thus the minimizer is obtained by finding critical point, for which we need to take derivative of $\|\mathcal{K}U - \mathbf{P}_{k+1}\|^2$ w.r.t. U:

$$\frac{\partial}{\partial U}\langle \mathcal{K}U - \mathbf{P}_{k+1}, \mathcal{K}U - \mathbf{P}_{k+1} \rangle = h^2(2\mathcal{K}^*\mathcal{K}U - 2\mathcal{K}^*\mathbf{P}_{k+1}).$$

So the second line can be equivalently written as

$$\lambda(U_{k+1} - A) - \mathcal{K}^*\mathbf{Q}_k + \sigma\mathcal{K}^*\mathcal{K}U_{k+1} - \sigma\mathcal{K}^*\mathbf{P}_{k+1} = 0$$

which is

$$(\lambda I + \sigma\mathcal{K}^*\mathcal{K})U_{k+1} = \lambda A + \mathcal{K}^*\mathbf{Q}_k + \sigma\mathcal{K}^*\mathbf{P}_{k+1}.$$