An arbitrarily high-order moment limiter for the discontinuous Galerkin method on unstructured meshes

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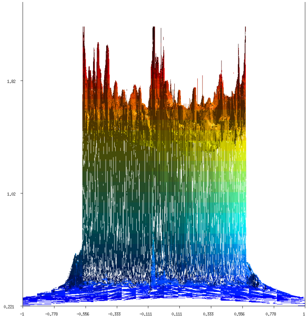
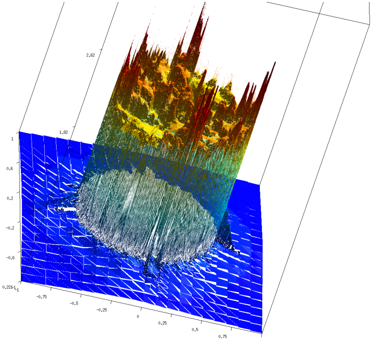
$$\mathbf{u}_t + \nabla \cdot \mathbf{F}(\mathbf{u}) = 0$$

Oscillatory solutions in the presence of discontinuities:





Noh test case with a second order numerical approximation

Limiters in two and three dimensions

- TVD in two dimensions is difficult to define.
- The number of derivatives that must be limited increases.
- No clear directions on unstructured meshes.
- Very few concrete theoretical results for higher than second order.
- Many limiters in two and three dimensions are complicated or ad hoc, without provable stability properties.

Goals

- Development of limiters for DG in two and three dimensions on unstructured meshes that are easy to implement, fast to execute, and
- with some theoretical backing: conditions on second order accuracy and stability (LMP).

The discontinuous Galerkin method

$$\frac{\partial}{\partial t}\mathbf{u} + \nabla \cdot \mathbf{F}(\mathbf{u}) = 0 \text{ on } \Omega \tag{1}$$

Partition the domain Ω into a mesh of elements Ω_i and define numerical solution $\mathbf{U}_i$ on cell Ω_i of polynomial order approximation p

$$\mathbf{U}_i = \sum_k \mathbf{c}_{i,k}\varphi_k.$$

$\mathbf{c}_{i,k}$	kth solution coefficient, or moment, on cell Ω_i.
φ_k	kth orthonormal basis function.

The DG method for a scalar conservation law (1) is

$$\frac{d}{dt}c_{i,k} = \int_{\Omega_i} \mathbf{F}(U_i)\cdot\nabla\varphi_k d\Omega_i - \sum_j \int_{\partial\Omega_{i,j}} \mathbf{F}(U_i, U_j)\cdot\mathbf{n}_{i,j}\varphi_k dl.$$

$\mathbf{F}(\mathbf{U}_i, \mathbf{U}_j)$	numerical flux function
$\partial\Omega_{i,j}$	edge shared by elements Ω_i and Ω_j

Moment Limiter

General approach

- Limit solution moments by reducing their magnitudes: $c_{i,k} \to l_{i,k} c_{i,k},\ 0 \le l_{i,k} \le 1$
- Original solution

$$U_i = \sum_k c_{i,k} \varphi_k.$$

- Limited solution

$$\tilde{U}_i = \sum_k l_{i,k} c_{i,k} \varphi_k.$$

- Note: no need for expensive function evaluations

Moment Limiter $p = 1$

- One-dimensional mimmod limiter

$$\tilde{c}_{i,1} = \text{minmod}\Big(c_{i+1,0} - c_{i,0}, c_{i,1}, c_{i,0} - c_{i-1,0}\Big).$$

- Two-dimensional minmod limiter ($p = 1$)

$$\tilde{c}_{i,1} = \text{minmod}\left(\delta\frac{U^f_{i,1} - \overline{U}^n_i}{2}, c_{i,1}, \delta\frac{\overline{U}^n_i - U^b_{i,1}}{2}\right),$$

$$\tilde{c}_{i,2} = \text{minmod}\left((4-2\delta)\frac{U^f_{i,2} - \overline{U}^n_i}{4\sqrt{3}}, c_{i,2}, (4-2\delta)\frac{\overline{U}^n_i - U^b_{i,2}}{4\sqrt{3}}\right).$$

- $\overline{U}^n_i$- solution average, $U^f_{i,1} - \overline{U}^n_i$ - forward difference, $\overline{U}^n_i - U^b_{i,2}$ - backward difference, δ and other coefficients - appropriate scaling factors.

Outline

Questions:

- In what directions to limit?
- How to find proper forward and backward values?
- How to find suitable scaling factors?
- How to be LMP and maintain second order accuracy?
- Time step restriction?

Outline:

- Sketch of the derivation for $p = 1$ in 2D on nonuniform conforming grids
- Extension to $p > 2$ in 2D on nonuniform conforming grids
- Extension to nonconforming meshes (arbitrary p)
- Extension to 3D

Propagation of the solution averages

DG scheme for propagation of the solution coefficient corresponding to the constant basis function $(k = 0)$

$$\frac{d}{dt}c_{i,0} = -\frac{1}{\det J_i}\sum_{j\in N_i^e}\int_{\partial\Omega_{i,j}} \varphi_0 \mathbf{F}(U_i, U_j)\cdot \mathbf{n}_{i,j}\ dl.$$

Multiplying the above by φ_0 and using $\overline{U}_i = c_{i,0}\varphi_0$, we obtain

$$\frac{d}{dt}\overline{U}_i = -\frac{1}{|\Omega_i|}\sum_{j\in N_i^e}\int_{\partial\Omega_{i,j}} \mathbf{F}(U_i, U_j)\cdot \mathbf{n}_{i,j}\ dl,$$

where $|\Omega_i|$ is the area or volume of the cell. With one forward Euler time step:

$$\overline{U}_i^{n+1} = \overline{U}_i^{n} - \frac{\Delta t}{|\Omega_i|}\sum_{j\in N_i^e}\int_{\partial\Omega_{i,j}} \mathbf{F}(U_i^n, U_j^n)\cdot \mathbf{n}_{i,j}\ dl.$$

Fully discrete scheme for the solution averages

For a nonlinear flux

$$\overline{U}_i^{n+1} = \overline{U}_i^n - \Delta t \sum_{j \in N_i^e, j \neq i} \frac{1}{2} \frac{|\partial \Omega_{i,j}|}{|\Omega_i|} \sum_{q=1,2} \mathbf{F}^*(U_i^n(\mathbf{x}_{i,j,q}), U_j^n(\mathbf{x}_{i,j,q})) \cdot \mathbf{n}_{i,j},$$

and for a linear flux

$$\overline{U}_i^{n+1} = \overline{U}_i^n - \Delta t \sum_{j \in N_i^e, j \neq i} \frac{|\partial \Omega_{i,j}|}{|\Omega_i|} \mathbf{F}^*(U_i^n(\mathbf{x}_{i,j}), U_j^n(\mathbf{x}_{i,j})) \cdot \mathbf{n}_{i,j},$$

where $x_{i,j}$ and $x_{i,j,q}$ are quadrature points on the edge shared by Ω_i and Ω_j.

General approach to proof for triangles

1. Limit the solution

$$U_i^n(\mathbf{x}) = c_{i,0}\varphi_0 + c_{i,1}\varphi_1 + c_{i,2}\varphi_2$$

   to obtain

$$\tilde{U}_i^n(\mathbf{x}) = c_{i,0}\varphi_0 + l_{i,1}c_{i,1}\varphi_1 + l_{i,2}c_{i,2}\varphi_2$$

2. Substitute $\tilde{U}_i^n(\mathbf{x})$ into the scheme for the DG solution averages

$$\overline{U}_i^{n+1} = \overline{U}_i^n - \Delta t \sum_{j \in N_i^e, j \neq i} \frac{|\partial\Omega_{i,j}|}{|\Omega_i|} \mathbf{F}(\tilde{U}_i^n(\mathbf{x}_{i,j}), \tilde{U}_j^n(\mathbf{x}_{i,j})) \cdot \mathbf{n}_{i,j},$$

General approach to proof for triangles

3. Derive conditions on $l_{i,1}$, $l_{i,2}$, and Δt such that the scheme can be written as a convex combination

$$\overline{U}_i^{n+1} = \overline{U}_i + \sum d_j (U_j^n - \overline{U}_i),$$

   where U_j are understood to be solution means and values in the neighborhood of Ω_i at the previous time step t^n.

4. The local maximum principle

$$\min_{j \in \mathcal{N}_i} \overline{U}_j^n \leq \overline{U}_i^{n+1} \leq \max_{j \in \mathcal{N}_i} \overline{U}_j^n,$$

   immediately follows.

5. Stability of solution in the maximum norm follows.

$p = 1$ approximation

U_i written as a linear combination of basis functions of degree $<= 1$

$$U_i = c_{i,0}\varphi_0 + c_{i,1}\varphi_1 + c_{i,2}\varphi_2$$

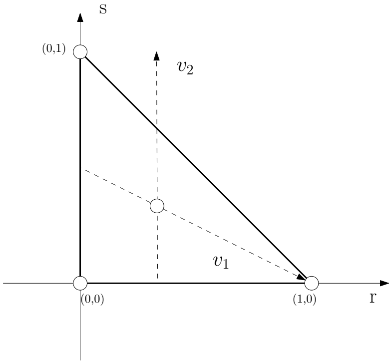
basis functions

$$\varphi_0 = \sqrt{2}, \quad \varphi_1 = -2 + 6r,$$
$$\varphi_2 = -2\sqrt{3} + 2\sqrt{3}r + 4\sqrt{3}s$$

Decoupling linear coefficients

Consider $v_1 = (1, -1/2),\ v_2 = (0, 1)$

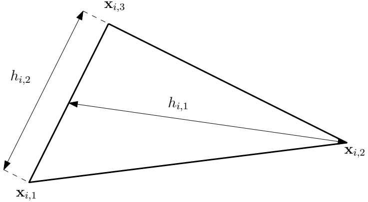
$$c_{i,1} = \nabla_{rs} U_i \cdot v_1 = \nabla_{rs} U_i \cdot \left[1, -\frac{1}{2}\right]$$
$$c_{i,2} = \nabla_{rs} U_i \cdot v_2 = \nabla_{rs} U_i \cdot [0, 1]$$



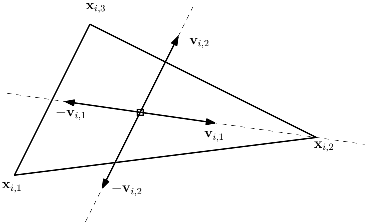
Directional derivatives in physical space

The linear moments on triangles can be written

$$c_{i,1} = \frac{h_{i,1}}{6} D_{\mathbf{v}_{i,1}} U_i \quad \text{and} \quad c_{i,2} = \frac{h_{i,2}}{4\sqrt{3}} D_{\mathbf{v}_{i,2}} U_i.$$



(a) Cell dependent size parameters $h_{i,1}$, $h_{i,2}$.



(b) Limiting directions $\mathbf{v}_{i,1}$ and $\mathbf{v}_{i,2}$.

Size parameters and limiting directions.

Moment limiters on unstructured meshes of triangles

- The limited moments are

$$l_{i,1}c^n_{i,1} = l_{i,1}\frac{h_{i,1}}{6}D_{\mathbf{v}_{i,1}}U^n_i \quad \text{and} \quad l_{i,2}c^n_{i,2} = l_{i,2}\frac{h_{i,2}}{4\sqrt{3}}D_{\mathbf{v}_{i,2}}U^n_i.$$

- We limit the solution coefficients by comparing them to reconstructed forward and backward differences

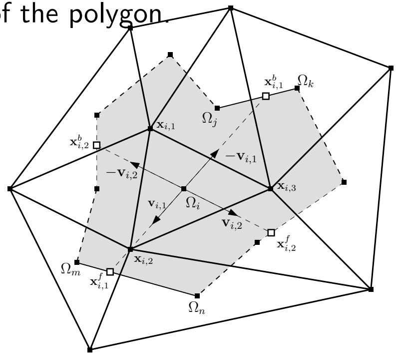
$$l^f_{i,1}c^n_{i,1} = l^f_{i,1}\frac{h_{i,1}}{6}\frac{U^f_{i,1}-\overline{U}_i}{d^f_{i,1}} \quad \text{or} \quad l^b_{i,1}c^n_{i,1} = l^b_{i,1}\frac{h_{i,1}}{6}\frac{\overline{U}_i-U^b_{i,1}}{d^b_{i,1}},$$

$$l^f_{i,2}c^n_{i,2} = l^f_{i,2}\frac{h_{i,2}}{4\sqrt{3}}\frac{U^f_{i,2}-\overline{U}_i}{d^f_{i,2}} \quad \text{or} \quad l^b_{i,2}c^n_{i,2} = l^b_{i,2}\frac{h_{i,2}}{4\sqrt{3}}\frac{\overline{U}_i-U^b_{i,2}}{d^b_{i,2}},$$

  where $d^f_{i,1}$ is the distance from the forward reconstruction point $\mathbf{x}^f_{i,1}$ and the cell centroid $\mathbf{x}_i$.
- Substitute these expressions back in to the DG scheme for the propagation of the solution averages.

Reconstruction neighborhood in two dimensions

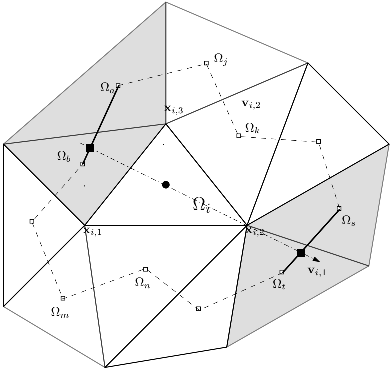
- Form a polygon from the centroids of the elements in the vertex neighborhood.
- Find the intersection of the lines defined by $(\mathbf{x}_i, \mathbf{v}_{i,1})$ and $(\mathbf{x}_i, \mathbf{v}_{i,2})$ and the edges of the polygon.



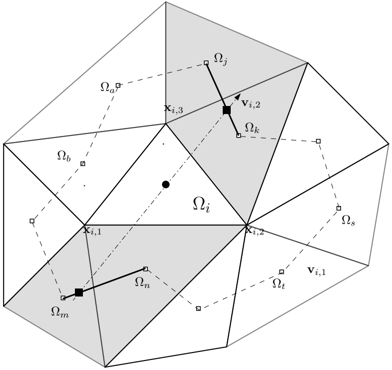
Interpolation neighborhood with forward and backward interpolation points $\mathbf{x}^f_{i,1}$, $\mathbf{x}^f_{i,2}$, $\mathbf{x}^b_{i,1}$, $\mathbf{x}^b_{i,1}$.

On a triangular element

Apply 1D limiter on the stencils along $\mathbf{v}_{i,1}$ and $\mathbf{v}_{i,1}$ independently to reconstruct the directional derivatives



(a) Reconstruction stencil along $\mathbf{v}_{i,1}$



(b) Reconstruction stencil along $\mathbf{v}_{i,2}$

Limiter for second order approximation

Reconstruction in $\mathbf{v}_{i,2}$ direction:

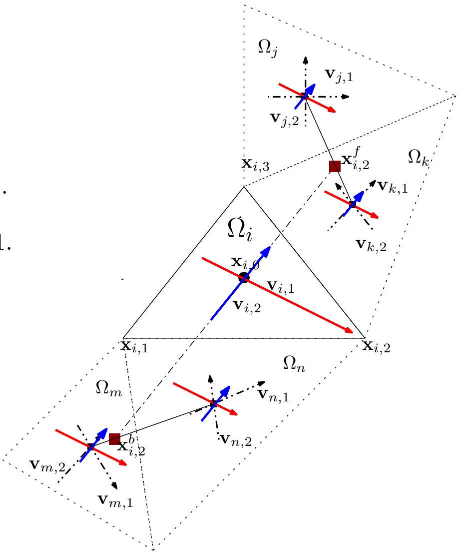
$$U^f_{i,2} = \beta^f_{i,2}\overline{U}^n_m + (1-\beta^f_{i,2})\overline{U}_n \text{ with } 0 \le \beta^f_{i,2} \le 1.$$

$$U^b_{i,2} = \beta^b_{i,2}\overline{U}^n_j + (1-\beta^b_{i,2})\overline{U}_k, \text{ with } 0 \le \beta^b_{i,2} \le 1.$$

Limiter:

$$c_{i,1} = \text{minmod}\left(\frac{U^f_{i,1} - \overline{U}^n_i}{2}, c_{i,1}, \frac{\overline{U}^n_i - U^b_{i,1}}{2}\right)$$

$$c_{i,2} = \text{minmod}\left(\frac{U^f_{i,2} - \overline{U}^n_i}{2\sqrt{3}}, c_{i,2}, \frac{\overline{U}^n_i - U^b_{i,2}}{2\sqrt{3}}\right)$$



Full set of constraints

After expressing the solution coefficients in terms of reconstructed forward and backward differences, we can derive the constraints on $l_{i,1}$, $l_{i,2}$. There is one constraint per outflow edge of the limited element Ω_i

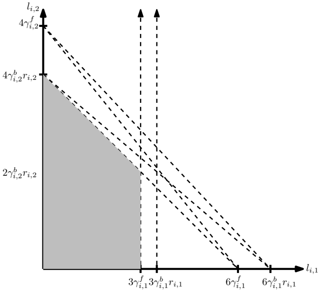
$$\frac{l_{i,1}}{6\gamma^f_{i,1}} + \frac{l_{i,2}}{4\gamma^b_{i,2} r_{i,2}} \leq 1 \quad \text{and} \quad \frac{l_{i,1}}{6\gamma^b_{i,1} r_{i,1}} + \frac{l_{i,2}}{4\gamma^f_{i,2}} \leq 1 \quad \text{if} \quad s = 1,$$

$$\frac{l_{i,1}}{6\gamma^f_{i,1}} + \frac{l_{i,2}}{4\gamma^f_{i,2}} \leq 1 \quad \text{and} \quad \frac{l_{i,1}}{6\gamma^b_{i,1} r_{i,1}} + \frac{l_{i,2}}{4\gamma^b_{i,1} r_{i,2}} \leq 1 \quad \text{if} \quad s = 2,$$

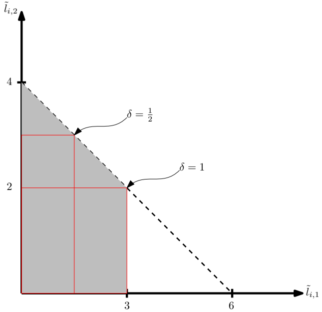
$$\frac{l_{i,1}}{3\gamma^b_{i,1} r_{i,1}} \leq 1 \quad \text{and} \quad \frac{l_{i,1}}{3\gamma^f_{i,1}} \leq 1 \quad \text{if} \quad s = 3,$$

where s is the face number, the $\gamma_{i,k}$ are geometrical constants and $r_{i,k}$ are the ratio of forward to backward reconstructed differences for $k = 1, 2$.

Stability Region (shaded)



(a) The full set of inequalities.



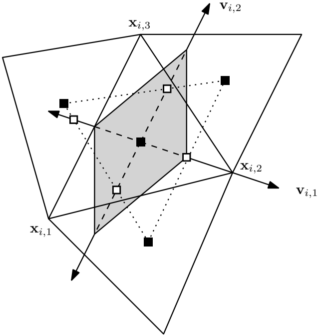
(b) Simplified set of inequalities.

$$\tilde{c}_{i,1} = \text{minmod}\left(\delta\frac{U^f_{i,1} - \overline{U}^n_i}{2}, c_{i,1}, \delta\frac{\overline{U}^n_i - U^b_{i,1}}{2}\right),$$

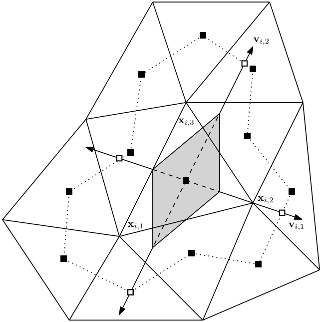
$$\tilde{c}_{i,2} = \text{minmod}\left((4-2\delta)\frac{U^f_{i,2} - \overline{U}^n_i}{4\sqrt{3}}, c_{i,2}, (4-2\delta)\frac{\overline{U}^n_i - U^b_{i,2}}{4\sqrt{3}}\right).$$

Requirement for second order accuracy

The interpolation points must lie a minimum distance from the centroid, i.e., outside the gray triangle.



(a) Edge neighborhood.



(b) Vertex neighborhood.

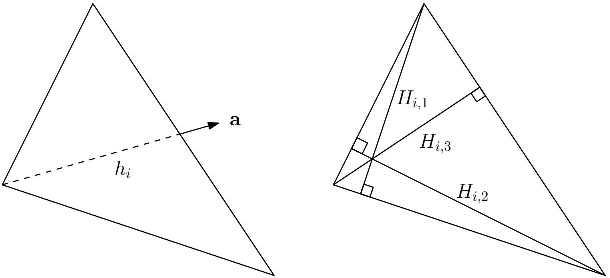
Interpolation points must lie outside the grayed diamond.

Time step restriction

Restrains also lead to a restriction on the time step size

$$\Delta t \leq \frac{1}{4} \min_i \frac{h_i}{||\mathbf{a}||}$$

where h_i is the cell width in the direction of flow, $\mathbf{a}$.



(a) Cell width in direction of flow.

(b) Minimum cell width.

Summary

- Local maximum principle in the means

$$m_i^n = \min_{j\in V_i} \overline{U}_j^n \le \overline{U}_i^{n+1} \le \max_{j\in V_i} \overline{U}_j^n = M_i^n$$

  V_i - a neighborhood of Ω_i, $\overline{U}_i^n$, $\overline{U}_j^n$ are cell averages.
- Moment limiter

$$c_{i,1} = \text{minmod}\left(\frac{U_{i,1}^f - \overline{U}_i^n}{2}, c_{i,1}, \frac{\overline{U}_i^n - U_{i,1}^b}{2}\right)$$

$$c_{i,2} = \text{minmod}\left(\frac{U_{i,2}^f - \overline{U}_i^n}{2\sqrt{3}}, c_{i,2}, \frac{\overline{U}_i^n - U_{i,2}^b}{2\sqrt{3}}\right)$$

- DGM with the moment limiter satisfies LMP under the time step restriction

$$\Delta t \le \frac{1}{4}\min_i \frac{h_i}{||\mathbf{a}||}$$

Moment limiters on unstructured meshes of triangles

Verification of CFL

We solve the linear advection equation with

$$u_0(x,y) = \begin{cases} 1 \text{ if } \max(|x|,|y|) \le \frac{1}{4} \\ 0 \text{ elsewhere.} \end{cases}$$

1/CFL	Minimum	Maximum
2	-3.97e-01	1.14
3	-4.83-02	1.00062
3.5	-4.31e-03	1
4	**0**	**1**

(a) Forward Euler.

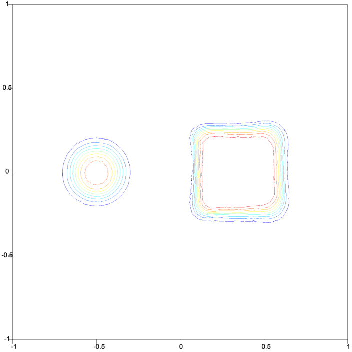
1/CFL	Minimum	Maximum
2	0	1
3	0	1
3.5	0	1
4	**0**	**1**

(b) RK2.

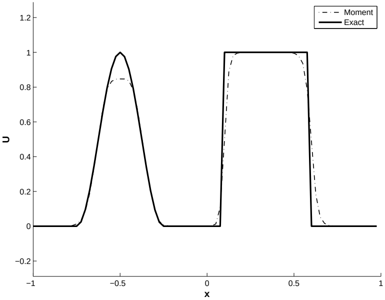
Minimum and maximum cell averages for various CFL numbers.

The derived bounds on the time step restrictions are tight.

Solutions on unstructured triangular and Cartesian meshes are nearly the same.



(a) Isolines with moment limiter.



(b) Profile of solution on the line $y = 0$.

Rotating shapes at $T = 1$.

Higher orders of approximation

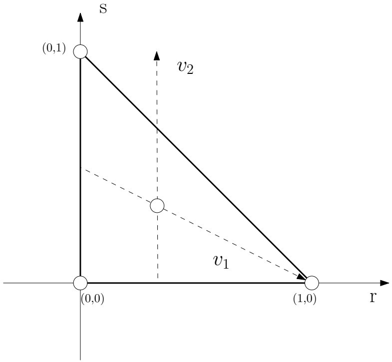
Spatial approximation:

$$U_i = \sum_{l+k=0}^{p} c_{i,k}^{l} \varphi_k^l(r,s),$$

$c_{i,k}^{l}$: moments, φ_k^l : basis functions,
p : degree of approximation

Dubiner basis φ_k^l :

$$\varphi_k^l(r,s) = 2^k L_k(a)(1-r)^k P_l^{2k+1,0}(b),$$
$$a = \frac{2s}{1-r} - 1, \quad b = 2r - 1.$$



Reference triangular element Ω_0

pth degree approximation

Denote jth directional derivative, $0 \leq j \leq p$, by

$$D^j_{v_1} U_i = \frac{\partial^j}{\partial v_1^j} U_i, \quad D^j_{v_2} U_i = \frac{\partial^j}{\partial v_2^j} U_i,$$

$$\begin{pmatrix} D^p_{v_2} U_i \\ \beta D^{p-1}_{v_2} D^1_{v_1} U_i \\ \beta^2 D^{p-2}_{v_2} D^2_{v_1} U_i \\ \vdots \\ \beta^p D^p_{v_1} U_i \end{pmatrix} = \begin{bmatrix} \theta_{0,0} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \theta_{1,1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \theta_{2,0} & 0 & \theta_{2,2} & 0 & 0 & 0 & 0 & \cdots \\ & & & & \vdots & & & \\ & & & & \cdots & \theta_{p,p-2} & 0 & \theta_{p,p} \end{bmatrix} \begin{pmatrix} c^0_{i,p} \\ c^1_{i,p-1} \\ c^2_{i,p-2} \\ \vdots \\ c^p_{i,0} \end{pmatrix}$$

$\beta = \frac{\sqrt{5}}{2}$, $\theta_{i,j}$: constants

High-order solution coefficients do not fully decouple

Limiting with $p > 1$

Reconstructing $D^{p-1}_{\mathbf{v}_{i,2}} D^1_{\mathbf{v}_{i,2}} U_i$:

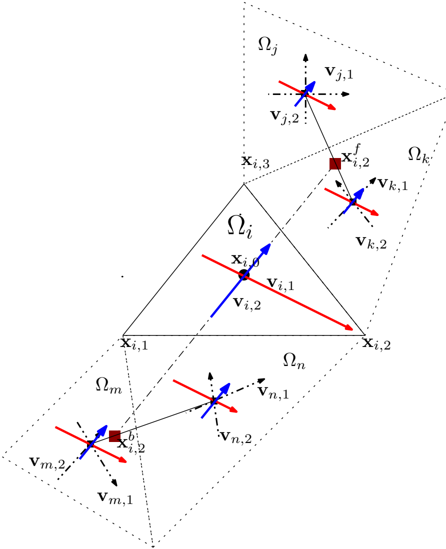
$$\tilde{D}^{p-1}_{\mathbf{v}_{i,1}} \tilde{D}^1_{\mathbf{v}_{i,2}} U_i = m(D^{p-1}_{\mathbf{v}_{i,1}} D^1_{\mathbf{v}_{i,2}} U_i, l^f_{i,2} D_+, l^b_{i,2} D_-),$$

$$D_+ = \frac{1}{\Delta \mathbf{x}^f} \left(D^{p-1}_{\mathbf{v}_{i,1}} U^f_i - D^{p-1}_{\mathbf{v}_{i,1}} U_i|_{\mathbf{x}_{i,0}} \right),$$

$$D_- = \frac{1}{\Delta \mathbf{x}^b} \left(D^{p-1}_{\mathbf{v}_{i,1}} U_i|_{\mathbf{x}_{i,0}} - D^{p-1}_{\mathbf{v}_{i,1}} U^b_i \right),$$

$$1 \le l^f_{i,2} \le \frac{\Delta \mathbf{x}^f}{h_{i,2}}(2p-1), \; 1 \le l^b_{i,2} \le \frac{\Delta \mathbf{x}^b}{h_{i,2}}(2p-1),$$

$$\Delta \mathbf{x}^f = |\mathbf{x}^f_{i,2} - \mathbf{x}_{i,0}|, \quad \Delta \mathbf{x}^b = |\mathbf{x}^b_{i,2} - \mathbf{x}_{i,0}|$$

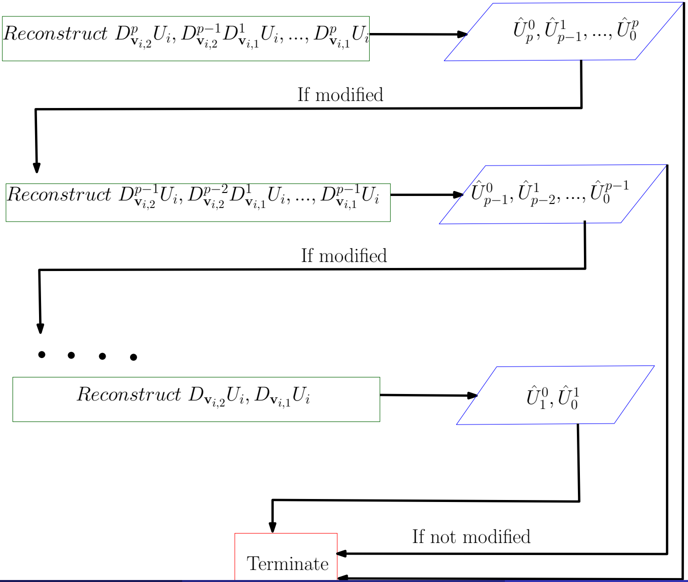


Interpolation:

$$D^{p-1}_{\mathbf{v}_{i,1}} U^f_i = \beta^f_{i,2} D^{p-1}_{\mathbf{v}_{i,1}} U_8 + (1 - \beta^f_{i,2}) D^{p-1}_{\mathbf{v}_{i,1}} U_9,$$

$$D^{p-1}_{\mathbf{v}_{i,1}} U^b_i = \beta^b_{i,2} D^{p-1}_{\mathbf{v}_{i,1}} U_2 + (1 - \beta^b_{i,2}) D^{p-1}_{\mathbf{v}_{i,1}} U_3$$

Hierarchical Limiting



Moment Limiter with $p = 2$

Limiting of 2nd order coefficients

$$\begin{pmatrix} h_{i,2}^2 D_{\mathbf{v}_{i,2}}^2 U_i(\mathbf{x}_{i,0}) \\ h_{i,2} h_{i,1} D_{\mathbf{v}_{i,2}} D_{\mathbf{v}_{i,1}} U_i(\mathbf{x}_{i,0}) \\ h_{i,1}^2 D_{\mathbf{v}_{i,1}}^2 U_i(\mathbf{x}_{i,0}) \end{pmatrix} = \begin{bmatrix} 12\sqrt{30} & 0 & 0 \\ 0 & 30\sqrt{2} & 0 \\ -\sqrt{30} & 0 & 20\sqrt{6} \end{bmatrix} \begin{pmatrix} c_{i,2}^0 \\ c_{i,1}^1 \\ c_{i,0}^2 \end{pmatrix}.$$

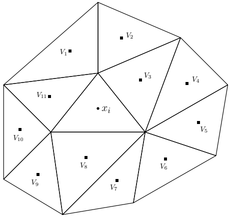
Limit directional derivatives using 4 minmod limiters, update the solution coefficients

$$\begin{aligned} c_{i,2}^0 &= \frac{h_{i,2}^2}{12\sqrt{30}} D_{\mathbf{v}_{i,2}}^2 U_i, \\ c_{i,1}^1 &= \frac{h_{i,2} h_{i,1}}{30\sqrt{2}} D_{\mathbf{v}_{i,2}} D_{\mathbf{v}_{i,1}} U_i, \\ c_{i,0}^2 &= \frac{h_{i,1}^2}{20\sqrt{6}} D_{\mathbf{v}_{i,1}}^2 U_i + \frac{h_{i,2}^2}{240\sqrt{6}} D_{\mathbf{v}_{i,2}}^2 U_i. \end{aligned}$$

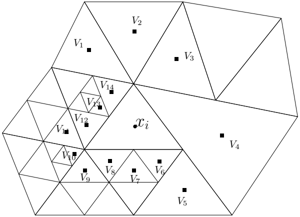
If unlimited, stop. If limited, continue to $p = 1$ limiter

Limiting on adaptively refined nonconforming meshes

- Adaptive computations allow effective use of computational resource, e.g., to better capture the fine features of the solution
- Nonconforming meshes arise in simulations employing run-time adaptive mesh refinement
- Most limiting techniques are designed for conforming meshes, i.e., meshes where an edge is shared between only two elements
- Applying traditional techniques involve projecting solution to bigger/smaller elements, which is computationally expensive

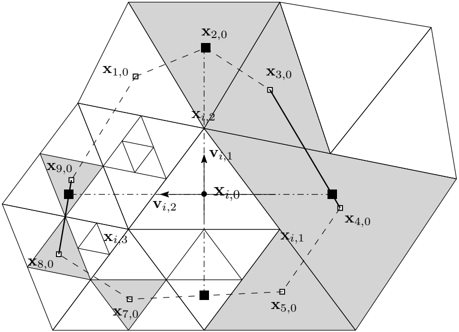


(a) Conforming mesh



(b) Nonconforming mesh

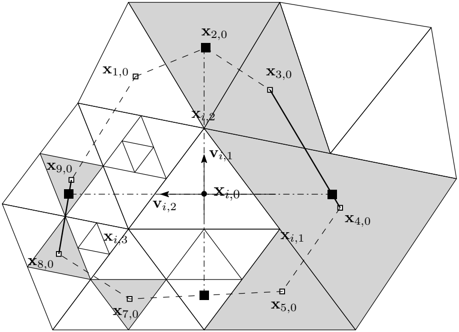
Limiting on nonconforming meshes



Reconstruction neighborhood for moment limiter

- Reconstruct the neighborhood of an element using vertex-element connectivity of its vertices
- Find interpolation points along v_1 and v_2
- Limit the solution coefficients by applying 1D limiters along $\mathbf{v}_1$ and $\mathbf{v}_2$

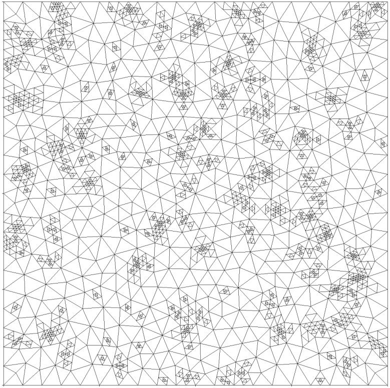
Limiting on nonconforming meshes



Reconstruction neighborhood for moment limiter

- More elements of varying sizes in the neighborhood
- Update the neighborhood with every mesh refinement
- Find a neighborhood not violating geometrical constrains
- Minimize the cost and storage of mesh connectivity info
- Race conditions on GPUs: simultaneously update neighborhoods
- New time step restriction

Advecting hill



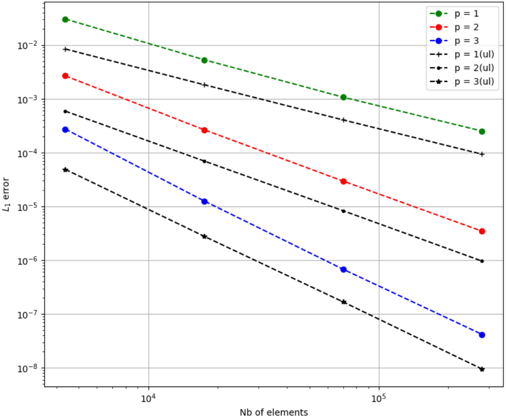
Initial nonconforming mesh

Linear advection

$$u_t + a_1 u_x + a_2 u_y = 0,$$

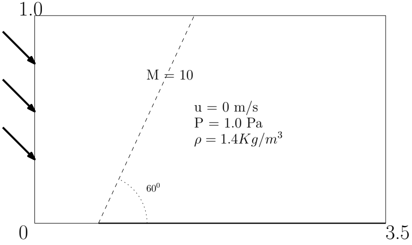
- $(a_1, a_2) = (1, 1)$
- $u_0(x, y) = 2.5\exp(-\frac{r^2}{2x0.15^2})$
- $r = |(x, y) - (-0.25, -0.25)|_2$
- $\Omega = [-1, 1] \times [-1, 1]$
- $T = 0.5$

Advecting hill

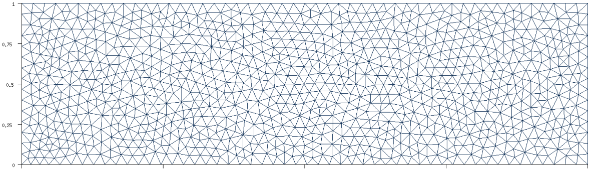


Log-log plot of L_1 error vs. number of elements

Double Mach reflection

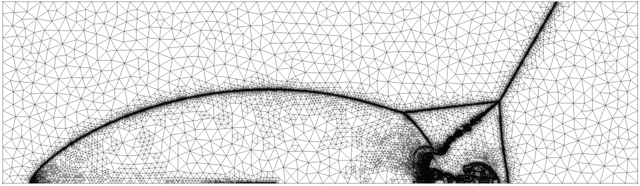


(a) Initial condition



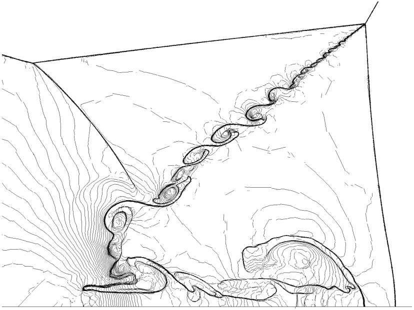
(b) Initial mesh consisting of 1953 triangular elements

(a) Density isolines

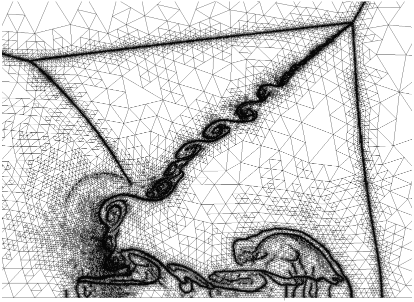


(b) Final adapted mesh

Density isolines at $t = 0.2$ for double Mach test case, with second order approximation and moment limiter, on a final mesh with 236,910 triangles and 7 levels of refinement

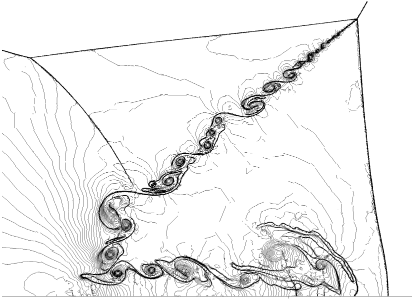


(a) Zoom-in near slipline

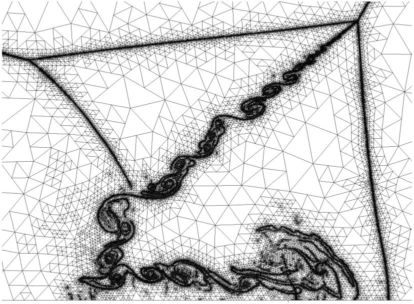


(b) Final adapted mesh

Density isolines at $t = 0.2$ for double Mach test case, with second order approximation and moment limiter, on a final mesh with 236,910 triangles and 7 levels of refinement



(a) Zoom-in near slipline.



(b) Final adapted mesh.

Density isolines at $t = 0.2$ for double Mach test case, with fourth order approximation and moment limiter, on a final mesh with 222,969 triangles and 7 levels of refinement

Comparing the performance of limiters on adaptively refined mesh

	Moment limiter	Barth-Jesperson limiter
Total run time	1747.68s	1984.40s
Limiting time	71.39s	311.5s
Setting up neighbourhood	48.56s	18.94s
Total limiting time	119.95s	330.44s

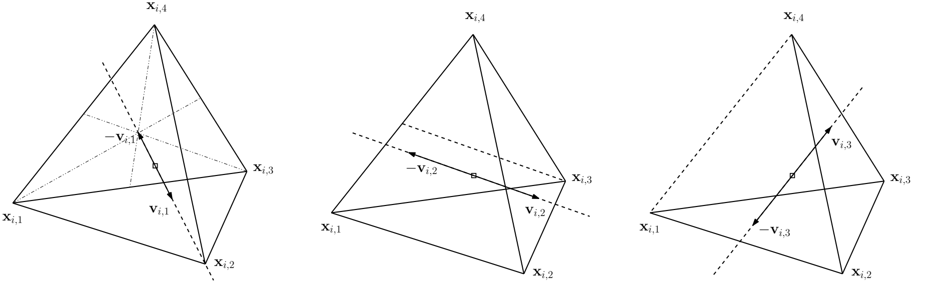
Wall clock timings (in seconds) for simulation of double mach test case until $t = 0.2$, with a second order approximation, an initial unstructured mesh of 1953 elements, 7 levels of refinement, and the mesh adapted every 10 timestep

Limiting time with moment limiter $\approx$ **25%** of (Limiting time with Barth-Jesperson limiter)

Moment limiter on tetrahedral meshes

The linear moments on tetrahedra can be written

$$c_{i,1}^n = \frac{h_{i,1}}{4\sqrt{10}} D_{\mathbf{v}_{i,1}} U_i^n, \quad c_{i,2}^n = \frac{h_{i,2}}{6\sqrt{5}} D_{\mathbf{v}_{i,2}} U_i^n, \quad \text{and} \quad c_{i,3}^n = \frac{h_{i,3}}{4\sqrt{15}} D_{\mathbf{v}_{i,3}} U_i^n.$$



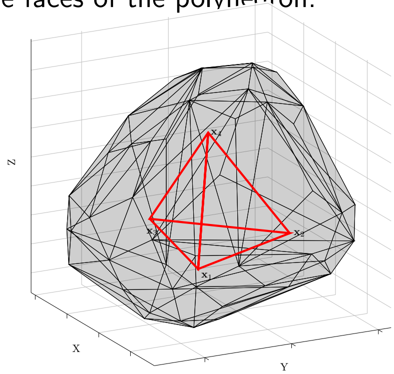
(a) Limiting direction $\mathbf{v}_{i,1}$. (b) Limiting direction $\mathbf{v}_{i,2}$. (c) Limiting direction $\mathbf{v}_{i,3}$.

Limiting directions on tetrahedra.

Moment limiters on unstructured meshes

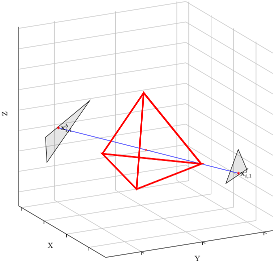
Reconstruction neighborhood in three dimensions

- Compute the convex hull of the centroids vertex neighbors and form a polyhedron.
- Find the intersection of the lines defined by $(\mathbf{x}_i, \mathbf{v}_{i,1})$, $(\mathbf{x}_i, \mathbf{v}_{i,2})$, and $(\mathbf{x}_i, \mathbf{v}_{i,3})$ and the faces of the polyhedron.

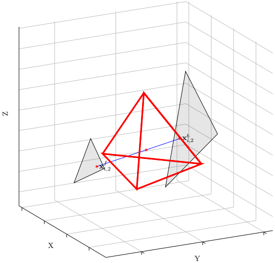


Interpolation in three dimensions.

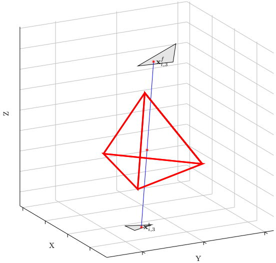
Using linear interpolation, reconstruct the numerical solution at the interpolation points using the solution averages of the elements that form the interpolation planes.



(a) Interpolation planes for $c_{i,1}$.



(b) Interpolation planes for $c_{i,2}$.



(c) Interpolation planes for $c_{i,3}$.

Interpolation in three dimensions.

Moment limiters on unstructured meshes of tetrahedra

Limiters that result from an analysis in three dimensions on tetrahedra are

$$c_{i,1} = \text{minmod}\left(\frac{U^f_{i,1} - \overline{U}_i}{\sqrt{10}}, c_{i,1}, \frac{\overline{U}^n_i - U^b_{i,1}}{\sqrt{10}}\right),$$

$$c_{i,2} = \text{minmod}\left(\frac{U^f_{i,2} - \overline{U}_i}{2\sqrt{5}}, c_{i,2}, \frac{\overline{U}^n_i - U^b_{i,2}}{2\sqrt{5}}\right),$$

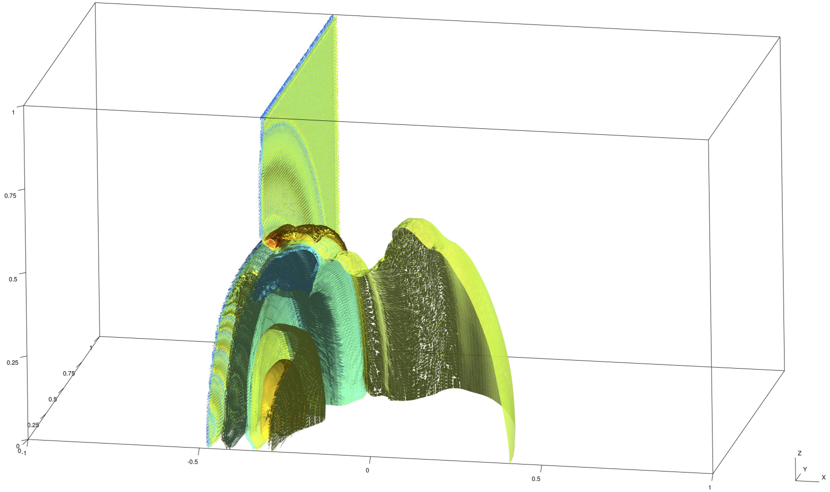
$$c_{i,3} = \text{minmod}\left(\frac{U^f_{i,3} - \overline{U}_i}{2\sqrt{15}}, c_{i,3}, \frac{\overline{U}^n_i - U^b_{i,2}}{2\sqrt{15}}\right).$$

We store for each element Ω_i

- Pointers to the 18 elements involved in the interpolation stencil.
- 12 interpolation weights.

Moment limiters on unstructured meshes of tetrahedra

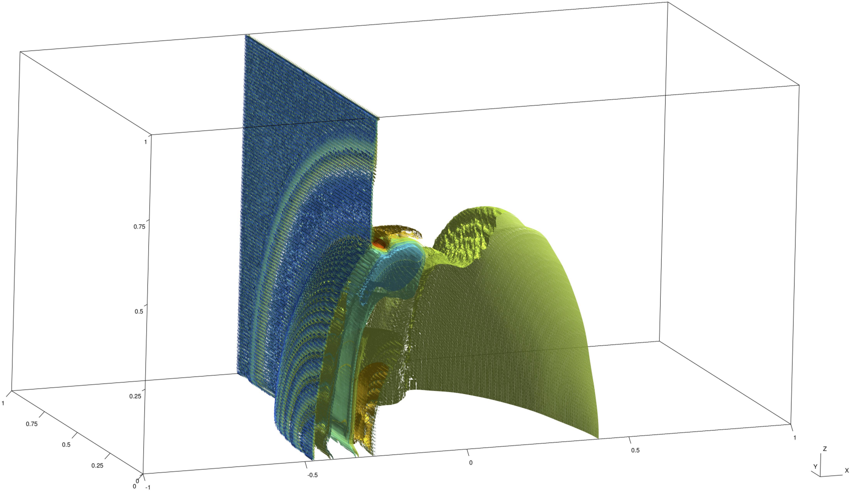
Mach 10 shock interacting with a bubble



Isosurfaces of density after shock-bubble interaction.

Moment limiters on unstructured meshes of tetrahedra

Mach 10 shock interacting with a bubble



Isosurfaces of density after shock-bubble interaction.

Conclusion

- Arbitrary high-order moment limiter on conforming and nonconforming meshes
- Limiter is applied directly to solution coefficients (moments)
- Hierarchical limiting solution coefficients (moments) starting with the highest, thus preventing overlimiting
- Simple and fixed reconstruction stencil
- Proven stability property for $p = 1$ case
- The limiter takes only about fifteen percent of the total computing time
- DG method equipped with the proposed limiter retains the $p + 1$ convergence rate for smooth solutions and eliminates spurious oscillations in the presence of discontinuities