# Stability analysis of the Eulerian-Lagrangian finite volume methods for nonlinear hyperbolic equations in one space dimension

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# Outline

Introduction

Eulerian-Lagrangian finite volume scheme

Stability analysis

Numerical experiments

Conclusion

# Outline

We consider the following nonlinear hyperbolic equation

$$\begin{cases} u_t + f(u)_x = 0. \\ u(x,0) = u_0 \end{cases}$$

The finite volume scheme on Eulerian mesh can be written as

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \frac{\Delta t}{\Delta x}\left(\hat{f}_{j-\frac{1}{2}} - \hat{f}_{j+\frac{1}{2}}\right).$$

Consider the Burger's equation, i.e. $f(u) = u^2/2$, with Lax-Friedrichs flux. To preserve the maximum-principle of the first-order scheme, we need $\Delta t \leq \frac{\Delta x}{\max|u_0|}$.

# Linear problems

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- Eulerian-Lagrangian method: Approximate the local characteristic speed by linear functions.
- Both methods are mostly used for linear problems.
- No previous works can handle nonlinear problems with shocks.

# Outline

We consider the following nonlinear hyperbolic equation

$$\begin{cases} u_t + f(u)_x = 0. \\ u(x,0) = u_0 \end{cases}$$

subject to periodic boundary condition and assume $b \le u_0 \le a$. We give a partition of the computational domain $\Omega = [0, 2\pi]$ as

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi,$$

and denote $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ as the cells with length $\Delta x$. Let $t^n$ be the $n-$th time level and denote $\Delta t$ as the time step size.

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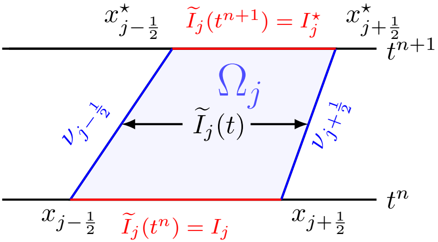


Figure: The space-time region.

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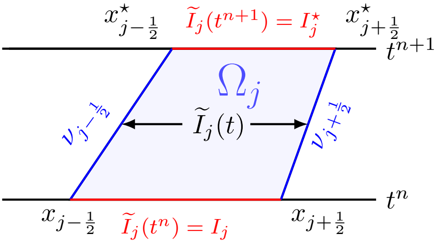


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Given the numerical approximations $u$ at $t^n$, we take

$$\nu = \frac{[f]}{[u]}.$$

# Space-time domain

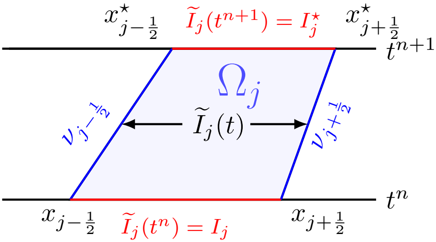


Figure: The space-time region.

If the characteristics intersect before $t = t^{n+1}$, then $I_j$ is defined as a **troubled cell** and the time that the intersection appears $t^\star_j$ satisfies

$$x_{j-\frac{1}{2}} + \nu_{j-\frac{1}{2}} t^\star = x_{j+\frac{1}{2}} + \nu_{j+\frac{1}{2}} t^\star.$$

Otherwise,

$$\Delta x^\star_j = \Delta x + \nu_{j+\frac{1}{2}} \Delta t - \nu_{j-\frac{1}{2}} \Delta t.$$

The semi-discrete scheme can be written as

$$\frac{d}{dt}\int_{\widetilde{I}_j(t)} u \, dx + F|_{\widetilde{x}_{j+\frac{1}{2}}(t)} - F|_{\widetilde{x}_{j-\frac{1}{2}}(t)} = 0.$$

where

$$F_{j\pm\frac{1}{2}}(u) \doteq f(u) - \nu_{j\pm\frac{1}{2}} u.$$

It is easy to verify that

$$[F]_{j+\frac{1}{2}} = [f]_{j+\frac{1}{2}} - \nu_{j+\frac{1}{2}}[u]_{j+\frac{1}{2}} = 0.$$

With Euler forward time discretization, we have

$$\Delta x_j^\star u^\star - \Delta x u + \Delta t \left( F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}} \right) = 0.$$

- For each troubled cell $I_i$, we construct the **influence region** of $I_i$ (5 or 6 cells) and merge them. If there is another troubled cell between $I_{i-1}$ and $I_{i+1}$, the influence region can be selected based on either one, and the selected cell is called an **Effective troubled cell (ETC)**.

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- Keep the original numerical fluxes at the interfaces of the influence region.
- Update the numerical approximations.
- After we obtain the numerical approximations on the next time level, we map the mesh to the original background uniform mesh by $L^2$ projection.

# Outline

# A basic lemma

We consider (1) $f(u) = \frac{u^2}{2}$, (2) first-order scheme, (3) the troubled cells are isolated.

## Lemma

*Suppose the characteristics do not intersect and* $\{\Omega_j\}_{j=1}^{N}$ *is the partition of the space-time domain* $\Omega \times [t^n, t^{n+1}]$, *then the first-order numerical approximation satisfies*

$$u^\star = u.$$

*Hence the scheme is total-variation-diminishing (TVD) and Maximum-principle-preserving (MPP).*

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- The modification keeps the physical bounds.
- The modification does not increase the total variation.
- In general we merge 5 cells. (exceptions may apply)

# Influence region

Suppose $I_j$ is an ETC, and the numerical approximations on cell $I_i$, $i = j-3, \cdots, j+3$ are $s_\ell$, $z_\ell$, $z_1$, $z_2$, $z_3$, $z_r$, $s_r$, respectively. Assume the initial condition is bounded by $a \geq u_0 \geq b$.

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- Let the updated numerical approximations be $r_1$, $r_2$, $r_3$.
- $r_1 + r_2 + r_3 = z_1 + z_2 + z_3$.
- $TV(s_\ell, z_\ell, z_1, z_2, z_3, z_r, s_r) \geq TV(s_\ell, z_\ell, r_1, r_2, r_3, z_r, s_r)$.

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- $b \geq r_1 \geq r_2 \geq r_3 \geq a$.

# Minimum total variation

We want to find $\tilde{z}_1$, $\tilde{z}_2$, $\tilde{z}_3$ yielding minimum total variation.

The admissible set is

$$G = \left\{ (z_1, z_2, z_3) : a \geq z_1 \geq z_2 \geq z_3 \geq b,\ z_1 \geq z_3 + \frac{2}{\lambda} \right\}, \quad \lambda = \frac{\Delta t}{\Delta x}.$$

## Theorem

*Let $z_\ell, z_1, z_2, z_3, z_r \in [b, a]$ be the numerical approximations within five adjacent cells from left to right, then we can define $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3$ in the admissible set $G$ such that $TV(z_\ell, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3, z_r) \leq TV(z_\ell, z_1, z_2, z_3, z_r)$. In addition, the chosen $\tilde{z}_1$, $\tilde{z}_2$, $\tilde{z}_3$ satisfy $\tilde{z}_1 \geq \frac{a+b}{2}$, $\tilde{z}_3 \leq \frac{a+b}{2}$ and $\tilde{z}_1 = \tilde{z}_3 + \frac{2}{\lambda}$.*

# A preliminary result

Then we can update $\tilde{z}_i$, $i = 1, 2, 3$ to obtain $r_i$, $i = 1, 2, 3$ such that the characteristics originated from the cell interfaces do not intersect within one time step. Moreover, the total variation does not increase.

This algorithm can yield a time step size $\Delta t < \frac{C\Delta x}{b-a}$, with $C = 3$, which theoretically guarantees the MPP and TVD properties. This time step larger than $\frac{\Delta x}{\max\{|a|,|b|\}}$

Consider the initial condition

$$u_0(x) = \begin{cases} 2, & x \leq 0, \\ -0.6, & 0 < x \leq \Delta x, \\ -2, & \text{otherwise}, \end{cases}$$

where $x \in [-\pi, \pi]$. We set final time $T = 3$ and $N = 100$. We choose $C = 3.4$, $3.6$ and $3.9$ ($CFL = 1.7$, $1.8$, and $1.95$) to compute the total variations at each time step.

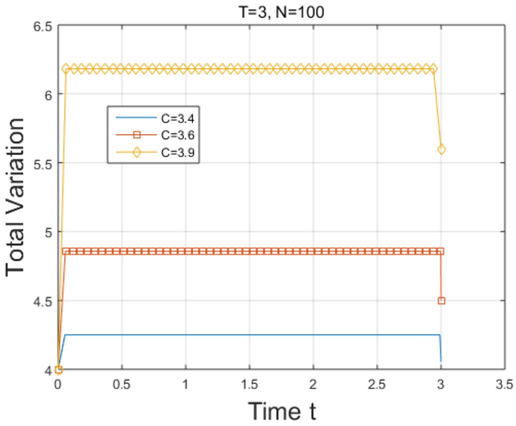


Figure: The numerical solution at $T = 3$, $N = 100$.

# Definition of the influence region

## Definition

Suppose $I_j$ is an ETC, and the numerical approximations on cell $I_i$, $i = j-3, \cdots, j+3$ are $s_\ell$, $z_\ell$, $z_1$, $z_2$, $z_3$, $z_r$, $s_r$, respectively. Assume the initial condition is bounded by $a \geq u_0 \geq b$, Then the influence region is defined as follows:

1. If $s_r + z_r < \frac{a+3b}{2}$, $A = z_1 + z_2 + z_3 \geq \frac{7a+5b}{4}$, then the influence region contains $I_i$, $i = j-2 \cdots, j+3$.
2. If $s_\ell + z_\ell > \frac{3a+b}{2}$, $A = z_1 + z_2 + z_3 \leq \frac{5a+7b}{4}$, then the influence region contains $I_i$, $i = j-3 \cdots, j+2$.
3. In all other cases, the influence region contains $I_i$, $i = j-2, \cdots, j+2$.

## Theorem

*Suppose the numerical approximations are within the interval* $[b, a]$, *and* $I_j$ *is an ETC. The numerical approximations on* $I_{j-3}, \cdots, I_{j+3}$ *are* $s_\ell$, $z_\ell$, $z_1$, $z_2$, $z_3$, $z_r$, $s_r$, *respectively, with* $z_1 \geq z_2 \geq z_3$ *and* $z_1 \geq z_3 + \frac{2}{\lambda}$. *The influence region is given in the previous slides. If we take*

$$\lambda = \frac{c}{a-b}, \quad C = 4$$

*then we can find* $r_\ell$, $r_1$, $r_2$, $r_3$, $r_r \in [b, a]$ *defined in cells* $I_{j-2}, \cdots, I_{j+2}$, *respectively, without changing the numerical approximations on the boundary cells in the influence region, such that the proposed new numerical approximations satisfy*

$$\sum_{j=\ell,1,2,3,r} z_j = \sum_{j=\ell,1,2,3,r} r_j,$$

$$TV(s_\ell, z_\ell, z_1, z_2, z_3, z_r, s_r) \geq TV(s_\ell, r_\ell, r_1, r_2, r_3, r_r, s_r).$$

*Moreover, the characteristics originated from* $x_{i+\frac{1}{2}}$, $i = j-3, \cdots, j+2$ *do not intersect within one time step and the characteristics speeds at the boundaries of the influence region keep the same.*

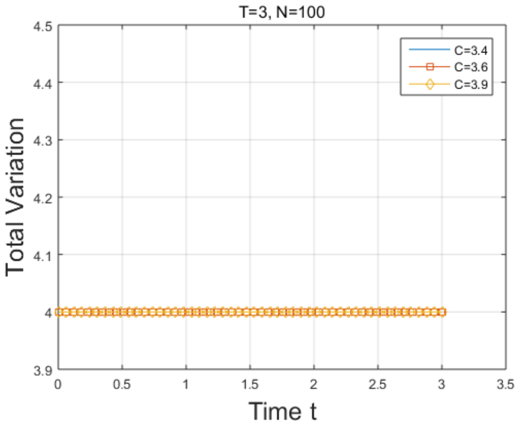


Figure: The numerical solution at $T = 3$ and $N = 100$.

# High-order extension

High-order spatial discretization can be obtained by using the minmod limiter to the reconstructed function.

It is not easy to apply the SSP RK methods since the space-time domain is partitioned based on the numerical approximation at time level $n$. Therefore, the partition may not work for the second stage in the SSP RK methods.

# Shock-shock interaction

We consider the cells on the right to the influence region. Suppose the 5 cells, with the ETC as the center, in the influence region and the 5 cells on the right are given as $r_\ell, r_1, r_2, r_3, r_r, s_1, s_2, s_3, s_4, s_5$ from left to right, where $r's$ are the updated numerical approximations in the influence region. Then $r_r$ is not a troubled cell. Therefore, the troubled cells can only be $s_1$, $s_2$ or $s_3$. The procedure is given as follows:

1. If $s_2$ is a troubled cell, then $s_2$ is regarded as an ETC. The cells to be merged also include $s_i$, $i = 1, 2, 3, 4$, and probably $s_5$ depending on the influence region of $s_2$.

2. If $s_2$ is not a troubled cell, but $s_3$ is an ETC. We will show that the influence region of $s_3$ does not contain $r_r$, then we also merge cells $s_i$, $i = 1, 2, 3, 4, 5$.

3. If $s_2$ is not a troubled cell, but $s_1$ is a troubled cell, then $s_1$ is regarded as an ETC. Then $s_3$ is not a troubled cell. The cells to be merged also include $s_i$, $i = 1, 2, 3$, and probably $s_4$ depending on the influence region of $s_1$.

# Outline

Introduction

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Numerical experiments

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We first test the method with continuous initial value $u_0(x) = \sin(x)$, where $x \in [0, 2\pi]$. We apply a periodic boundary condition. We test the methods when the solution evolves up $T = 1.3$ (after shock) where the shock is located at $x = \pi$.

We take

$$\Delta t = \frac{C}{\max\{u_0\} - \min\{u_0\}} \cdot \Delta x, \quad 0 < C < 4,$$

where $u_0$ is the initial conditions.

The $CFL$ number is

$$CFL = \frac{\Delta t}{\Delta x} \cdot \max|f'(u)|.$$

This is example, C=2*CFL

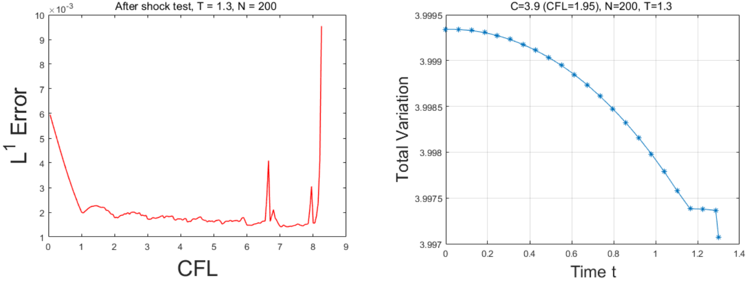



Figure: CFL vs. error plot and total variation over time with $T = 1.3$, $N = 200$, $C = 3.9$.

# Shock wave

We consider a Riemann problem with initial condition

$$u_0(x) = \begin{cases} 2, & x \leq 0, \\ -1, & \text{otherwise}, \end{cases}$$

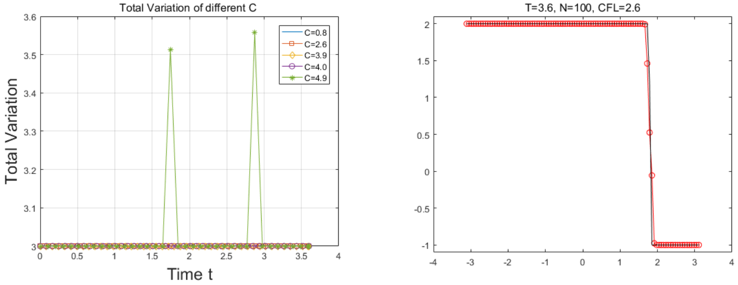



Figure: Total variation and the numerical solutions (C=3.9).

# Rarefaction wave

We consider a Riemann problem with initial condition

$$u_0(x) = \begin{cases} -1, & x \leq 0, \\ 1, & \text{otherwise}, \end{cases}$$

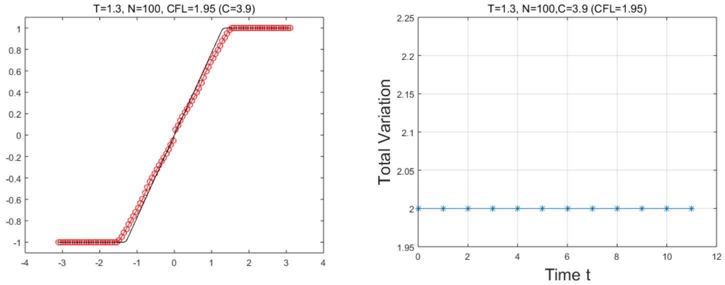
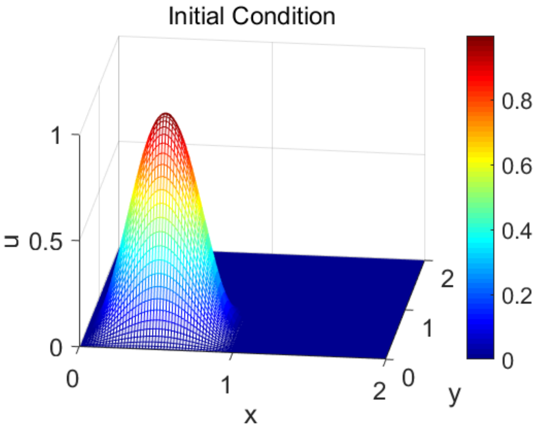



Figure: The numerical solution at $T = 1.3$ and total variation over time. $N = 100$, and $CFL = 1.95$ $(C = 3.9)$.

# Two dimensional problems

We take the initial condition as



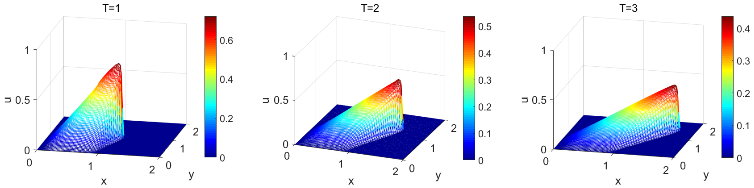



Figure: $N_x = N_y = 100$, $C = 3.8$.

Consider the 2D Burgers' equation with the Riemann Initial condition:

$$u_0(x) = \begin{cases} 1, & (x,y) \in (0,0.5] \times (0,0.5], \\ 2, & (x,y) \in (-0.5,0] \times [0,0.5), \\ 3, & (x,y) \in [-0.5,0) \times [-0.5,0), \\ 4, & (x,y) \in (0,0.5) \times (-0.5,0). \end{cases}$$

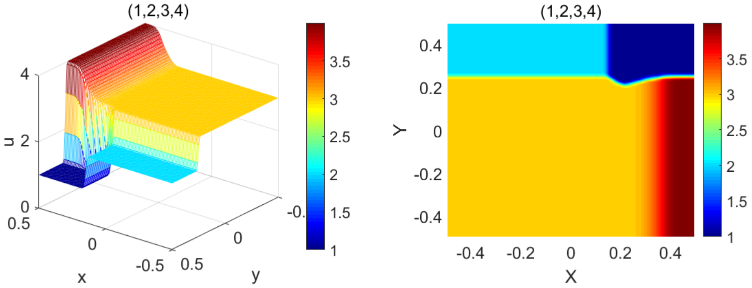



Figure: $T = 0.1,\ N_x = N_y = 100,\ CFL = 8.6.$

# Outline

In this talk, we designed a novel Eulerian-Lagrangian finite volume method. With special merging strategies, the numerical algorithm is theoretically proved to be TVD and MPP under the condition that $\Delta t \leq \frac{4\Delta x}{a-b}$, where $a$ and $b$ are the maximum and minimum values of the initial condition.