Optimally Stable High-order DG Schemes for Multi-D Linear Hyperbolic Systems

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Outline

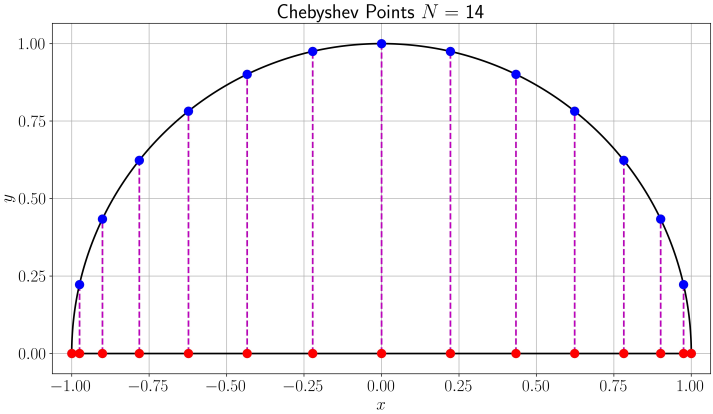
Outline

Spectral differention

Ill-conditioning due to mesh non-uniformity

Chebyshev points:

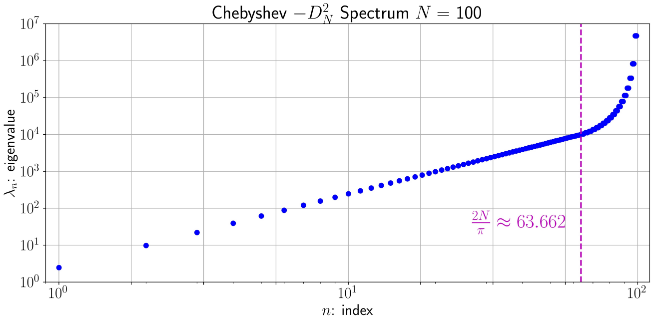
$$x_k = \cos\left(\frac{k\pi}{N}\right) \quad \text{for} \quad k = 0, 1, \ldots, N$$



Spectral differention

Ill-conditioning due to mesh non-uniformity

Spectrum of the second derivative matrix:



- Spurious eigenvalues for large n (modes localized near boundaries):

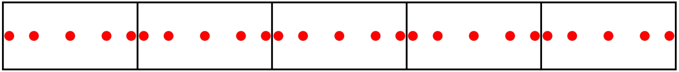
$$\text{small-to-moderate } n: \quad \lambda_n = \mathcal{O}\left(n^2\right), \qquad \text{large } n: \quad \lambda_n = \mathcal{O}\left(n^4\right)$$

- e.g., explicit time-stepping for $q_{,t} = q_{,x,x}$ requires: $\Delta t = \mathcal{O}\left(N^{-4}\right)$

Discontinuous Galerkin methods

Ill-conditioning due to sub-mesh non-uniformity

DG-FEM is a type of spectral element method:



- Consider the simple advection equation:

$$q_{,t} + u\, q_{,x} = 0$$

- Explicit methods (fixed stencil) must satisfy the CFL condition:

$$\Delta t = \mathcal{O}\left(\Delta x\right)$$

- Due to the non-uniformity in the sub-mesh, DG stability satisfies:

$$\Delta t \leq C \cdot \frac{\Delta x}{M_{\text{deg}}}$$

- Well-known issue for Runge-Kutta DG, Lax-Wendroff DG, ...

Motivating application

Linear kinetic models

Linear kinetic equations

$$f_{,t}\left(t,\underline{x},\underline{\Omega}\right)+\underline{\Omega}\cdot\nabla_{\underline{x}} f\left(t,\underline{x},\underline{\Omega}\right)+\sigma_t f\left(t,\underline{x},\underline{\Omega}\right)=\frac{\sigma_s}{4\pi}\int_{\mathbb{S}^2} f\left(t,\underline{x},\underline{\Omega}'\right)\, d\Omega'$$

Reduction to systems of linear PDEs:

e.g., P_N approximation: $\quad f\left(t,\underline{x},\underline{\Omega}\right)\approx\sum_{\ell=0}^{N}\sum_{m=\ell}^{\ell} F_\ell^m\left(t,\underline{x}\right)\, Y_\ell^m(\mu,\varphi)$

$$\underline{F}_{,t}+\underline{\underline{A}}^x\,\underline{F}_{,x}+\underline{\underline{A}}^y\,\underline{F}_{,y}+\underline{\underline{A}}^z\,\underline{F}_{,z}=\underline{\underline{C}}\,\underline{F}$$

Goals of this work:

- High-order explicit methods (DG-FEM)
- Compact stencils (3-point in 1D, 9-point in 2D, 27-point in 3D)
- Time-steps restrictions at or near the CFL limit

Motivating application

Kinetic Vlasov models

Vlasov-Maxwell equations

$$\begin{aligned} f_{,t} + \underline{v} \cdot \nabla_{\underline{x}} f - \left(\underline{E} + \underline{v} \times \underline{B}\right) \cdot \nabla_{\underline{v}} f &= 0 \\ \underline{E}_{,t} - \nabla_{\underline{x}} \times \underline{B} &= -\underline{J} \\ \underline{B}_{,t} + \nabla_{\underline{x}} \times \underline{E} &= -\underline{0} \\ \nabla_{\underline{x}} \cdot \underline{E} &= \rho \\ \nabla_{\underline{x}} \cdot \underline{B} &= 0 \end{aligned}$$

Goals of this work:

- High-order explicit methods (DG-FEM)
- Compact stencils (3-point in 1D, 9-point in 2D, 27-point in 3D)
- Time-steps restrictions at or near the CFL limit
- Unsplit time-stepping and exact mass, momentum, & energy conservation

Outline

Lax-Wendroff discontinuous Galerkin
[Qiu, Dumbser, and Shu, 2005]

For simplicity consider 1D advection:

$$q_{,t} + u q_{,x} = 0, \qquad \text{let} \quad t = t^{n+\frac{1}{2}} + \tau \frac{\Delta t}{2} \quad \text{and} \quad x = x_i + \xi \frac{\Delta x}{2}$$

$$q_{,\tau} + \nu q_{,\xi} = 0, \qquad \nu = \frac{u \Delta t}{\Delta x}$$

Taylor series in time:

$$q(1,\xi) = q(-1,\xi) + 2 q_{,\tau}(-1,\xi) + 2 q_{,\tau,\tau}(-1,\xi) + \frac{4}{3} q_{,\tau,\tau,\tau}(-1,\xi) + \cdots$$

$$q(1,\xi) = q(-1,\xi) - 2\nu q_{,\xi}(-1,\xi) + 2\nu^2 q_{,\xi,\xi}(-1,\xi) - \frac{4}{3}\nu^3 q_{,\xi,\xi,\xi}(-1,\xi) + \cdots$$

Multiply by test function $\underline{\phi}$ and integrate over $\xi \in [-1,1]$:

$$\underline{Q_i^{n+1}} = \underline{Q_i^n} - \int_{-1}^{1} \underline{\phi}\, F_{,\xi}\, d\xi, \qquad F^{\text{LxW}}(q^h) = \nu q^h - \nu^2 q^h_{,\xi} + \frac{2}{3}\nu^3 q^h_{,\xi,\xi} + \cdots$$

$$\underline{Q_i^{n+1}} = \underline{Q_i^n} + \int_{-1}^{1} \underline{\phi}_{,\xi}\, F^{\text{LxW}}(q^h)\, d\xi - \left[\underline{\phi}(1)\, \mathcal{F}^{\text{LxW}}_{i+\frac{1}{2}} - \underline{\phi}(-1)\, \mathcal{F}^{\text{LxW}}_{i-\frac{1}{2}} \right]$$

Reformulating LxW-DG

[Gassner, Dumbser, Hindenlang, and Munz, 2011]

Predictor step:

- Solution from the previous time step:

$$q(t^n, x) \approx q_i^n := \underline{\Phi}^T \underline{Q_i^n}, \qquad \underline{\Phi} \in \mathbb{R}^M$$

- Solution in each space-time element, $\mathcal{T}_i^{n+\frac{1}{2}}$ $(M^\star = M(M+1)/2)$:

$$q(t,x)\Big|_{\mathcal{T}_i^{n+\frac{1}{2}}} \approx w_i^{n+\frac{1}{2}} := \underline{\Psi}^T \underline{W_i^{n+\frac{1}{2}}}, \qquad \underline{\Psi} \in \mathbb{R}^{M^\star}$$

- Multiply by test function, integrate-by-parts in time (but not space):

$$\iint \underline{\Psi} \left[\underline{\Psi}_{,\tau} + \nu \underline{\Psi}_{,\xi}\right]^T \underline{W_i^{n+\frac{1}{2}}} \, d\tau \, d\xi + \int \underline{\Psi}_{|_{\tau=-1}} \underline{\Psi}^T_{|_{\tau=-1}} \underline{W_i^{n+\frac{1}{2}}} \, d\xi = \int \underline{\Psi}_{|_{\tau=-1}} \underline{\Phi}^T \underline{Q_i^n} \, d\xi$$

$$\Longrightarrow \quad \underline{\underline{L}}^0 \, \underline{W_i^{n+\frac{1}{2}}} = \underline{\underline{T}} \, \underline{Q_i^n} \quad \Longrightarrow \quad \underline{W_i^{n+\frac{1}{2}}} = \left(\underline{\underline{L}}^0\right)^{-1} \underline{\underline{T}} \, \underline{Q_i^n}$$

Reformulating LxW-DG
[Gassner, Dumbser, Hindenlang, and Munz, 2011]

Corrector step:

- Multiply equation by $\underline{\Phi} \in \mathbb{R}^M$ and integrate over $[\tau, \xi] \in [-1,1] \times [-1,1]$:

$$\frac{1}{2} \int \underline{\Phi} \left(\int q_{,\tau} \, d\tau \right) d\xi + \frac{\nu}{2} \iint \underline{\Phi} \, q_{,\xi} \, d\xi \, d\tau = \underline{0}$$

$$\underline{Q_i^{n+1}} = \underline{Q_i^n} - \frac{\nu}{2} \iint \underline{\Phi} \, q_{,\xi} \, d\xi \, d\tau$$

$$\underline{Q_i^{n+1}} = \underline{Q_i^n} + \frac{\nu}{2} \iint \underline{\Phi}_{,\xi} \, q \, d\xi \, d\tau - \frac{1}{2} \int \left[\underline{\Phi}(1) \, \mathcal{F}_{i+\frac{1}{2}} - \underline{\Phi}(-1) \, \mathcal{F}_{i-\frac{1}{2}} \right] d\tau$$

- Replace q and $\mathcal{F}$ above by the predicted solution
- This results in the update:

$$\underline{Q_i^{n+1}} = \underline{Q_i^n} + \underline{\underline{C^0}} \, \underline{W_i^{n+\frac{1}{2}}} + \underline{\underline{C^-}} \, \underline{W_{i-1}^{n+\frac{1}{2}}}$$

- After some algebra, this can be written as

$$\underline{Q_i^{n+1}} = \underline{\underline{M^0}} \, \underline{Q_i^n} + \underline{\underline{M^-}} \, \underline{Q_{i-1}^n}$$

Stability analysis of LIDG

[Guthrey and R, 2019]

von Neumann stability analysis

- Fourier ansatz:

$$\underline{Q_i^{n+1}} = \underline{\widehat{Q}^{n+1}}\, e^{\sqrt{-1}\,k\,i} \qquad \text{and} \qquad \underline{Q_i^{n}} = \underline{\widehat{Q}^{n}}\, e^{\sqrt{-1}\,k\,i}$$

- This results in the following system:

$$\underline{\widehat{Q}^{n+1}} = \left(\underline{\underline{M^0}} + e^{-\sqrt{-1}\,k} \underline{\underline{M^-}} \right) \underline{\widehat{Q}^{n}} = \mathcal{M}(\nu, k)\, \underline{\widehat{Q}^{n}}$$

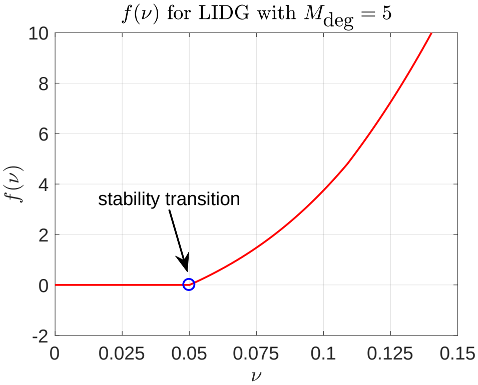
- We define the following function:

$$f(\nu) := \max_{0 \le k \le 2\pi} \rho\left(\mathcal{M}(\nu, k)\right) - 1$$

- Using the Bisection Method, find root ν_{stable} such that $f(\nu_{\text{stable}}) \approx 0$

Order	1	2	3	4	5	6
ν_{stable}	1.000	0.333	0.171	0.104	0.070	0.050

Stability analysis of LIDG
[Guthrey and R, 2019]



von Neumann stability analysis

$$f(\nu) := \max_{0 \leq k \leq 2\pi} \rho\left(\mathcal{M}(\nu, k)\right) - 1$$

Outline

Fully-implicit space-time DG

Basic method

Fully-implicit:

- Solution in each space-time element, $\mathcal{T}_i^{n+\frac{1}{2}}$ $(M^\star = M(M+1)/2)$:

$$q(t,x)\Big|_{\mathcal{T}_i^{n+\frac{1}{2}}} \approx w_i^{n+\frac{1}{2}} := \underline{\Psi}^T \underline{W_i^{n+\frac{1}{2}}}$$

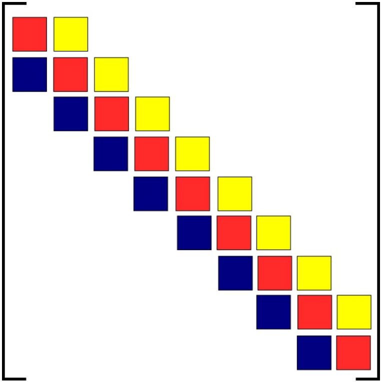
- Multiply by test function, integrate-by-parts in **both** space and time:

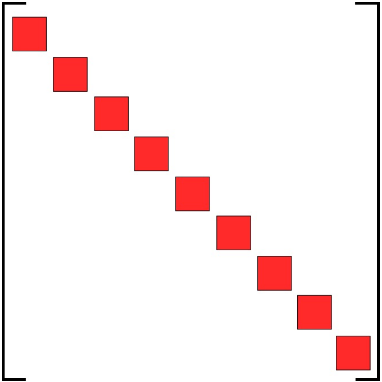
$$\begin{aligned}
&\iint \underline{\Psi} \left(\underline{\Psi}_{,\tau} + \nu \underline{\Psi}_{,\xi}\right)^T \underline{W_i^{n+\frac{1}{2}}} \, d\xi \, d\tau \\
&+ \int \underline{\Psi}(-1,\xi) \left[\underline{\Psi}(-1,\xi)^T \underline{W_i^{n+\frac{1}{2}}} - \underline{\Phi}^T \underline{Q_i^n}\right] d\xi \\
&- \int \underline{\Psi}(\tau,1) \left[\nu \underline{\Psi}(\tau,1)^T \underline{W_i^{n+\frac{1}{2}}} - \mathcal{F}_{i+\frac{1}{2}}\right] d\tau \\
&+ \int \underline{\Psi}(\tau,-1) \left[\nu \underline{\Psi}(\tau,-1)^T \underline{W_i^{n+\frac{1}{2}}} - \mathcal{F}_{i-\frac{1}{2}}\right] d\tau = \underline{0}
\end{aligned}$$

Fully-implicit space-time DG

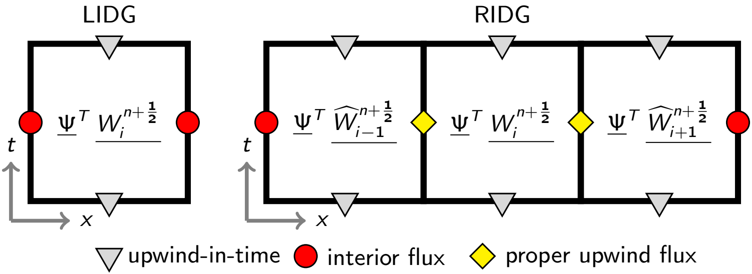
Matrix structure

Fully-implicit vs locally-implicit:





Regionally-implicit space-time DG
[Guthrey and R, 2019]



- Prediction step results in a block 3×3 system:

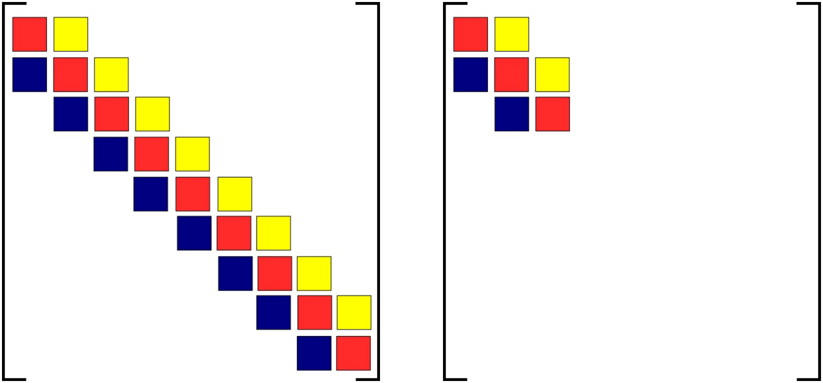
$$\begin{bmatrix} \underline{\underline{L}}^0 + \underline{\underline{L}}^- & \underline{\underline{X}}^- & \\ \underline{\underline{X}}^+ & \underline{\underline{L}}^0 + \underline{\underline{L}}^- + \underline{\underline{L}}^+ & \underline{\underline{X}}^- \\ & \underline{\underline{X}}^+ & \underline{\underline{L}}^0 + \underline{\underline{L}}^+ \end{bmatrix} \begin{bmatrix} \underline{\widehat{W}}_{i-1}^{n+\frac{1}{2}} \\ \underline{W}_{i}^{n+\frac{1}{2}} \\ \underline{\widehat{W}}_{i+1}^{n+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \underline{\underline{T}}\,\underline{Q}_{i-1}^n \\ \underline{\underline{T}}\,\underline{Q}_{i}^n \\ \underline{\underline{T}}\,\underline{Q}_{i+1}^n \end{bmatrix}$$

- Same corrector as LIDG

Regionally-implicit space-time DG

Matrix structure

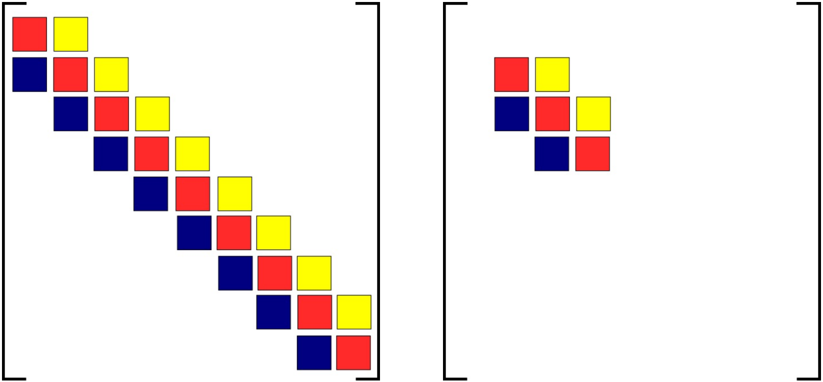
Fully-implicit vs regionally-implicit:



Regionally-implicit space-time DG

Matrix structure

Fully-implicit vs regionally-implicit:



Stability analysis of RIDG
[Guthrey and R, 2019]

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- This results in the following system:

$$\underline{\widehat{Q}^{n+1}} = \left(\underline{\underline{M_0}} + e^{-\sqrt{-1}\,k}\, \underline{\underline{M_{-1}}} + e^{-2\sqrt{-1}\,k}\, \underline{\underline{M_{-2}}} \right) \underline{\widehat{Q}^{n}} = \mathcal{M}(\nu, k)\, \underline{\widehat{Q}^{n}}$$

- We define the following function:

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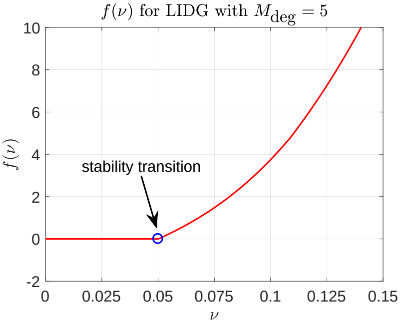
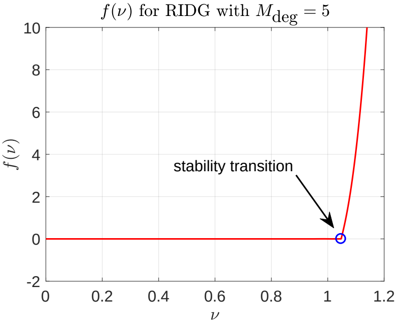
- Using the Bisection Method, find root ν_{stable} such that $f(\nu_{\text{stable}}) \approx 0$

Order	1	2	3	4	5	6
LIDG	1.000	0.333	0.171	0.104	0.070	0.050
RIDG	1.000	1.168	1.135	1.097	1.066	1.047

Stability analysis of RIDG
[Guthrey and R, 2019]





von Neumann stability analysis

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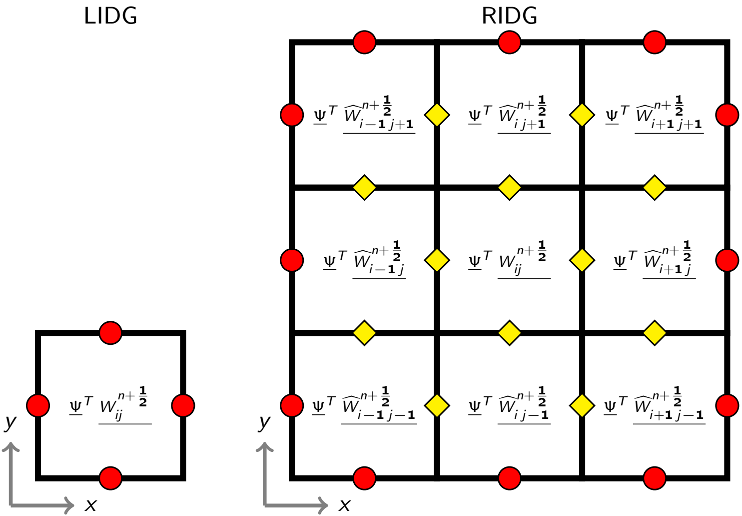
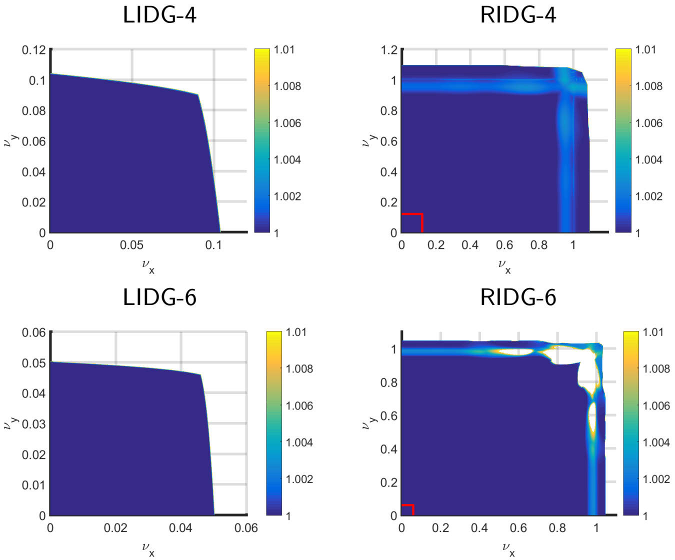
Regionally-implicit space-time DG

Extension to 2D



Regionally-implicit space-time DG

Extension to 2D



Regionally-implicit space-time DG

Extension to 2D

Simplified prediction step procedure [Hu and R., in prep]:

- In 2D prediction step is block 9×9 system
- Reduce complexity by breaking into series of 1D solves

$$\underline{Q^n_{ij}} \longrightarrow \left\{ \underline{W^x_{ij}}, \quad \underline{W^y_{ij}} \right\} \longrightarrow \underline{W^{n+\frac{1}{2}}_{ij}}$$

- In 3D prediction step is block 27×27 system
- Reduce complexity by breaking into series of 1D and 2D solves

$$\underline{Q^n_{ijk}} \longrightarrow \left\{ \underline{W^x_{ijk}}, \quad \underline{W^y_{ijk}}, \quad \underline{W^z_{ijk}} \right\}$$
$$\longrightarrow \left\{ \underline{W^{xy}_{ijk}}, \quad \underline{W^{yz}_{ijk}}, \quad \underline{W^{xz}_{ijk}} \right\} \longrightarrow \underline{W^{n+\frac{1}{2}}_{ijk}}$$

- In 4D prediction step is block 81×81 system

RIDG for Vlasov-Poisson

Vlasov-Poisson system

- Some simplifying assumptions:
  1. Two species: ions $(+)$ & electrons $(-)$
  2. Slow moving charges $\Longrightarrow$ electrostatics
  3. Track electrons, assume fixed background ions
- Electrons are described by a probability density function:

$$f(t, \underline{x}, \underline{v}) : \mathbb{R}_{\geq 0} \times \mathbb{R}^D \times \mathbb{R}^V \mapsto \mathbb{R}_{\geq 0}$$

- Moments of $f(t, \underline{x}, \underline{v})$ correspond to various physical observables:

$$\rho(t, \underline{x}) := \int f \, d\underline{v}, \quad \rho \underline{u}(t, \underline{x}) := \int \underline{v} \, f \, d\underline{v}, \quad \mathcal{E}(t, \underline{x}) := \frac{1}{2} \int \|\underline{v}\|^2 \, f \, d\underline{v}$$

- The Vlasov-Poisson system:

$$f_{,t} + \underline{v} \cdot \nabla_{\underline{x}} f + \underline{E} \cdot \nabla_{\underline{v}} f = 0,$$

$$\underline{E} = -\nabla_{\underline{x}} \phi, \quad -\nabla^2_{\underline{x}} \phi = \rho(t, \underline{x}) - \rho_0$$

RIDG for Vlasov-Poisson

Vlasov-Poisson system

- **Characteristics (Vlasov is an advection equation in phase space):**

$$\left(\underline{x}\left(t; \underline{x}, \underline{v}, s\right), \, \underline{v}\left(t; \underline{x}, \underline{v}, s\right)\right) \quad \Longrightarrow \quad \frac{d\underline{x}}{dt} = \underline{v}(t), \qquad \frac{d\underline{v}}{dt} = \underline{E}\left(t, \underline{x}(t)\right),$$

$$f\left(t, \underline{x}, \underline{v}\right) = f_0\left(\underline{x}\left(0; t, \underline{x}, \underline{v}\right), \, \underline{v}\left(0; t, \underline{x}, \underline{v}\right)\right)$$

- **Maximum principle:**

$$0 \leq f\left(t, \underline{x}, \underline{v}\right) \leq \max_{(\underline{x}, \underline{v})} f_0\left(\underline{x}, \underline{v}\right)$$

- **Conserved functional:**

$$\frac{d}{dt} \int_{\underline{x}} \int_{\underline{v}} G(f) \, d\underline{v} \, d\underline{x} = 0, \quad L^p \text{ norm:} \quad G(f) = |f|^p, \quad \text{entropy:} \quad G(f) = -f \ln f$$

- **Conservation laws:**

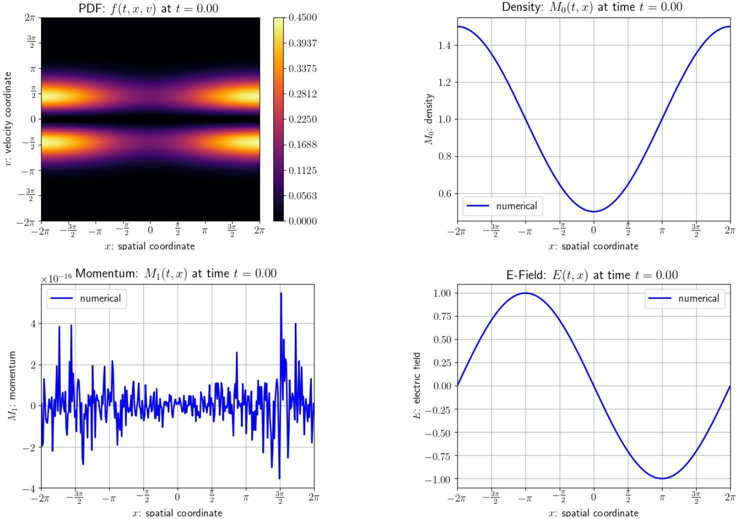
Mass: $\rho_{,t} + \nabla_{\underline{x}} \cdot \left(\rho \underline{u}\right) = 0$

Momentum: $\left(\rho \underline{u}\right)_{,t} + \nabla_{\underline{x}} \cdot \left(\rho \underline{u}\, \underline{u}^T + \underline{\underline{\mathbb{P}}} + \rho_0 \phi \underline{\underline{\mathbb{I}}} - \underline{\underline{\mathbb{T}}}\right) = \underline{0}$

Energy: $\left(\mathcal{E} + \frac{1}{2}\|\underline{E}\|^2\right)_{,t} + \nabla_{\underline{x}} \cdot \left(\underline{\mathcal{F}} + \phi \rho_0 \underline{u}_0\right) = 0$

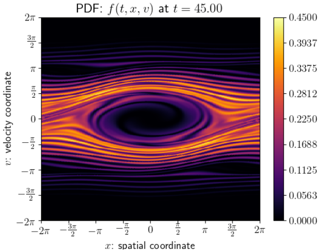
RIDG for Vlasov-Poisson

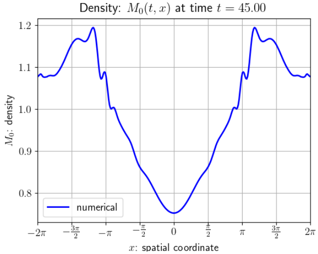
[Vaughan, 2021], [R and Vaughan, in prep]

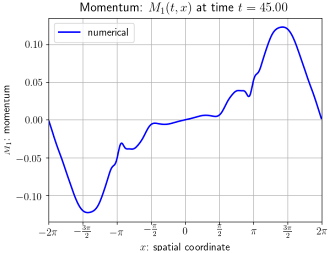


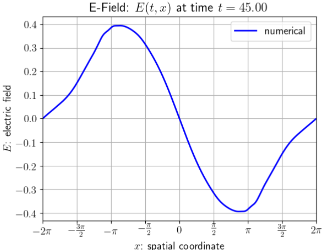
RIDG for Vlasov-Poisson

[Vaughan, 2021], [R and Vaughan, in prep]





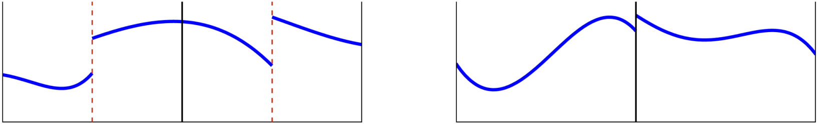




Outline

Semi-Lagrangian DG-FEM

Reconstruct-Evolve-Average



- 1D advection:

$$q_{,t} + u q_{,x} = 0, \qquad q(t,x) : \mathbb{R}_{\geq 0} \times \mathbb{R} \mapsto \mathbb{R}$$

$$q^h\left(0, x_i + 0.5 \Delta x\, \xi\right) \Big|_{\mathcal{T}_i} = \sum_{\ell=1}^{M} Q_i^{(\ell)}(0)\, \varphi^{(\ell)}(\xi)$$

- Update formula: $\nu = u \Delta t / \Delta x, \quad 0 \leq \nu_{\text{stable}} \leq 1$

$$Q_i^{(\ell)}\left(t^{n+1}\right) = \frac{1}{2} \sum_{k=1}^{M} Q_{i-1}^{(k)}\left(t^n\right) \int_{-1}^{-1+2\nu} \varphi^{(k)}(\xi + 2 - 2\nu)\, \varphi^{(\ell)}(\xi)\, d\xi$$

$$+ \frac{1}{2} \sum_{k=1}^{M} Q_i^{(k)}\left(t^n\right) \int_{-1+2\nu}^{1} \varphi^{(k)}(\xi - 2\nu)\, \varphi^{(\ell)}(\xi)\, d\xi$$

SLDG vs. Lax-Wendroff DG

Second order comparison (accuracy)

Semi-Lagrangian DG:

$$Q_i^{(1)n+1} = Q_i^{(1)n} - \nu \left(\left[Q_i^{(1)n} + \sqrt{3}\, Q_i^{(2)n} \right] - \left[Q_{i-1}^{(1)n} + \sqrt{3}\, Q_{i-1}^{(2)n} \right] \right) + \sqrt{3}\, \nu^2 \left(Q_i^{(2)n} - Q_{i-1}^{(2)n} \right)$$

$$Q_i^{(2)n+1} = Q_i^{(2)n} + \sqrt{3}\, \nu \left(\left[Q_i^{(1)n} - \sqrt{3}\, Q_i^{(2)n} \right] - \left[Q_{i-1}^{(1)n} + \sqrt{3}\, Q_{i-1}^{(2)n} \right] \right) - \sqrt{3}\, \nu^2 \left(Q_i^{(1)n} - Q_{i-1}^{(1)n} - 2\sqrt{3}\, Q_{i-1}^{(2)n} \right) + 2\nu^3 \left(Q_i^{(2)n} - Q_{i-1}^{(2)n} \right)$$

Lax-Wendroff DG:

$$Q_i^{(1)n+1} = Q_i^{(1)n} - \nu \left(\left[Q_i^{(1)n} + \sqrt{3}\, Q_i^{(2)n} \right] - \left[Q_{i-1}^{(1)n} + \sqrt{3}\, Q_{i-1}^{(2)n} \right] \right) + \sqrt{3}\, \nu^2 \left(Q_i^{(2)n} - Q_{i-1}^{(2)n} \right)$$

$$Q_i^{(2)n+1} = Q_i^{(2)n} + \sqrt{3}\, \nu \left(\left[Q_i^{(1)n} - \sqrt{3}\, Q_i^{(2)n} \right] - \left[Q_{i-1}^{(1)n} + \sqrt{3}\, Q_{i-1}^{(2)n} \right] \right) - 3\, \nu^2 \left(Q_i^{(2)n} - Q_{i-1}^{(2)n} \right)$$

SLDG vs. Lax-Wendroff DG

Second order comparison (stability)

Semi-Lagrangian DG:

$$\begin{bmatrix} \widehat{Q}^1 \\ \widehat{Q}^2 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 - 6\nu + 6\nu^2 \end{bmatrix} \begin{bmatrix} \widehat{Q}^1 \\ \widehat{Q}^2 \end{bmatrix}^{n}$$

$$g(\nu) = \max\left\{1, \left|1 - 6\nu + 6\nu^2\right|\right\} \quad \Longrightarrow \quad g(\nu) \equiv 1 \quad \forall \, \nu \in [0, 1]$$

Lax-Wendroff DG:

$$\begin{bmatrix} \widehat{Q}^1 \\ \widehat{Q}^2 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 - 6\nu \end{bmatrix} \begin{bmatrix} \widehat{Q}^1 \\ \widehat{Q}^2 \end{bmatrix}^{n}$$

$$g(\nu) = \max\left\{1, \left|1 - 6\nu\right|\right\} \quad \Longrightarrow \quad g(\nu) \equiv 1 \quad \forall \, \nu \in \left[0, \frac{1}{3}\right]$$

Maximum-Taylor DG

[R and Van Fleet, in prep]

- **Observation:** SLDG = LxW-DG + (some high-order corrections)
- Extra terms needed for improved stability (but not convergence order)
- SLDG cannot be directly applied to multi-D linear systems:

$$\underline{Q}_{,t} + \underline{\underline{A}}^x \underline{Q}_{,x} + \underline{\underline{A}}^y \underline{Q}_{,y} + \underline{\underline{A}}^z \underline{Q}_{,z} = \underline{0}$$

- **Question:** how can we generate missing terms for multi-D linear systems?
- **Idea:** include more terms in the Taylor expansion:

$$\textbf{LxW-DG:} \quad \underline{q_i^{n+1}}(\xi) = \underline{q_i^n}(\xi) - \sum_{s=1}^{M} \frac{(-2)^s}{s!} \underline{\underline{\mathbb{A}}}^s \, \partial_\xi^s \underline{q_i^n}(\xi)$$

$$\textbf{maxT-DG:} \quad \underline{q_i^{n+1}}(\xi) = \underline{q_i^n}(\xi) - \sum_{s=1}^{2M-1} \frac{(-2)^s}{s!} \underline{\underline{\mathbb{A}}}^s \, \partial_\xi^s \underline{q_i^n}(\xi)$$

- **Idea:** LxW-DG does one int-by-parts, instead do the max ints-by-parts

Maximum-Taylor DG

[R and Van Fleet, in prep]

Maximum-Taylor DG 1D Update:

$$\left[\underline{Q_i^k}\right]^{n+1} = \underline{Q_i^k} + \sum_{s=1}^{M} \underline{I_i^{sk}} + \sum_{s=1}^{2M-1} \underline{F_i^{sk}}$$

$$\underline{I_i^{sk}} = \underline{\underline{\mathbb{A}}}^s \sum_{p=1}^{M} \mu_{skp} \, \underline{Q_i^p} \quad \text{and} \quad \underline{F_i^{sk}} = \sum_{\ell=1}^{s} \left[\beta^2_{s\ell k} \, \underline{\mathcal{F}^{s\ell}_{i-\frac{1}{2}}} - \alpha^2_{s\ell k} \, \underline{\mathcal{F}^{s\ell}_{i+\frac{1}{2}}} \right]$$

$$\underline{\mathcal{F}^{s\ell}_{i-\frac{1}{2}}} = \underline{\underline{\mathbb{A}}}^{s+} \sum_{p=1}^{M} \alpha^1_{s-\ell+1\,p} \, \underline{Q^p_{i-1}} + \underline{\underline{\mathbb{A}}}^{s-} \sum_{p=1}^{M} \beta^1_{s-\ell+1\,p} \, \underline{Q^p_i}$$

- Exactly equivalent to SLDG applied to each characteristic component
- Therefore, stable up to CFL $= 1$
- Achieves optimal stability and is high-order: $\mathcal{O}\left(\Delta x^M\right)$
- maxT-DG fixes LxW-DG by correcting face fluxes

Maximum-Taylor DG

Extension to multi-D

- Maximum int-by-parts becomes tedious (automated scripts):
- For systems, products are non-commutative, e.g.:

$$q_{,t} + u q_{,x} + v q_{,y} = 0 \quad \Longrightarrow \quad \underline{q}_{,t} + \underline{\underline{A}}\,\underline{q}_{,x} + \underline{\underline{B}}\,\underline{q}_{,y} = 0$$

$$u^2 v \quad \Longrightarrow \quad \frac{1}{3}\left(\underline{\underline{\mathbb{A}}}^2\,\underline{\underline{\mathbb{B}}} + \underline{\underline{\mathbb{A}}}\,\underline{\underline{\mathbb{B}}}\,\underline{\underline{\mathbb{A}}} + \underline{\underline{\mathbb{B}}}\,\underline{\underline{\mathbb{A}}}^2\right)$$

- Despite derivation complexity, final method is cheap & simple:

$$\begin{aligned} \underline{Q}_{ij}^{n+1} = & \underline{\underline{U}}_{i-1j+1}\,\underline{Q}^n_{i-1j+1} + \underline{\underline{U}}_{ij+1}\,\underline{Q}^n_{ij+1} + \underline{\underline{U}}_{i+1j+1}\,\underline{Q}^n_{i+1j+1} \\ & + \underline{\underline{U}}_{i-1j}\,\underline{Q}^n_{i-1j} + \underline{\underline{U}}_{ij}\,\underline{Q}^n_{ij} + \underline{\underline{U}}_{i+1j}\,\underline{Q}^n_{i+1j} \\ & + \underline{\underline{U}}_{i-1j-1}\,\underline{Q}^n_{i-1j-1} + \underline{\underline{U}}_{ij-1}\,\underline{Q}^n_{ij-1} + \underline{\underline{U}}_{i+1j-1}\,\underline{Q}^n_{i+1j-1} \end{aligned}$$

- Scalar stability is optimal, system stability is near-optimal:

$$\max\left(\frac{|u|\Delta t}{\Delta x}, \frac{|v|\Delta t}{\Delta y}\right) \leq 1 \quad \text{and} \quad \max\left(\frac{\rho\left(A\right)\Delta t}{\Delta x}, \frac{\rho\left(B\right)\Delta t}{\Delta y}\right) \leq 1, \frac{4}{5}, \frac{1}{2}, \frac{1}{3}$$

Numerical example

Wave equation

- Consider the wave equation

$$q_{,t,t} = q_{,x,x} + q_{,y,y}, \quad x \in (-5,5), \quad y \in (-5,5), \quad t \in (0,2)$$

$$\rho_0(x,y) = \sqrt{\frac{2\pi^2}{25}} \sin\left(\frac{\pi}{5}(x+y)\right)$$

$$u_0(x,y) = v_0(x,y) = -\frac{\pi}{5}\cos\left(\frac{\pi}{5}(x+y)\right)$$

- Double periodic boundary conditions
- Let $\rho = q_{,t}$, $u = -q_{,x}$, and $v = -q_{,y}$:

$$\underline{q}_{,t} + \underline{\underline{A}}\,\underline{q}_{,x} + \underline{\underline{B}}\,\underline{q}_{,y} = \underline{0}$$

$$\underline{q} = \begin{bmatrix} \rho \\ u \\ v \end{bmatrix}, \quad \underline{\underline{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \underline{\underline{B}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Numerical example

Wave equation

- Convergence table for ρ with CFL $= 0.45$

# elems	L^2 error	order
25	$2.6880294738 \times 10^{-4}$	-
50	$3.3537868500 \times 10^{-5}$	3.00269
100	$4.1677002990 \times 10^{-6}$	3.00847
200	$5.1835616412 \times 10^{-7}$	3.00724

- Convergence table for $\sqrt{u^2 + v^2}$ with CFL $= 0.45$

# elems	L^2 error	order
25	$5.2666062443 \times 10^{-4}$	-
50	$6.6279264530 \times 10^{-5}$	2.99024
100	$8.2937082427 \times 10^{-6}$	2.99847
200	$1.0366137699 \times 10^{-6}$	3.00014

Outline

Conclusions & future work

Summary

- RK-DG and LxW-DG suffer from linear stability restrictions $\ll$ CFL limit
- Method # 1: regionally-implicit DG (RIDG)
  - RIDG is a modification of LxW-DG
  - Linear stability constraint: CFL ≈ 1
- Method #2: maximum-Taylor DG (maxT-DG)
  - maxT-DG is a modification that blends SLDG & LxW-DG
  - Linear stability constraint: CFL $= 1$ (1D systems, 2D scalar)
  - Linear stability constraint: CFL $\leq 1, \frac{4}{5}, \frac{1}{2}, \frac{1}{3}$ (2D systems)

Ongoing & future work

- Application to linear kinetic transport models
- Application to kinetic models of plasma