Robust discontinuous Galerkin methods for incompressible fluid flows

Guosheng Fu

Department of Applied and Computational Mathematics and Statistics
University of Notre Dame

CCAM workshop on hyperbolic conservation laws @ Purdue University

May 9-10, 2022

Outline I

Outline

The Navier-Stokes equations

Consider the incompressible Navier-Stokes equations:

$$\begin{aligned} \partial_t \boldsymbol{u} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p - \nu \nabla^2 \boldsymbol{u} &= \boldsymbol{f}, && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0, && \text{in } \Omega, \\ \boldsymbol{u} &= 0, && \text{on } \partial\Omega, \end{aligned}$$

with initial condition

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x) \quad \forall x \in \Omega.$$

- $\boldsymbol{u}$: velocity, p: pressure, $1 \gg \nu \geq 0$: viscosity.
- $\Omega \subset \mathbb{R}^2$: a polygonal domain.
- The initial velocity $\boldsymbol{u}_0(x)$ is assumed to be divergence-free.

(very) Brief background on numerical methods for NS

Sequential methods

- Vorticity based formulation

  FD: (Fromm, 63), Spectral (Orszag, 71), Compact FD/FE (E & Liu, 95–01)...
- Fractional step methods (Projection methods)

  (Chorin,68),(Temam,69), (Guermond, Minev, Shen,06)...

Pro: efficient solver   Con: numerical boundary conditions

Coupled methods

- Finite elements/discontinuous Galerkin

  (Crouzeix & Raviart, 73), (Hood & Taylor, 74), (Arnold & Brezzi, 84)...
  (Arnold & Qin, 92), (Zhang, 05), (Cockburn, Kanschat, Schötzau, 04-09)...
- Splines/IGA

  (Awanou & Lai, 05), (Evans & Hughes, 13)...
- Least-squares (Hughes et al. 89)...

Pro: numerical boundary conditions   Con: expensive solver

Goal: a fast and accurate scheme (velocity-only formulation)

- Velocity $\boldsymbol{u}$ lies in the divergence-free space

$$\boldsymbol{V}_0 = \{\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) : \ \nabla \cdot \boldsymbol{w} = 0\}$$

- Leray's NS equation:

$$\partial_t \boldsymbol{u} = \mathbb{P}\left(-\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}\right),$$

  where $\mathbb{P}$ projects a vector onto its solenoidal part (Helmholtz decomposition)

- (Leray-Hopf) weak solution of NS: find $\boldsymbol{u}(t) \in \boldsymbol{V}_0$ s.t.

$$(\partial_t \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u} \otimes \boldsymbol{u}, \nabla \boldsymbol{v}) - (\nu \nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}_0. \tag{1}$$

  Pressure does not enter into the weak formulation!

Goal: a fast and accurate scheme (velocity-only formulation)

- Velocity $\boldsymbol{u}$ lies in the divergence-free space

$$\boldsymbol{V}_0 = \{\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) : \quad \nabla \cdot \boldsymbol{w} = 0\}$$

- Leray's NS equation:

$$\partial_t \boldsymbol{u} = \mathbb{P}\left(-\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}\right),$$

where $\mathbb{P}$ projects a vector onto its solenoidal part (Helmholtz decomposition)

- (Leray-Hopf) weak solution of NS: find $\boldsymbol{u}(t) \in \boldsymbol{V}_0$ s.t.

$$(\partial_t \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u} \otimes \boldsymbol{u}, \nabla \boldsymbol{v}) - (\nu \nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}_0. \tag{1}$$

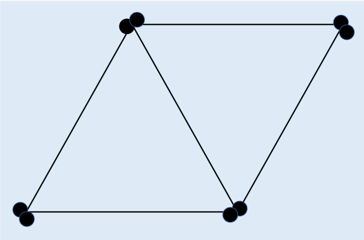
Pressure does not enter into the weak formulation!

Obstacle: construction of finite element subspaces for $\boldsymbol{V}_0$

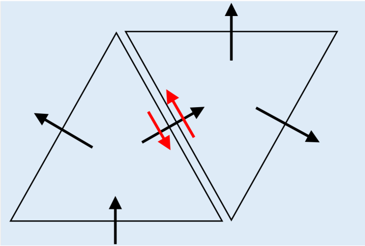
The div-free DG approach (Fu, CMAME, 19)

- Relax tangential conformity of velocity field
- Use $H(\mathrm{div})$-conforming, and divergence-free finite elements:

$$\boldsymbol{V}_{h,0}^{\mathrm{div}} = \{\boldsymbol{w} \in H_0(\mathrm{div}, \Omega) : \quad \boldsymbol{w}|_T \in \boldsymbol{\mathcal{P}}^k(T), \nabla \cdot \boldsymbol{w} = 0 \;\; \forall T \in \mathcal{T}_h\}$$



H^1-conforming

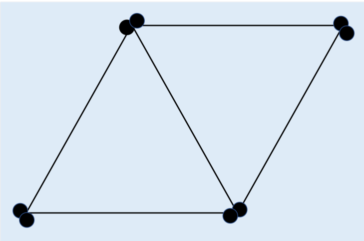


$H(\mathrm{div})$-conforming

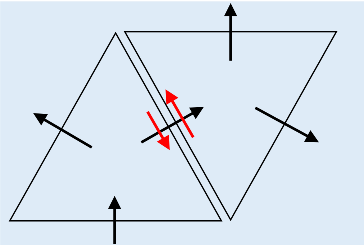
The div-free DG approach (Fu, CMAME, 19)

- Relax tangential conformity of velocity field
- Use $H(\mathrm{div})$-conforming, and divergence-free finite elements:

$$\boldsymbol{V}_{h,0}^{\mathrm{div}} = \{\boldsymbol{w} \in H_0(\mathrm{div}, \Omega): \quad \boldsymbol{w}|_T \in \boldsymbol{\mathcal{P}}^k(T), \nabla \cdot \boldsymbol{w} = 0 \ \ \forall T \in \mathcal{T}_h\}$$



H^1-conforming
$+$ divergene-free ☹



$H(\mathrm{div})$-conforming
$+$ divergence-free ☺

The div-free DG approach (Fu, CMAME, 19)

- The div-free DG scheme: find $\boldsymbol{u}_h \in \boldsymbol{V}_{h,0}^{\mathrm{div}}$ such that

$$(\partial_t \boldsymbol{u}_h, \boldsymbol{v}) = \mathcal{C}_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}) - \nu \mathcal{B}_h(\boldsymbol{u}_h, \boldsymbol{v}) + (\boldsymbol{f}, \boldsymbol{v}), \ \forall \boldsymbol{v} \in \boldsymbol{V}_{h,0}^{\mathrm{div}} \tag{2}$$

  - Nonlinear convective operator (**upwinding DG**):

$$\mathcal{C}_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}) := \sum_{T \in \mathcal{T}_h} \int_T (\boldsymbol{u}_h \otimes \boldsymbol{u}_h) : \nabla \boldsymbol{v} \, \mathrm{dx} - \int_{\partial T} \underbrace{(\boldsymbol{u}_h \cdot \boldsymbol{n}) \boldsymbol{u}_h^-}_{\text{upwinding flux}} \cdot \boldsymbol{v} \, \mathrm{ds}$$

  - Second-order viscous term (**symmetric interior penalty DG**):

$$\mathcal{B}_h(\boldsymbol{u}_h, \boldsymbol{v}) := \sum_{T \in \mathcal{T}_h} \int_T \nabla \boldsymbol{u}_h : \nabla \boldsymbol{v} \, \mathrm{dx} - \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla \boldsymbol{u}_h\}\!\} [\![\boldsymbol{v} \otimes \boldsymbol{n}]\!] \, \mathrm{ds}$$
$$- \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla \boldsymbol{v}\}\!\} [\![\boldsymbol{u}_h \otimes \boldsymbol{n}]\!] \, \mathrm{ds} + \int_F \frac{\alpha k^2}{h} [\![\boldsymbol{u}_h \otimes \boldsymbol{n}]\!] [\![\boldsymbol{v} \otimes \boldsymbol{n}]\!] \, \mathrm{ds}.$$

The div-free DG scheme: properties

(1) Exact mass conservation: $\nabla\cdot\boldsymbol{u}_h = 0$ pointwise

(2) Natural *upwinding* discretization of convection term:
no need of additional stabilization or skew-symmetrization

$$\mathcal{C}_h(\boldsymbol{u}_h;\boldsymbol{v}_h,\boldsymbol{v}_h) = \frac{1}{2}\sum_{F\in\mathcal{F}_h}\int_F |\boldsymbol{u}_h\cdot\boldsymbol{n}|(|[\![\boldsymbol{v}_h]\!]|^2)\,\mathrm{ds} \geq 0,$$

(3) Energy-stability:
minimal amount of numerical dissipation

$$\frac{1}{2}\partial_t(\boldsymbol{u}_h(t),\boldsymbol{u}_h(t)) = -\underbrace{\mathcal{C}_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{u}_h)}_{\text{num. disp.}} - \underbrace{\nu\mathcal{B}_h(\boldsymbol{u}_h,\boldsymbol{u}_h)}_{\text{phy. disp.}} \leq 0$$

(4) Pressure robustness (John et al., SIAM Rev., 17)
velocity estimates independent of pressure regularity

(5) High-order accuracy

Time discretization: explicit Runge-Kutta

- The semi-discrete scheme (2) can be written as

$$M(\partial_t \boldsymbol{u}_h) = L(\boldsymbol{u}_h),$$

where M is the mass matrix for the space $\boldsymbol{V}_{h,0}^{\mathrm{div}}$, and $L(\boldsymbol{u}_h)$ the spatial discretization operator.

- We use RK3 (Shu and Osher, JCP, 88) time stepping:

$$\begin{aligned}
M\boldsymbol{u}_h^{(1)} &= M\boldsymbol{u}_h^n + \Delta t L(\boldsymbol{u}_h^n),\\
M\boldsymbol{u}_h^{(2)} &= \frac{3}{4}M\boldsymbol{u}_h^n + \frac{1}{4}\left[M\boldsymbol{u}_h^{(1)} + \Delta t L(\boldsymbol{u}_h^{(1)})\right],\\
M\boldsymbol{u}_h^{n+1} &= \frac{1}{3}M\boldsymbol{u}_h^n + \frac{2}{3}\left[M\boldsymbol{u}_h^{(2)} + \Delta t L(\boldsymbol{u}_h^{(2)})\right],
\end{aligned}$$

- In each time step, three **mass matrix inversions** are needed.

Time discretization: explicit Runge-Kutta

- The semi-discrete scheme (2) can be written as

$$M(\partial_t \boldsymbol{u}_h) = L(\boldsymbol{u}_h),$$

  where M is the mass matrix for the space $\boldsymbol{V}_{h,0}^{\mathrm{div}}$, and $L(\boldsymbol{u}_h)$ the spatial discretization operator.

- We use RK3 (Shu and Osher, JCP, 88) time stepping:

$$\begin{aligned}
M\boldsymbol{u}_h^{(1)} &= M\boldsymbol{u}_h^n + \Delta t L(\boldsymbol{u}_h^n),\\
M\boldsymbol{u}_h^{(2)} &= \frac{3}{4}M\boldsymbol{u}_h^n + \frac{1}{4}\left[M\boldsymbol{u}_h^{(1)} + \Delta t L(\boldsymbol{u}_h^{(1)})\right],\\
M\boldsymbol{u}_h^{n+1} &= \frac{1}{3}M\boldsymbol{u}_h^n + \frac{2}{3}\left[M\boldsymbol{u}_h^{(2)} + \Delta t L(\boldsymbol{u}_h^{(2)})\right],
\end{aligned}$$

- In each time step, three **mass matrix inversions** are needed.

apply M^{-1} $\Leftrightarrow$ elliptic Poisson solver (fast!)

Example 1: double shear layer problem (Euler) (Bell et al., 87)

Consider the incompressible Euler equations ($\nu = 0$)

- Periodic domain: $[0, 2\pi] \times [0, 2\pi]$
- Initial condition:

$$u_1(x, y, 0) = \begin{cases} \tanh((y - \pi/2)/\rho) & y \le \pi \\ \tanh((3\pi/2 - y)/\rho) & y > \pi \end{cases},$$

$$u_2(x, y, 0) = \delta \sin(x),$$

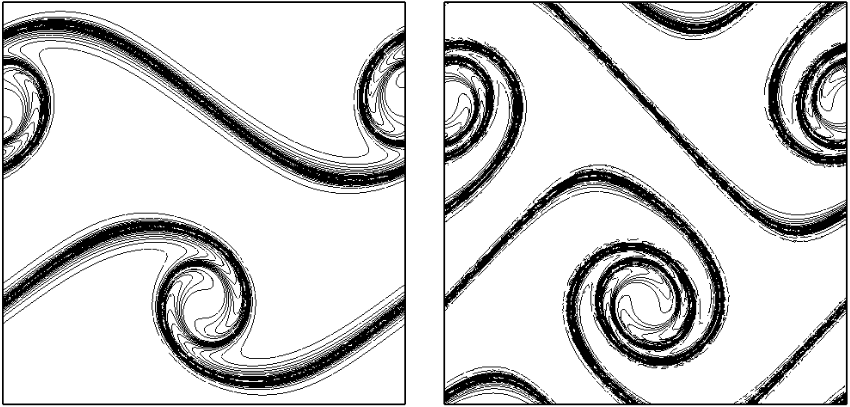
with $\rho = \pi/15$ and $\delta = 0.05$.

Example 1: double shear layer problem (Euler)

- Polynomial deg. $k = 3$, triangular mesh $h = 2\pi/80$

Vorticity contour lines, $\omega_h = \nabla_h \times \boldsymbol{u}_h$

t=6 **t=8**

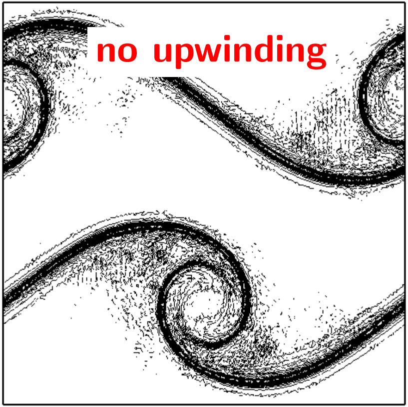


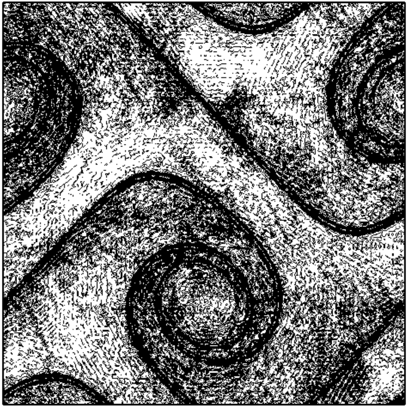
Example 1: double shear layer problem (Euler)

- Polynomial deg. $k = 3$, triangular mesh $h = 2\pi/80$

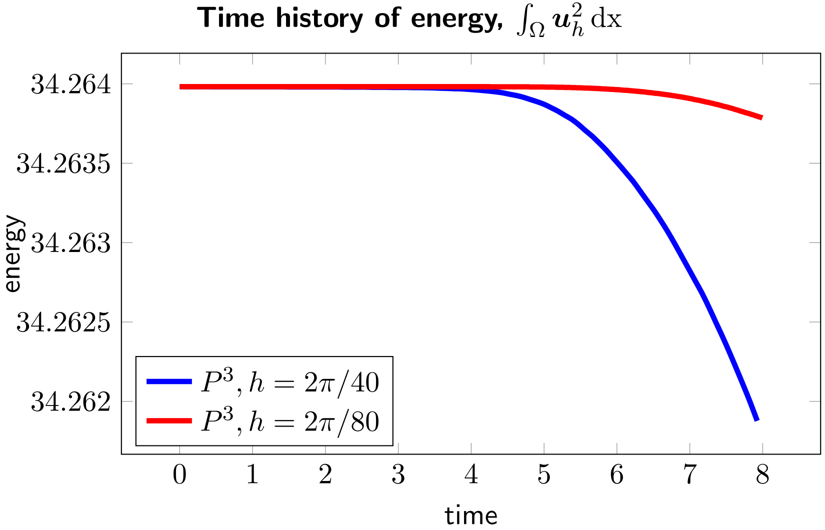
Vorticity contour lines, $\omega_h = \nabla_h \times \boldsymbol{u}_h$

t=6 **t=8**





Example 1: double shear layer problem (Euler)

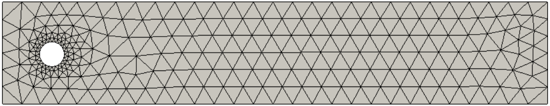


Example 2: flow around a cylinder (NS) (Schäfer et al., 96)

Consider the Navier-Stokes equations with Reynolds number $\mathrm{Re} = 100$

- Domain: a rectangular channel without an almost vertically centered circular obstacle

$$\Omega := [0, 2.2] \times [0, 0.41] \backslash \{\|(x, y) - (0.2, 0.2)\|_2 \leq 0.05\}.$$



- Inflow boundary, $\Gamma_{in} := \{x = 0\}$:

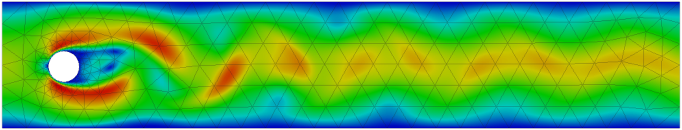
$$\boldsymbol{u}(0, y, t) = (6\, y(0.41 - y)/0.41^2, 0),$$

  Outflow boundary, $\Gamma_{out} := \{x = 2.2\}$: $(-\nu \nabla \boldsymbol{u} + pI)\boldsymbol{n} = 0$,
  No-slip boundary, $\Gamma_{wall} = \partial\Omega \backslash (\Gamma_{in} \cup \Gamma_{out})$: $\boldsymbol{u} = 0$.
- The quantities of interest: (maximal and minimal) drag and lift forces cD , cL that act on the disc:

$$[c_D, c_L] = 20 \int_{\Gamma_o} (\nu \nabla \boldsymbol{u} - pI)\boldsymbol{n} \mathrm{ds}$$

Example 2: flow around a cylinder (NS)

- The flow turns into a time-periodic behavior with a vortex shedding behind the cylinder. A typical solution for velocity magnitude:



	#dof local	#dof global	$\max c_D$	$\min c_D$	$\max c_L$	$\min c_L$
k=2	5 368	2 316	3.132939	3.074858	0.935284	-0.884771
k=3	7 808	3 088	3.229686	3.170424	0.969323	-0.965982
k=4	10 736	3 860	3.226865	3.163545	0.986497	-1.018691
www.featflow.de		167 232	3.22662	3.16351	0.98620	-1.02093
$Q_2 - P_1^{\text{disc}}$		667 264	3.22711	3.16426	0.98658	-1.02129

Implementation: basis functions for $\boldsymbol{V}_{h,0}^{\mathrm{div}}$

de Rham complex property (2D)

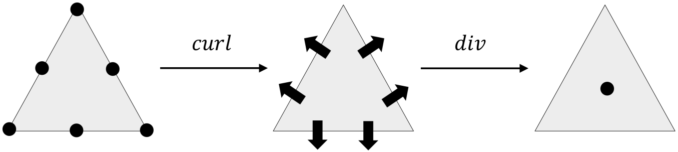
There holds $\boldsymbol{V}_{h,0}^{\mathrm{div}} = \nabla \times \Phi_h^{k+1}$, with $\Phi_h^{k+1} \in H_0^1(\Omega)$ given by

$$\Phi_h^{k+1} := \{\psi \in H_0^1(\Omega) : \quad \psi|_T \in \mathcal{P}^{k+1}(T), \ \forall T \in \mathcal{T}_h\}.$$

$\Rightarrow$ mass matrix for $\boldsymbol{V}_{h,0}^{\mathrm{div}}$ equals stiffness matrix for Φ_h^{k+1}.

- Mass matrix inversion $\Leftrightarrow$ Poisson solver (fast)
- 3D basis construction is much more complicated

$H^1 - \mathcal{P}^2(T)$ $\qquad$ $H(\mathrm{div}) - \boldsymbol{\mathcal{P}}^1(T)$ $\qquad$ $L^2 - \mathcal{P}^0(T)$



Alternative formulation: the mixed Poisson Solver

- Relax the divergence-free condition in velocity space:

$$\boldsymbol{V}_{h,0}^{\rm div} \longrightarrow \boldsymbol{V}_h^{\rm div} = \{\boldsymbol{w} \in H_0(\mathrm{div},\Omega): \ \ \boldsymbol{w}|_T \in \boldsymbol{\mathcal{P}}^k(T) \ \ \forall T \in \mathcal{T}_h\}$$

- Use a Lagrange multiplier to reinforce the divergence-free condition:

$$W_h = \nabla \cdot \boldsymbol{V}_h^{\rm div} = \{w \in L_0^2(\Omega): \ \ w|_T \in \mathcal{P}^{k-1}(T) \ \ \forall T \in \mathcal{T}_h\}$$

Forward Euler+div-free

Find $\boldsymbol{u}_h^{n+1} \in \boldsymbol{V}_{h,0}^{\rm div}$ such that

$$(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) = F^n(\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_{h,0}^{\rm div}$$

where

$$F^n(\boldsymbol{v}_h) = (\boldsymbol{u}_h^n, \boldsymbol{v}_h) + \Delta t\, \mathcal{C}(\boldsymbol{u}_h^n; \boldsymbol{u}_h^n, \boldsymbol{v}_h) - \Delta t\, \nu \mathcal{B}_h(\boldsymbol{u}_h^n, \boldsymbol{v}_h) + \Delta t(\boldsymbol{f}^n, \boldsymbol{v}_h)$$

$\Rightarrow$

"Relaxed" div-free

Find $(\widetilde{\boldsymbol{u}}_h^{n+1}, p_h^{n+1}) \in \boldsymbol{V}_h^{\rm div} \times W_h$ such that

$$\begin{aligned}
(\widetilde{\boldsymbol{u}}_h^{n+1}, \boldsymbol{v}_h) - (\frac{p_h^{n+1}}{\Delta t}, \nabla \cdot \boldsymbol{v}_h) &= F^n(\boldsymbol{v}_h), \\
(\nabla \cdot \widetilde{\boldsymbol{u}}_h, w_h) &= 0,
\end{aligned}$$

for all $(\boldsymbol{v}_h, w_h) \in \boldsymbol{V}_h^{\rm div} \times W_h$.

Alternative formulation: hybrid-mixed Poisson solver

- Relax the divergence-conformity condition in velocity space:

$$\boldsymbol{V}_{h,0}^{\text{div}} \longrightarrow \boldsymbol{V}_{h,0}^{\text{div,dg}} := \{\boldsymbol{w} \in \boldsymbol{L}^2(\Omega) : \ \boldsymbol{w}|_T \in \boldsymbol{\mathcal{P}}^k(T), \nabla \cdot \boldsymbol{w} = 0 \ \ \forall T \in \mathcal{T}_h\}$$

- Use a Lagrange multiplier to reinforce normal continuity:

$$M_h = \{\widehat{w} \in L^2(\mathcal{F}_h) : \ \widehat{w}|_F \in \mathcal{P}^k(F) \ \ \forall F \in \mathcal{F}_h\} \quad \text{(hybrid space)}$$

"Relaxed" div-conformity ⇒ SPD system for the hybrid unknowns

Find $(\boldsymbol{u}_h^{n+1}, \widehat{p}_h^{n+1}) \in \boldsymbol{V}_{h,0}^{\text{div,dg}} \times M_h$ such that

$$\begin{aligned}
(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) + \sum_{T \in \mathcal{T}_h} \langle \frac{\widehat{p}_h^{n+1}}{\Delta t}, \boldsymbol{v}_h \cdot \boldsymbol{n} \rangle_{\partial T} &= F^n(\boldsymbol{v}_h), && \longleftarrow \text{local solver for } \boldsymbol{u}_h^{n+1} \\
\sum_{T \in \mathcal{T}_h} \langle \boldsymbol{u}_h^{n+1} \cdot \boldsymbol{n}, \widehat{w}_h \rangle_{\partial T} &= 0, && \longleftarrow \text{global solver for } \widehat{p}_h^{n+1}
\end{aligned}$$

for all $(\boldsymbol{v}_h, \widehat{w}_h) \in \boldsymbol{V}_{h,0}^{\text{div,dg}} \times M_h$.

Outline

The incompressible, resistive MHD equations

Consider the incompressible, resistive MHD equations:

$$\begin{aligned} \partial_t \boldsymbol{u} + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p - \nu\nabla^2\boldsymbol{u} &= \underbrace{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}}_{\text{Lorentz force}}, \\ \nabla\cdot\boldsymbol{u} &= 0, \\ \partial_t\boldsymbol{B} - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) + \eta\nabla\times(\nabla\times\boldsymbol{B}) &= 0, \end{aligned}$$

- $\boldsymbol{u}$: velocity, p: pressure, $\boldsymbol{B}$: magnetic filed.
- $\nu \geq 0$: viscosity, $\eta \geq 0$: resistivity.
- Both velocity $\boldsymbol{u}$ and magnetic field $\boldsymbol{B}$ are exactly divergence-free

The incompressible, resistive MHD equations

Consider the incompressible, resistive MHD equations:

$$\partial_t \boldsymbol{u} + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p - \nu\nabla^2\boldsymbol{u} = \underbrace{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}}_{\text{Lorentz force}},$$

$$\nabla\cdot\boldsymbol{u} = 0,$$

$$\partial_t \boldsymbol{B} - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) + \eta\nabla\times(\nabla\times\boldsymbol{B}) = 0,$$

- $\boldsymbol{u}$: velocity, p: pressure, $\boldsymbol{B}$: magnetic filed.
- $\nu \geq 0$: viscosity, $\eta \geq 0$: resistivity.
- Both velocity $\boldsymbol{u}$ and magnetic field $\boldsymbol{B}$ are exactly divergence-free

  $\Longrightarrow$ use divergence-free space $\boldsymbol{V}_{h,0}^{\mathrm{div}}$ to approximate both quantities

The MHD equations in conservation form

- Vector calculus identities

$$
\begin{aligned}
(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} &= (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \frac{1}{2}\nabla(\boldsymbol{B} \cdot \boldsymbol{B}), \\
\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) &= \nabla \cdot (\boldsymbol{B} \otimes \boldsymbol{u} - \boldsymbol{u} \otimes \boldsymbol{B}), \\
\nabla \times (\nabla \times \boldsymbol{B}) &= -\nabla^2 \boldsymbol{B} + \nabla(\nabla \cdot \boldsymbol{B}),
\end{aligned}
$$

- Denoting $p_{\text{tot}} = p + \frac{1}{2}\boldsymbol{B} \cdot \boldsymbol{B}$, and using the fact that $\nabla \cdot \boldsymbol{B} = 0$,

  $\Longrightarrow$ the incompressible MHD equations in conservation form:

$$
\begin{aligned}
\partial_t \boldsymbol{u} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{B} \otimes \boldsymbol{B}) + \nabla p_{\text{tot}} - \nu \nabla^2 \boldsymbol{u} &= 0, \\
\nabla \cdot \boldsymbol{u} &= 0, \\
\partial_t \boldsymbol{B} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{B} - \boldsymbol{B} \otimes \boldsymbol{u}) - \eta \nabla^2 \boldsymbol{B} &= 0,
\end{aligned}
$$

The MHD equations in conservation form

- Vector calculus identities

$$
\begin{aligned}
(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} &= (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \frac{1}{2}\nabla(\boldsymbol{B} \cdot \boldsymbol{B}), \\
\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) &= \nabla \cdot (\boldsymbol{B} \otimes \boldsymbol{u} - \boldsymbol{u} \otimes \boldsymbol{B}), \\
\nabla \times (\nabla \times \boldsymbol{B}) &= -\nabla^2 \boldsymbol{B} + \nabla(\nabla \cdot \boldsymbol{B}),
\end{aligned}
$$

- Denoting $p_{\text{tot}} = p + \frac{1}{2}\boldsymbol{B} \cdot \boldsymbol{B}$, and using the fact that $\nabla \cdot \boldsymbol{B} = 0$,

  $\Longrightarrow$ the incompressible MHD equations in conservation form:

$$
\begin{aligned}
\partial_t \boldsymbol{u} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{B} \otimes \boldsymbol{B}) + \nabla p_{\text{tot}} - \nu \nabla^2 \boldsymbol{u} &= 0, \\
\nabla \cdot \boldsymbol{u} &= 0, \\
\partial_t \boldsymbol{B} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{B} - \boldsymbol{B} \otimes \boldsymbol{u}) - \eta \nabla^2 \boldsymbol{B} &= 0,
\end{aligned}
$$

- Energy stability:

$$
\frac{1}{2}\partial_t \int_\Omega (\boldsymbol{u}^2 + \boldsymbol{B}^2)\mathrm{dx} = -\int_\Omega (\nu \nabla \boldsymbol{u} : \nabla \boldsymbol{u} + \eta \nabla \boldsymbol{B} : \nabla \boldsymbol{B})\mathrm{dx} \le 0
$$

The div-free DG scheme (Fu, JSC, 19)

- The div-free DG scheme: find $\boldsymbol{u}_h, \boldsymbol{B}_h \in \boldsymbol{V}_{h,0}^{\mathrm{div}}$ such that

$$
\begin{aligned}
(\partial_t \boldsymbol{u}_h, \boldsymbol{v}) = & -\mathcal{C}_{uu}(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}) + \mathcal{C}_{bb}(\boldsymbol{B}_h; \boldsymbol{B}_h, \boldsymbol{v}) - \nu \mathcal{B}_h(\boldsymbol{u}_h, \boldsymbol{v}), \\
(\partial_t \boldsymbol{B}_h, \boldsymbol{\phi}) = & -\mathcal{C}_{ub}(\boldsymbol{u}_h; \boldsymbol{B}_h, \boldsymbol{\phi}) + \mathcal{C}_{bu}(\boldsymbol{B}_h; \boldsymbol{u}_h, \boldsymbol{\phi}) - \eta \mathcal{B}_h(\boldsymbol{B}_h, \boldsymbol{\phi}),
\end{aligned}
$$

  for all test function $\boldsymbol{v}, \boldsymbol{\phi} \in \boldsymbol{V}_{h,0}^{\mathrm{div}}$.

  - Second-order viscous/resistive terms: **SIP DG**, $\mathcal{B}_h(\cdot, \cdot)$
  - Nonlinear convective terms: **upwinding DG**, next 2 slides

$$
\begin{aligned}
\mathcal{C}_{uu} &\leftrightarrow \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}), \qquad & \mathcal{C}_{bb} &\leftrightarrow \nabla \cdot (\boldsymbol{B} \otimes \boldsymbol{B}), \\
\mathcal{C}_{ub} &\leftrightarrow \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{B}), \qquad & \mathcal{C}_{bu} &\leftrightarrow \nabla \cdot (\boldsymbol{B} \otimes \boldsymbol{u}).
\end{aligned}
$$

Upwinding DG for convection (I)

The four convection terms

$$\underbrace{\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u})}_{\text{vel. driven}}, \nabla \cdot (\boldsymbol{B} \otimes \boldsymbol{B}), \qquad \underbrace{\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{B})}_{\text{vel. driven}}, \nabla \cdot (\boldsymbol{B} \otimes \boldsymbol{u})$$

$$\mathcal{C}_{uu}(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}) := \sum_{T \in \mathcal{T}_h} \int_T -(\boldsymbol{u}_h \otimes \boldsymbol{u}_h) : \nabla \boldsymbol{v} \, \mathrm{dx} + \int_{\partial T} (\boldsymbol{u}_h \cdot \boldsymbol{n}) \boldsymbol{u}_h^- \cdot \boldsymbol{v} \, \mathrm{ds},$$

$$\mathcal{C}_{ub}(\boldsymbol{u}_h; \boldsymbol{B}_h, \boldsymbol{\phi}) := \sum_{T \in \mathcal{T}_h} \int_T -(\boldsymbol{u}_h \otimes \boldsymbol{B}_h) : \nabla \boldsymbol{\phi} \, \mathrm{dx} + \int_{\partial T} (\boldsymbol{u}_h \cdot \boldsymbol{n}) \boldsymbol{B}_h^- \cdot \boldsymbol{\phi} \, \mathrm{ds},$$

Stability property:

$$\mathcal{C}_{uu}(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{u}_h) = \frac{1}{2} \sum_{F \in \mathcal{F}_h} \int_F |\boldsymbol{u}_h \cdot \boldsymbol{n}| [\![\boldsymbol{u}_h]\!]^2 \, \mathrm{ds} \geq 0,$$

$$\mathcal{C}_{ub}(\boldsymbol{u}_h; \boldsymbol{B}_h, \boldsymbol{B}_h) = \frac{1}{2} \sum_{F \in \mathcal{F}_h} \int_F |\boldsymbol{u}_h \cdot \boldsymbol{n}| [\![\boldsymbol{B}_h]\!]^2 \, \mathrm{ds} \geq 0,$$

Upwinding DG for convection (II)

The four convection terms

$$\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}), \underbrace{\nabla \cdot (\boldsymbol{B} \otimes \boldsymbol{B})}_{\text{mag. driven}}, \qquad \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{B}), \underbrace{\nabla \cdot (\boldsymbol{B} \otimes \boldsymbol{u})}_{\text{mag. driven}}$$

$$\mathcal{C}_{bb}(\boldsymbol{B}_h; \boldsymbol{B}_h, \boldsymbol{v}) := \sum_{T \in \mathcal{T}_h} \int_T -(\boldsymbol{B}_h \otimes \boldsymbol{B}_h) : \nabla \boldsymbol{v} \, \mathrm{dx} + \int_{\partial T} (\boldsymbol{B}_h \cdot \boldsymbol{n}) \widehat{\boldsymbol{B}}_h \cdot \boldsymbol{v} \, \mathrm{ds},$$

$$\mathcal{C}_{bu}(\boldsymbol{B}_h; \boldsymbol{u}_h, \boldsymbol{\phi}) := \sum_{T \in \mathcal{T}_h} \int_T -(\boldsymbol{B}_h \otimes \boldsymbol{u}_h) : \nabla \boldsymbol{\phi} \, \mathrm{dx} + \int_{\partial T} (\boldsymbol{B}_h \cdot \boldsymbol{n}) \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{\phi} \, \mathrm{ds},$$

with numerical fluxes: $\widehat{\boldsymbol{u}}_h = \{\!\{\boldsymbol{u}_h\}\!\} + \frac{1}{2}[\![\boldsymbol{B}_h]\!]$, $\widehat{\boldsymbol{B}}_h = \{\!\{\boldsymbol{B}_h\}\!\} + \frac{1}{2}[\![\boldsymbol{u}_h]\!]$.

Stability property:

$$\begin{aligned} &-\mathcal{C}_{bb}(\boldsymbol{B}_h; \boldsymbol{B}_h, \boldsymbol{u}_h) - \mathcal{C}_{bu}(\boldsymbol{B}_h; \boldsymbol{u}_h, \boldsymbol{B}_h) \\ &= \frac{1}{2} \sum_{F \in \mathcal{F}_h} \int_F |\boldsymbol{B}_h \cdot \boldsymbol{n}| ([\![\boldsymbol{u}_h]\!]^2 + [\![\boldsymbol{B}_h]\!]^2) \, \mathrm{ds} \geq 0, \end{aligned}$$

The div-free DG scheme: properties

(1) Exactly divergence-free: $\nabla\cdot\boldsymbol{u}_h = \nabla\cdot\boldsymbol{B}_h = 0$ pointwise

(2) Natural *upwinding* discretization of convection terms:
no need of additional stabilization or skew-symmetrization

(3) Energy-stability:
minimal amount of numerical dissipation

$$\frac{1}{2}\partial_t\left(\|\boldsymbol{u}_h\|^2 + \|\boldsymbol{B}_h\|^2\right) = \underbrace{-\frac{1}{2}\sum_{F\in\mathcal{F}_h}\int_F (|\boldsymbol{B}_h\cdot\boldsymbol{n}| + |\boldsymbol{u}_h\cdot\boldsymbol{n}|)([\![\boldsymbol{u}_h]\!]^2 + [\![\boldsymbol{B}_h]\!]^2)\,\mathrm{ds}}_{\text{num. disp.}}$$
$$\underbrace{-\Big(\nu\mathcal{B}_h(\boldsymbol{u}_h,\boldsymbol{u}_h) + \eta\mathcal{B}_h(\boldsymbol{B}_h,\boldsymbol{B}_h)\Big)}_{\text{phy. disp.}} \le 0$$

(4) Pressure robustness/High-order accuracy

(5) Efficient solver (coupled with explicit time integrator)

Example 1: Orszag-Tang vortex problem (MHD)

(Orszag and Tang, 98)

Consider the incompressible, resistive MHD equations

$$\begin{aligned}
\partial_t \boldsymbol{u} + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u} - \boldsymbol{B}\otimes\boldsymbol{B}) + \nabla p_{\text{tot}} - \nu\nabla^2\boldsymbol{u} &= 0, \\
\nabla\cdot\boldsymbol{u} &= 0, \\
\partial_t \boldsymbol{B} + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{B} - \boldsymbol{B}\otimes\boldsymbol{u}) - \eta\nabla^2\boldsymbol{B} &= 0,
\end{aligned}$$

with periodic boundary conditions on $\Omega = [0, 2\pi] \times [0, 2\pi]$.

- $\boldsymbol{u}$: velocity, $\boldsymbol{B}$: magnetic field.
- ν: viscosity, η: resistivity.
- Initial condition:

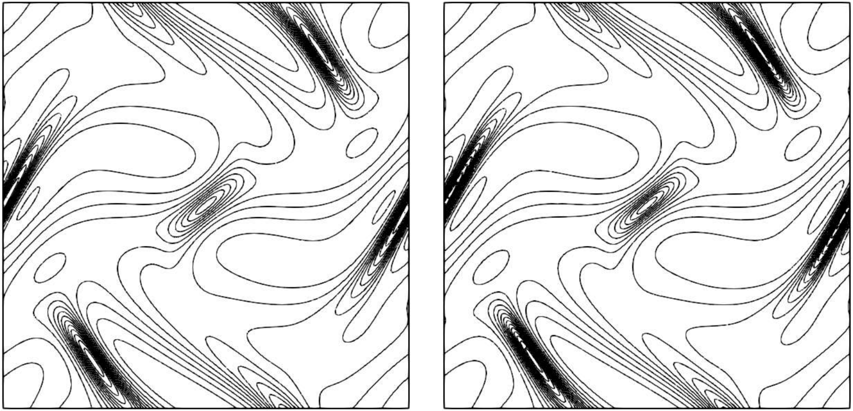
$$\begin{aligned}
u_1(x,y,0) &= -\sin(y), \quad u_2(x,y,0) = \sin(x), \\
B_1(x,y,0) &= -\sin(y), \quad B_2(x,y,0) = \sin(2x).
\end{aligned}$$

Example 1: Orszag-Tang vortex problem (MHD)

- Polynomial deg. $k = 3$, triangular mesh $h = 2\pi/80$

Vorticity contour lines at time t=1

viscous case $\nu = \eta = 0.005$ inviscid case $\nu = \eta = 0$

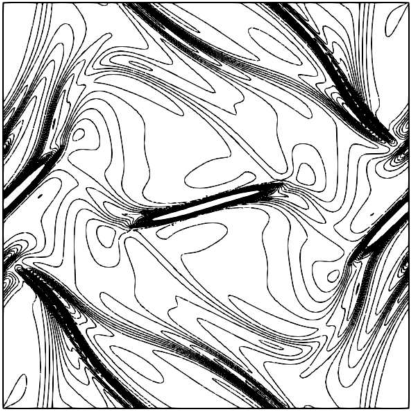


Example 1: Orszag-Tang vortex problem (MHD)

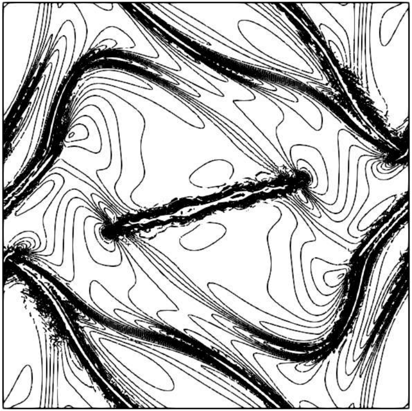
- Polynomial deg. $k = 3$, triangular mesh $h = 2\pi/80$

Vorticity contour lines at time t=2

viscous case $\nu = \eta = 0.005$

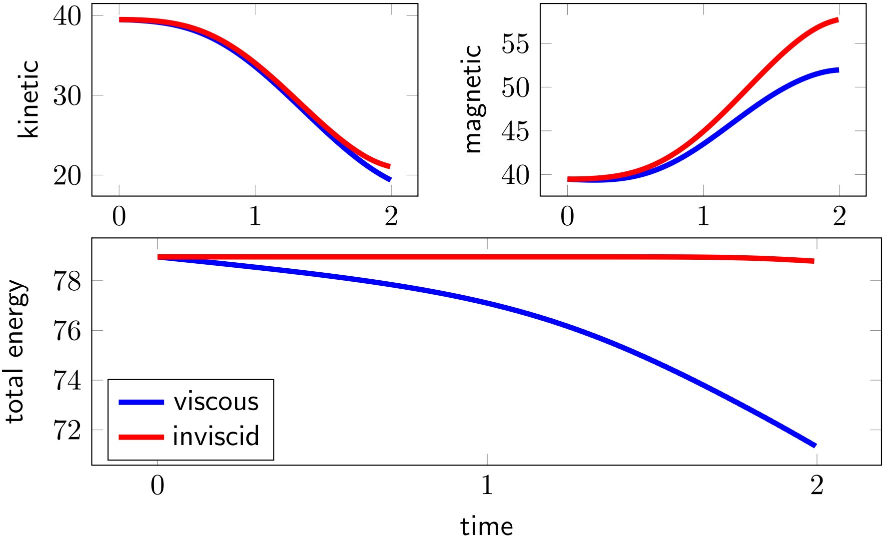


inviscid case $\nu = \eta = 0$



Example 1: Orszag-Tang vortex problem (MHD)

Time history of kinetic, magnetic, and total energy



Example 2: MHD Kelvin-Helmholtz instability (Frank et al., 96)

Consider incompressible,inviscid, ideal MHD $(\nu = \eta = 0)$

- Domain: $[0,1] \times [0,1]$. Periodic boundary condition in x-direction, and the no-flow boundary condition in y-direction.
- Initial conditions:

$$\boldsymbol{u}(x,y,0) = \left(-\frac{1}{2}\tanh(20(y-0.5)), 0\right) + 10^{-3}\nabla \times \psi(x,y),$$

$$\boldsymbol{B}(x,y,0) = (\frac{1}{M_A}, 0),$$

with corresponding stream function

$$\psi(x,y) = \exp(-400(y-0.5)^2)\cos(2\pi x).$$

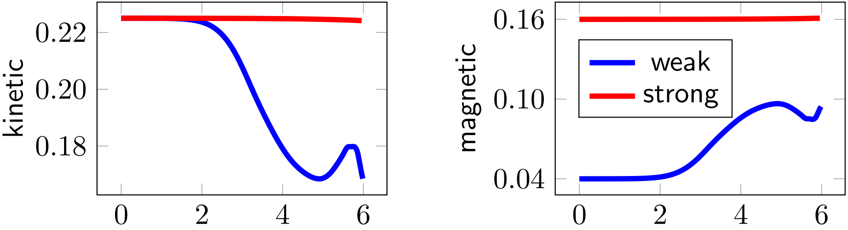
Here M_A is the Alfvénic Mach number.

Example 2: MHD Kelvin-Helmholtz instability

P^3 scheme on a 256×256 rectangular mesh. **Vorticity contour at t=6**

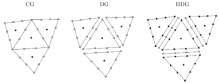
Hydrodynamic case ($M_A = \infty$) Weak magnetic case ($M_A = 5$) Strong magnetic case ($M_A = 2.5$)

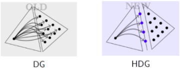
Time history of kinetic and magnetic energy



ALE div-free HDG for free surface flow (Fu, JCP, 20)

- **HDG:** a DG scheme with reduced DOFs coupling (more efficient linear system solver)





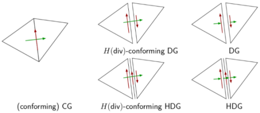


Figure 2.1.1: tangential and normal continuity for different methods

- **Test case:** solitary wave propagation

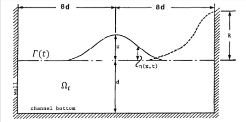


FIGURE 2. *The free-boundary domain.*

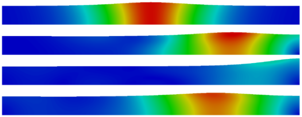


FIGURE 3. *Velocity magnitude (from left to right and top to bottom) at time* $t = 0, 4, 8,$ *and* 12.

	Duarte [13]	Ramaswamy [43]	HDG
Height	14.27	14.48	14.33
Time	7.7	7.6	7.65
Pressure	130	131.66	131.89

TABLE 2. *Comparison table: height, time and pressure.*

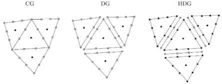
ALE-div-free-HDG with moving mesh (Fu, JCP 2020)

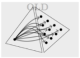
Features

- **ALE moving mesh** with node redistribution
- **Exact mass conservation:**
- **Upwinding (H)DG discretization** of nonlinear convection
- **Semi-discrete energy stability** with minimal amount of numerical dissipation
- **High-order accurate** (also low order)

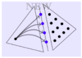
HDG for Incompressible two-phase flow

- **HDG:** a DG scheme with reduced DOFs coupling (more efficient linear system solver)



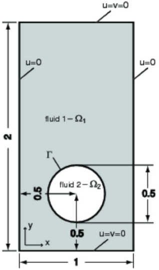


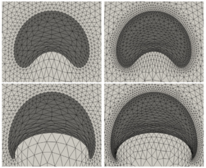
DG



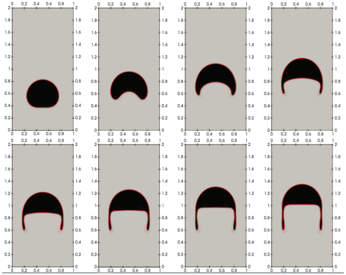
HDG

- **Test case 1:** a rising bubble problem





ALE-div-free-HDG with moving mesh (Fu, JCP 2020)



Div-free HDG for a Cahn-Hillard-NS phase-field model (Fu, CMAME 2020)

HDG for phase-field model of incompressible two-phase flow

- **Test case 2:** Rayleigh-Taylor instability

Div-free HDG for a Cahn-Hillard-NS phase-field model (Fu, CMAME 2020)

$$\begin{aligned} \partial_t\phi + \boldsymbol{u}\cdot\nabla\phi &= \nabla\cdot(M(\phi)\nabla\mu), \\ \mu &= \widetilde{\sigma}(\epsilon^{-1}W'(\phi) - \epsilon\triangle\phi), \\ \rho(\phi)(\partial_t(\boldsymbol{u}) + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u})) &= \nabla\cdot(2\nu(\phi)\mathbf{D}(\boldsymbol{u})) - \nabla p + \rho\boldsymbol{f} + \mu\nabla\phi, \\ \nabla\cdot\boldsymbol{u} &= 0, \end{aligned}$$

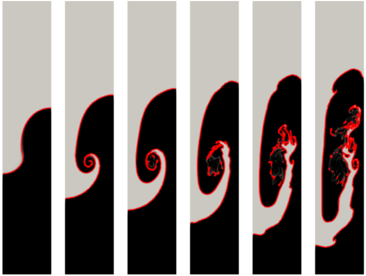


Figure: $Re = 5000$. *Contour of the RTI at time* $t = 1, 1.5, 1.75, 2.0, 2.25, 2.5$. $h = 2^{-7}, k = 2$. *Red contour line: the interface* $\phi_h = 0$.

- Embedded DG for the phase-field equations
- Divergence-free HDG for flow equation
- The divergence-free velocity space on 2D rectangular meshes:

$$\boldsymbol{V}_{h,0}^{div} = \nabla\times\{\xi\in H_0^1(\mathcal{T}_h): \quad \xi|_T\in\mathcal{Q}^{k+1}(T), \quad \forall T\in\mathcal{T}_h\}.$$

- Upwinding treatment for the convection terms
- No pressure approximation needed:
- Crank-Nicolson based implicit explicit time stepping.

Outline

The shallow water equations

- SWEs is a system of nonlinear balance law that has been widely used to model flow in the river, near shore ocean, and earth's atmosphere.
- The 2D inviscid SWEs takes the following form:

$$h_t + \nabla \cdot (h\boldsymbol{u}) = 0,$$
$$(h\boldsymbol{u})_t + \nabla \cdot (h\boldsymbol{u} \otimes \boldsymbol{u}) + \frac{1}{2} g \nabla (h^2) = -gh\nabla b,$$

- Main features of SWEs includes:
  1) **Entropy condition:** the total energy (entropy) does not increase over time.
  2) **Steady state and well-balanced property:** lake at rest steady state .
  3) **Positivity of water height:** .
  4) **Conservation property:** conservation for water height and discharge, and .

- **Goal:** design a DG scheme for SWEs that satisfies all these properties.

Entropy-stable DG for SWEs (Fu, arXiv:2201.13040)

- Main ideas: **(1)** work on a skew-symmetric formulation (for discharge), **(2)** use velocity and water height as the main approximation unknowns.

$$h_t + \nabla \cdot (h\boldsymbol{u}) = 0,$$

$$(h\boldsymbol{u})_t + \nabla \cdot (h\boldsymbol{u} \otimes \boldsymbol{u}) + gh\nabla(h+b) \boxed{- \frac{1}{2} h_t \boldsymbol{u} - \frac{1}{2}\nabla \cdot (h\boldsymbol{u})\boldsymbol{u}} = 0.$$

skew-symmetrization

- The designed DG spatial discretization is (1) **entropy stable**, (2) **locally conservative**, and (3) **well-balanced**.
- Combined with explicit SSP-RK temporal discretization, we can guarantee **positive** of water height with the help of a scaling limiter under a usual condition.
- Use characteristic-wise TVB limiter on troubled cells with the previous troubled-cell indicator to handle strong shocks.
- Apply the following **wetting/drying treatment** for velocity computation near dry areas:
    1) Identify dry cells as those with small water height cell average . Go back to piecewise constant approximations on these dry cells.
    2) Set a threshold velocity value , if the compute velocity in a cell is larger than , replace it with an average of the neighboring velocities.

Entropy-stable DG for SWEs (Fu, arXiv:2201.13040)

- **Test case 1:** circular dam break problem. (DG-P2 with RK3)

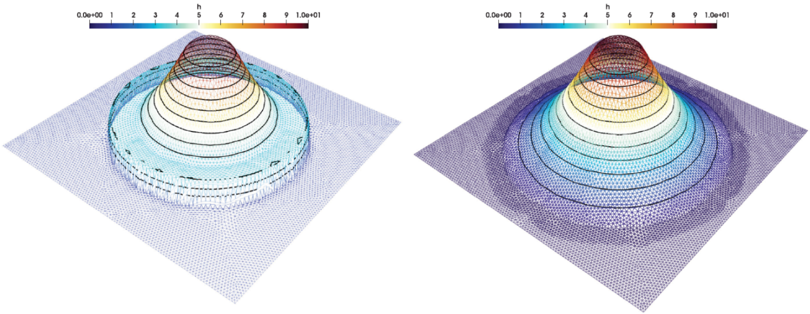


FIGURE 7. Example 4.9. Contour and surface plots of water height for the circular dam-break problem at $t = 0.69$. Left: web bed. 11 uniform contour lines from 2 to 9.4; Right dry bed. 12 uniform contour lines from 0.01 to 8.9.

Entropy-stable DG for SWEs (Fu, arXiv:2201.13040)

- **Test case 2:** dam break over three mounds. (DG-P2 with RK3)

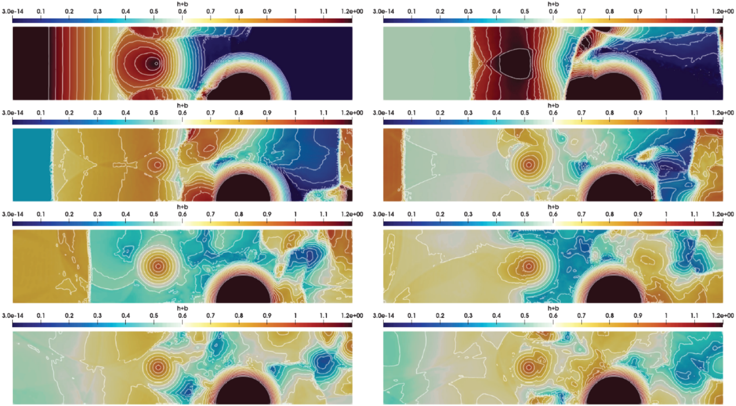


FIGURE 8. Example 4.10. Contour and surface plots of water surface for the dam-break problem on a closed channel. 20 uniform contour lines from 0 to 1.2. Left to right, top to bottom: $t = 5, 10, 15, 20, 25, 30, 35, 40$.

Outline

Conclusion and future work

(i) Proposed to solve incompressible flow problems using a **globally divergence-free** velocity space.

exact mass conservation, pressure-robustness ...

(ii) **Upwinding** treatment of the convection terms adds in necessary numerical dissipation that makes the scheme stable even in the convection-dominated regime without using any extra residual-based stabilization

(iii) Enforce **entropy stability** for SWEs by using water height and velocity as approximation unknowns.

Netgen/NGSolve

Conclusion and future work

(i) Proposed to solve incompressible flow problems using a **globally divergence-free** velocity space.

exact mass conservation, pressure-robustness ...

(ii) **Upwinding** treatment of the convection terms adds in necessary numerical dissipation that makes the scheme stable even in the convection-dominated regime without using any extra residual-based stabilization

(iii) Enforce **entropy stability** for SWEs by using water height and velocity as approximation unknowns.

Netgen/NGSolve

- Some future research plans:
  - Other incompressible multiphysics problems
  - Fluid flow on surfaces (e.g. SWE on sphere)
  - All Mach number compressible flow solver

Conclusion and future work

(i) Proposed to solve incompressible flow problems using a **globally divergence-free** velocity space.
exact mass conservation, pressure-robustness ...

(ii) **Upwinding** treatment of the convection terms adds in necessary numerical dissipation that makes the scheme stable even in the convection-dominated regime without using any extra residual-based stabilization

(iii) Enforce **entropy stability** for SWEs by using water height and velocity as approximation unknowns.

Netgen/NGSolve

- Some future research plans:
  - Other incompressible multiphysics problems
  - Fluid flow on surfaces (e.g. SWE on sphere)
  - All Mach number compressible flow solver

Thank you for your attention! Any questions?

Backup: Netgen/NGSolve

- **All in one**

Netgen/NGSolve provides the full workflow of finite element simulation: The constructive solid geometry module supports geometric modeling. Alternatively, geometric models can be imported from different formats. The Netgen mesh generator automatically generates high quality tetrahedal meshes. The NGSolve finite element library discretizes many physical models, and efficiently solves the arising systems of equations. The built-in visualization library allows fast and interactive visualization of the solution.

- **Flexible**

The Python frontend NGS-Py provides a flexible way to setup and combine various physical models. The input is provided in the natural mathematical language of variational formulations, where trial- and test-functions are chosen from all usual function spaces.

- **High-order Finite elements**

High order finite element spaces of H1, HDiv, HCurl, L2, skeleton types for all common cell types (segm, trig, quad, tet, prism, pyramid, hex). The order can be individually adapted for every edge, face, and cell of the mesh supporting **hp-adaptive** simulations. Netgen/NGSolve supports **surface PDEs** and **curved elements** of arbitrary order.

- **High performance/open source**