Structure Preserving Numerical Methods for Hyperbolic Systems of Conservation and Balance Laws

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Systems of Balance Laws

$$U_t + f(U)_x + g(U)_y = S(U)$$

Examples:

- Gas dynamics with pipe-wall friction
- Euler equations with gravity/friction
- shallow water equations with Coriolis forces

Applications:

- astrophysical and atmospheric phenomena in many fields including supernova explosions
- (solar) climate modeling and weather forecasting

$$U_t + f(U)_x + g(U)_y = \frac{1}{\varepsilon} S(U)$$

Examples:

- low Mach number compressible flows
- low Froude number shallow water flows
- diffusive relaxation in kinetic models

Applications:

- various two-phase flows such as bubbles in water
- unmostly incompressible flows with regions of high compressibility such as underwater explosions
- atmospheric flows

Systems of Balance Laws

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \boldsymbol{S}(\boldsymbol{U}) \quad \text{or} \quad \boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \frac{1}{\varepsilon}\boldsymbol{S}(\boldsymbol{U})$$

- Challenges: certain structural properties of these hyperbolic problems (conservation or balance law, equilibrium state, positivity, assymptotic regimes, etc.) are essential in many applications[1];
- Goal: to design numerical methods that are not only consistent with the given PDEs, but
  - preserve the structural properties at the discrete level – **well-balanced numerical methods**
  - remain accurate and robust in certain asymptotic regimes of physical interest – **asymptotic preserving numerical methods**

[1] LeFloch, 2014.

Well-Balanced (WB) Methods

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \boldsymbol{S}(\boldsymbol{U})$$

- In many physical applications, solutions of the system are small perturbations of the steady states;
- These perturbations may be smaller than the size of the truncation error on a coarse grid;
- To overcome this difficulty, one can use very fine grid, but in many physically relevant situations, this may be unaffordable;

Goal:

- to design a well-balanced numerical method, that is, the method which is capable of exactly preserving some steady state solutions;
- perturbations of these solutions will be resolved on a coarse grid in a non-oscillatory way.

Asymptotic Preserving (AP) Methods

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \frac{1}{\varepsilon}\boldsymbol{S}(\boldsymbol{U})$$

- Solutions of many hyperbolic systems reveal a multiscale character and thus their numerical resolution presence some major difficulties;
- Such problems are typically characterized by the occurance of a small parameter by $0 < \varepsilon \ll 1$;
- The solutions show a nonuniform behavior as $\varepsilon \to 0$;
- the type of the limiting solution is different in nature from that of the solutions for finite values of $\varepsilon > 0$.

Goal:

- asymptotic passage from one model to another should be preserved at the discrete level;
- for a fixed mesh size and time step, AP method should automatically transform into a stable discretization of the limiting model as $\varepsilon \to 0$.

Well-Balancing via Flux Globalization

Flux Globalization Approach[2]

$$\underbrace{\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x = \boldsymbol{S}(\boldsymbol{U})}_{\text{balance law}} \quad \Longrightarrow \quad \underbrace{\boxed{\boldsymbol{U}_t + \boldsymbol{K}(\boldsymbol{U})_x = \boldsymbol{0}}}_{\substack{\text{conservation law} \\ \text{with global flux}}}$$

where

$$\boldsymbol{K}(\boldsymbol{U}(x,t)) := \boldsymbol{f}(\boldsymbol{U}(x,t)) + \boldsymbol{R}(\boldsymbol{U}(x,t))$$

$$\boldsymbol{R}(\boldsymbol{U}(x,t)) := -\int^{x} \boldsymbol{S}(\boldsymbol{U}(\xi,t))\,\mathrm{d}\xi$$

The steady-state is then the solution $\boldsymbol{W}$ such that

$$\boldsymbol{f}(\boldsymbol{W})_x = \boldsymbol{S}(\boldsymbol{W}) \quad \Longrightarrow \quad \boxed{\boldsymbol{K}(\boldsymbol{W}) = \textbf{Const}}$$

- $\boldsymbol{U}$ is the conservative variable
- $\boldsymbol{W}$ is the equlibrium variable

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[2] Chertock, Herty, and Özcan, 2018.

Finite-Volume Method

$$\boldsymbol{U}_t + \boldsymbol{K}(\boldsymbol{U})_x = \boldsymbol{0}$$

- $\overline{\boldsymbol{U}}_j(t) \approx \dfrac{1}{\Delta x}\displaystyle\int_{C_j} \boldsymbol{U}(x,t)\,dx$: cell averages over $C_j := [x_{j-\frac12}, x_{j+\frac12})$

- $\boldsymbol{U}^-_{j+\frac12}(t)$ and $\boldsymbol{U}^+_{j+\frac12}(t)$: reconstructed point values at $x_{j+\frac12}$

$$\widetilde{\boldsymbol{U}}_j(x,t) = \overline{\boldsymbol{U}}_j(t) + (\boldsymbol{U}_x)_j(x - x_j), \quad x \in C_j$$

$$\boldsymbol{U}^+_{j+\frac12} := \overline{\boldsymbol{U}}_j + \frac{\Delta x}{2}(\boldsymbol{U}_x)_j, \quad \boldsymbol{U}^-_{j+\frac12} := \overline{\boldsymbol{U}}_j - \frac{\Delta x}{2}(\boldsymbol{U}_x)_j$$

- Semi-discrete FV method:

$$\frac{d}{dt}\overline{\boldsymbol{U}}_j(t) = -\frac{\boldsymbol{\mathcal{F}}_{j+\frac12}\left(\boldsymbol{U}^-_{j+\frac12}, \boldsymbol{U}^+_{j+\frac12}\right) - \boldsymbol{\mathcal{F}}_{j-\frac12}\left(\boldsymbol{U}^-_{j-\frac12}, \boldsymbol{U}^+_{j-\frac12}\right)}{\Delta x}$$

$\boldsymbol{\mathcal{F}}_{j\pm\frac12} \approx \boldsymbol{K}(\boldsymbol{U}_{j\pm\frac12}(t))$: numerical fluxes

Flux Globalization Approach

$$\boldsymbol{U}_t + \boldsymbol{K}(\boldsymbol{U})_x = \boldsymbol{0}$$

$$\boxed{\{\overline{\boldsymbol{U}}_j(t)\} \to \left\{\boldsymbol{U}^{\pm}_{j+\frac{1}{2}}(t)\right\} \to \left\{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_j(t+\Delta t)\}}$$

Semi-discrete FV method:

$$\frac{d}{dt}\overline{\boldsymbol{U}}_j(t) = -\frac{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}\left(\boldsymbol{U}^-_{j+\frac{1}{2}}, \boldsymbol{U}^+_{j+\frac{1}{2}}\right) - \boldsymbol{\mathcal{F}}_{j-\frac{1}{2}}\left(\boldsymbol{U}^-_{j-\frac{1}{2}}, \boldsymbol{U}^+_{j-\frac{1}{2}}\right)}{\Delta x}$$

Key Point: *The method is not necessary well-balanced in the sense that it will preserve steady states exactly, i.e.,* $\boldsymbol{K} = \textbf{Const}$

Flux Globalization Approach

$$\boldsymbol{U}_t + \boldsymbol{K}(\boldsymbol{U})_x = \boldsymbol{0}$$

Semi-discrete FV method:

$$\frac{d}{dt}\overline{\boldsymbol{U}}_j(t) = -\frac{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}\left(\boldsymbol{U}^-_{j+\frac{1}{2}},\boldsymbol{U}^+_{j+\frac{1}{2}}\right) - \boldsymbol{\mathcal{F}}_{j-\frac{1}{2}}\left(\boldsymbol{U}^-_{j-\frac{1}{2}},\boldsymbol{U}^+_{j-\frac{1}{2}}\right)}{\Delta x}$$

Key Idea: ***Reconstruct equilibrium variables $\boldsymbol{W}$, evolve conservative variables $\boldsymbol{U}$!***

$$\{\overline{\boldsymbol{U}}_j(t)\} \to \left\{\boldsymbol{U}^\pm_{j+\frac{1}{2}}(t)\right\} \to \left\{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_j(t+\Delta t)\}$$

$$\{\overline{\boldsymbol{U}}_j(t)\} \to \{\boldsymbol{W}_j(t)\} \to \left\{\boldsymbol{W}^\pm_{j+\frac{1}{2}}(t)\right\} \to \left\{\boldsymbol{U}^\pm_{j+\frac{1}{2}}(t)\right\} \to \left\{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_j(t+\Delta t)\}$$

Example – Gas dynamics with pipe-wall friction

$$
\begin{cases}
\rho_t + q_x = 0, \\
q_t + \left(c^2\rho + \dfrac{q^2}{\rho}\right)_x = -\mu \dfrac{q}{\rho}|q|,
\end{cases}
$$

- $\rho(x,t)$ is the density of the fluid
- $u(x,t)$ is the velocity of the fluid
- $q(x,t)$ is the momentum
- $\mu > 0$ is the friction coefficient (divided by the pipe cross section)
- $c > 0$ is the speed of sound

Gas dynamics with pipe-wall friction

Incorporate the source term into the global flux and solve the resulting system of conservation

$$\begin{cases} \rho_t + q_x = 0 \\ q_t + \left(hu^2 + \dfrac{g}{2}\rho^2\right)_x = -\mu\dfrac{q}{\rho}|q| \end{cases} \quad \Leftrightarrow \quad \begin{cases} \rho_t + q_x = 0 \\ q_t + K_x = 0 \end{cases}$$

Equilibrium variables:

$$q, \quad K := \frac{q^2}{h} + \frac{g}{2}\rho^2 + R, \quad R := \int^x \mu\frac{q}{\rho}|q|\mathrm{d}\xi$$

Steady states: $q \equiv \mathrm{Const}, \quad K \equiv \mathrm{Const}$

Flux Globalization Approach

$$\begin{cases}\rho_t + q_x = 0\\ q_t + K_x = 0\end{cases} \quad \Rightarrow \quad \boldsymbol{U} = (\rho, q)^\top, \quad \boldsymbol{W} = (q, K)^\top$$

$$K := \frac{q^2}{\rho} + \frac{g}{2}\rho^2 + R, \quad R := \int^x \mu \frac{q}{\rho}|q| \mathrm{d}\xi$$

Assume that at time $t = t^n$ we have for all $j = j_\ell, \ldots, j_r$:

$$\overline{\boldsymbol{U}}_j = (\overline{\rho}_j, \overline{q}_j)^\top$$

Algorithm:

$$\{\overline{\rho}_j, \overline{q}_j\}^n \xrightarrow{(1)} \{\overline{q}_j, K_j\}^n \xrightarrow{(2)} \left\{q^\pm_{j+\frac{1}{2}}, K^\pm_{j+\frac{1}{2}}\right\}^n$$
$$\xrightarrow{(3)} \left\{\rho^\pm_{j+\frac{1}{2}}, q^\pm_{j+\frac{1}{2}}\right\}^n \xrightarrow{(4)} \left\{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}\right\}^n \xrightarrow{(5)} \{\overline{\rho}_j, \overline{q}_j\}^{n+1}$$

Computaion of Equilibrium Variables

$$\left\{\bar{\rho}_j, \bar{q}_j\right\}^n \xrightarrow{(1)} \left\{\bar{q}_j, K_j\right\}^n, \quad j = j_\ell, \ldots, j_r$$

$$K_j := \frac{\bar{q}_j^2}{\bar{\rho}_j} + \frac{g}{2}\bar{\rho}_j^2 + R_j, \quad R(x) := g \int_{x_{j_\ell - \frac{1}{2}}}^{x} \mu \frac{q}{\rho} |q| \mathrm{d}\xi$$

Recursive formula for R at the cell interfaces starting with $x_{j_{\ell-\frac{1}{2}}}$:

$$R_{j_\ell - \frac{1}{2}} := 0$$

$$R_{j+\frac{1}{2}} = R_{j-\frac{1}{2}} + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mu \frac{q}{\rho} |q| \mathrm{d}x = R_{j-\frac{1}{2}} + \Delta x \, \mu \frac{\bar{q}_j}{\bar{\rho}_j} |\bar{q}_j|, \quad j = j_\ell, \ldots, j_r$$

$$R_j = \frac{1}{2}\left(R_{j-\frac{1}{2}} + R_{j+\frac{1}{2}}\right)$$

Numerical Tests

- Steady state initial data:

$$q(x,0) = q^* = 0.15 \quad \text{and} \quad K(x,0) = K^* = 0.4,$$

in a single pipe $x \in [0,1]$

- Perturbed initial data:

$$q(x,0) = q^* + \eta e^{-100(x-0.5)^2}, \quad K(x,0) = K^* = 0.4, \quad \eta > 0$$

in a single pipe $x \in [0,1]$

We compare the WB and NWB methods ...

Numerical Test – Steady state initial data

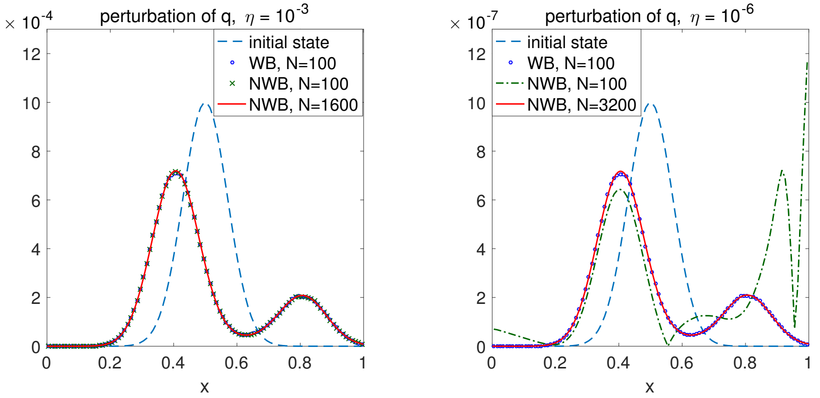
WB:

N	q	K
100	1.94E-18	7.77E-18
200	9.71E-19	9.71E-18
400	1.66E-18	9.57E-18
800	2.18E-18	1.18E-17

NWB:

N	q	rate	K	rate
100	1.29E-06	-	8.81E-07	-
200	3.30E-07	1.9668	2.25E-07	1.9692
400	8.34E-08	1.9843	5.69E-08	1.9834
800	2.09E-08	1.9965	1.43E-08	1.9924

Numerical Test – Perturbed initial data



Euler Equations with Gravity

$$
\begin{cases}
\rho_t + (\rho u)_x + (\rho v)_y = 0 \\
(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = -\rho\phi_x \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = -\rho\phi_y \\
E_t + (u(E+p))_x + (v(E+p))_y = -\rho(u\phi_x + v\phi_y)
\end{cases}
$$

- ρ is the density
- u, v are the x- and y-velocities
- E is the total energy
- p is the pressure; $E = \dfrac{p}{\gamma - 1} + \dfrac{\rho}{2}(u^2 + v^2)$
- ϕ is the gravitational potential

Euler Equations with Gravity[3]

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = -\rho \phi_x \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = -\rho \phi_y \\ E_t + (u(E+p))_x + (v(E+p))_y = -\rho(u\phi_x + v\phi_y) \end{cases}$$

Multiply the first (density) equation by ϕ and add to the last (energy) equation to obtain ...

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = -\rho \phi_x \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = -\rho \phi_y \\ (E + \rho\phi)_t + (u(E+\rho\phi+p))_x + (v(E+\rho\phi+p))_y = 0 \end{cases}$$

[3] Chertock, Cui, Kurganov, Özcan, and Tadmor, 2018.

Steady States

$$\begin{cases} \cancel{\rho_t} + (\rho u)_x + (\rho v)_y = 0 \\ \cancel{(\rho u)_t} + (\rho u^2 + p)_x + (\rho u v)_y = -\rho\phi_x \\ \cancel{(\rho v)_t} + (\rho u v)_x + (\rho v^2 + p)_y = -\rho\phi_y \\ \cancel{(E + \rho\phi)_t} + (u(E + \rho\phi + p))_x + (v(E + \rho\phi + p))_y = 0 \end{cases}$$

Plays an important role in modeling model astrophysical and atmospheric phenomena in many fields including supernova explosions, (solar) climate modeling and weather forecasting

Steady state solution:

$$u \equiv 0,\ v \equiv 0,\ K_x = p_x + \rho\phi_x \equiv 0,\ L_y = p_y + \rho\phi_y \equiv 0$$

$$K := p + Q, \quad Q(x,y,t) := \int^x \rho(\xi,y,t)\phi_x(\xi,y)\,d\xi$$

$$L := p + R, \quad R(x,y,t) := \int^y \rho(x,\eta,t)\phi_y(x,\eta)\,d\eta$$

2-D Well-Balanced Scheme

- Incorporate the source term into the flux:

$$K := p + Q, \quad Q(x,y,t) := \int^{y} \rho(\xi,y,t)\phi_x(\xi,y), d\xi$$

$$L := p + R, \quad R(x,y,t) := \int^{y} \rho(x,\eta,t)\phi_y(x,\eta), d\eta$$

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E + \rho\phi \end{pmatrix}_t + \boxed{\begin{pmatrix} \rho u \\ \rho u^2 + K \\ \rho u v \\ u(E + \rho\phi + p) \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + L \\ v(E + \rho\phi + p) \end{pmatrix}_y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}$$

- Define
  conservative variables: $\boldsymbol{U} := (\rho, \rho u, \rho v, E)^T$
  equilibrium variables: $\boldsymbol{W} := (\rho, K, L, E + \rho\phi)^T$
- Solve by the well-balanced scheme ...

Example — 2-D Isothermal Equilibrium Solution

- The ideal gas with $\gamma = 1.4$; domain $[0,1] \times [0,1]$
- The gravitational force is $\phi_y = \mathrm{g} = 1$
- The steady-state initial conditions are[4]

$$\rho(x,y,0) = 1.21e^{-1.21y}, \quad p(x,y,0) = e^{-1.21y},$$
$$u(x,y,0) \equiv v(x,y,0) \equiv 0$$

- Solid wall boundary conditions

$N \times N$	ρ	ρu	ρv	E
50 × 50	1.70E-016	0.00E+00	2.43E-016	5.97E-016
100 × 100	5.88E-017	0.00E+00	3.42E-016	5.31E-016
200 × 200	1.60E-016	0.00E+00	2.85E-016	5.33E-016

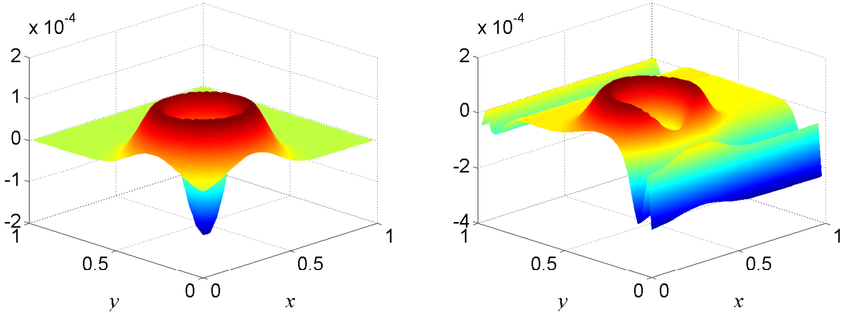
$N \times N$	ρ	ρu	ρv	E
50 × 50	1.05E-03	0.00E+00	5.72E-05	9.61E-05
100 × 100	4.02E-04	0.00E+00	2.07E-05	4.10E-05
200 × 200	1.63E-04	0.00E+00	7.11E-06	1.57E-05

[4] Xing and Shu, 2013.

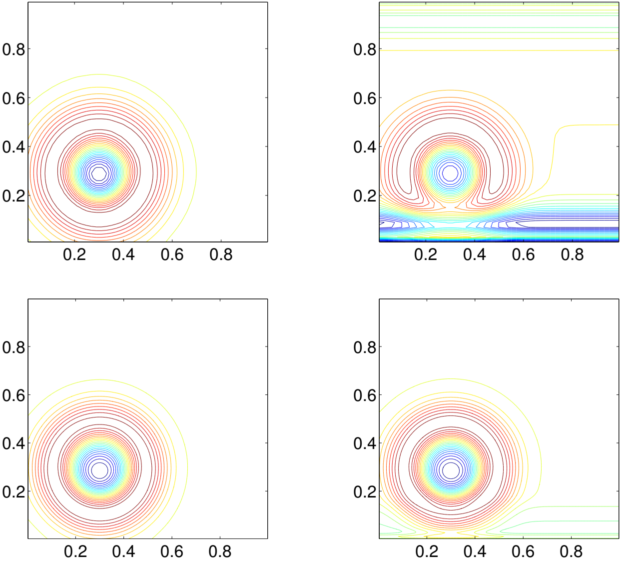
Perturbation

A small initial pressure perturbation:

$$p(x,y,0) = e^{-1.21y} + \eta e^{-121\left((x-0.3)^2+(y-0.3)^2\right)}, \quad \eta = 10^{-3}$$



50×50



WB : $50 \times 50,\ 200 \times 200$ NWB : $50 \times 50,\ 200 \times 200$

Shallow Water System with Coriolis Force[5]

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0 \\ (hu)_t + \left(hu^2 + \dfrac{g}{2}h^2\right)_x + (huv)_y = -ghB_x + fhv \\ (hv)_t + (huv)_x + \left(hv^2 + \dfrac{g}{2}h^2\right)_x = -ghB_y - fhu \end{cases}$$

- h: water height
- u, v: fluid velocity
- g: gravitational constant
- $B \equiv 0$ – bottom topography
- f – Coriolis parameter

[5] Chertock, Dudzinski, Kurganov, and Lukáčová-Medviďová, 2018.

Steady States

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0 \\ (hu)_t + \left(hu^2 + \frac{g}{2}h^2\right)_x + (huv)_y = -ghB_x + fhv \\ (hv)_t + (huv)_x + \left(hv^2 + \frac{g}{2}h^2\right)_y = -ghB_y - fhu \end{cases}$$

- "Lake at rest": $u \equiv 0,\ v \equiv 0,\ h + B \equiv \text{Const}$
- Geostrophic equlibria ("jets in the rotational frame") are both stationary and constant along the streamlines:

$$u \equiv 0,\ v_y \equiv 0,\ h_y \equiv 0,\ B_y \equiv 0,\ K \equiv \text{Const}$$

$$v \equiv 0,\ u_x \equiv 0,\ h_x \equiv 0,\ B_x \equiv 0,\ L \equiv \text{Const}$$

Here,

$$K := g(h + B - \frac{f}{g}v) \quad \text{and} \quad L := g(h + B + \frac{f}{g}u)$$

are the potential energies defined through the primitives of the Coriolis force

Example — 2-D Stationary Vortex

$$h(r,0) = 1+\varepsilon^2 \begin{cases} \frac{5}{2}(1+5\varepsilon^2)r^2 \\ \frac{1}{10}(1+5\varepsilon^2) + 2r - \frac{1}{2} - \frac{5}{2}r^2 + \varepsilon^2(4\ln(5r) + \frac{7}{2} - 20r + \frac{25}{2}r^2) \\ \frac{1}{5}(1 - 10\varepsilon + 4\varepsilon^2 \ln 2), \end{cases}$$

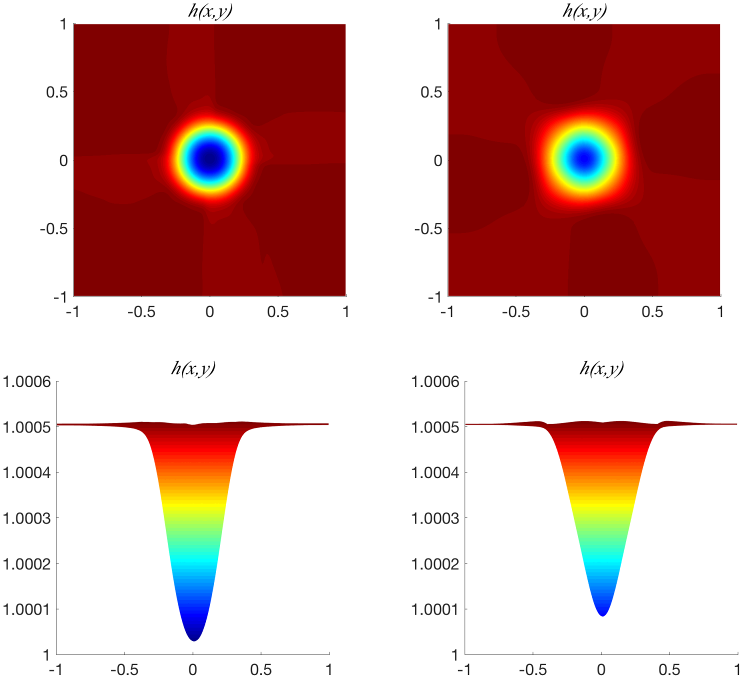
$$u(x,y,0) = -\varepsilon y \Upsilon(r), \quad v(x,y,0) = \varepsilon x \Upsilon(r), \quad \Upsilon(r) := \begin{cases} 5, & r < \frac{1}{5} \\ \frac{2}{r} - 5, & \frac{1}{5} \le r < \frac{2}{5} \\ 0, & r \ge \frac{2}{5}, \end{cases}$$

Domain: $[-1,1] \times [-1,1], \quad r := \sqrt{x^2+y^2}$

Boundary conditions: a zero-order extrapolation in both x- and y-directions[6]

Parameters: $B \equiv 0, \quad f = 1/\varepsilon$ and $g = 1/\varepsilon^2$ with $\varepsilon = 0.05$

[6] Audusse, Klein, Nguyen, and Vater, 2011.


h(x,y)
h(x,y)
h(x,y)
h(x,y)
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
1.0006
1.0005
1.0004
1.0003
1.0002
1.0001
1

Asymptotic Preserving Methods

Shallow Water System with Coriolis Force

$$
\begin{cases}
h_t + (hu)_x + (hv)_y = 0 \\
(hu)_t + \left(hu^2 + \dfrac{g}{2}h^2\right)_x + (huv)_y = -ghB_x + fhv \\
(hv)_t + (huv)_x + \left(hv^2 + \dfrac{g}{2}h^2\right)_x = -ghB_y - fhu
\end{cases}
$$

- h: water height
- u, v: fluid velocity
- g: gravitational constant
- $B \equiv 0$ – bottom topography
- f – Coriolis parameter

Dimensional Analysis

Introduce

$$\widehat{x} := \frac{x}{\ell_0}, \quad \widehat{y} := \frac{y}{\ell_0}, \quad \widehat{h} := \frac{h}{h_0}, \quad \widehat{u} := \frac{u}{w_0}, \quad \widehat{v} := \frac{v}{w_0}.$$

Substituting them into the SWE and dropping the hats in the notations, we obtain the dimensionless form:

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0, \\ (hu)_t + \left(hu^2 + \dfrac{1}{\varepsilon^2}\dfrac{h^2}{2}\right)_x + (huv)_y = \dfrac{1}{\varepsilon} hv, \\ (hv)_t + (huv)_x + \left(hv^2 + \dfrac{1}{\varepsilon^2}\dfrac{h^2}{2}\right)_y = -\dfrac{1}{\varepsilon} hu, \end{cases}$$

in which

$$\mathrm{Fr} := \frac{w_0}{\sqrt{gh_0}} = \varepsilon$$

is the reference Froude number

Numerical Challenges

Eigenvalues of the flux Jacobian:

$$\left\{u \pm \frac{1}{\varepsilon}\sqrt{h},\, u\right\} \quad \text{and} \quad \left\{v \pm \frac{1}{\varepsilon}\sqrt{h},\, v\right\}$$

This leads to the CFL condition

$$\Delta t_{\text{expl}} \leq \nu \cdot \min\left(\frac{\Delta x}{\max\limits_{u,h}\left\{|u| + \frac{1}{\varepsilon}\sqrt{h}\right\}}, \frac{\Delta y}{\max\limits_{v,h}\left\{|v| + \frac{1}{\varepsilon}\sqrt{h}\right\}}\right) = \mathcal{O}(\varepsilon \Delta_{\min}).$$

where $\Delta_{\min} := \min(\Delta x, \Delta y)$

- $0 < \nu \leq 1$ is the CFL number
- Numerical diffusion: $\mathcal{O}(\lambda_{max}\Delta x) = \mathcal{O}(\varepsilon^{-1}\Delta x)$.
- We must choose $\Delta x \approx \varepsilon$ to control numerical diffusion and the stability condition becomes

$$\Delta t = \mathcal{O}(\varepsilon^2)$$

Low Froude Number Flows

Low Froude number regime $(0 < \varepsilon \ll 1) \Longrightarrow$ very large propagation speeds

Explicit methods:

- very restrictive time and space dicretization steps, typically proportional to ε due to the CFL condition;
- too computationally expensive and typically impractical.

Implicit schemes:

- uniformly stable for $0 < \varepsilon < 1$;
- may be inconsistent with the limit problem;
- may provide a wrong solution in the zero Froude number limit.

Goal: to design robust numerical algorithms, whose accuracy and efficiency is independent of ε

Asymptotic-Preserving (AP) Methods[7]

Idea:

- asymptotic passage from one model to another should be preserved at the discrete level;
- for fixed mesh parameters δ, AP method should automatically transform into a stable discretization of the limiting model as $\varepsilon \to 0$.

$$
\begin{array}{ccc}
\mathcal{P}_\varepsilon^\delta & \xrightarrow{\ \delta \to 0\ } & \mathcal{P}_\varepsilon \\
\Big\downarrow {\scriptstyle \varepsilon \to 0} & & \Big\downarrow {\scriptstyle \varepsilon \to 0} \\
\mathcal{P}^\delta & \xrightarrow{\ \delta \to 0\ } & \mathcal{P}
\end{array}
$$

[7] Golse, Jin, and Levermore, 1999; Jin, 1999; Klar, 1999.

AP Sheme - Error Estimates Argument[8]

Assume that $\mathcal{P}_\varepsilon^\delta$ is an r-th order approximation to $\mathcal{P}_\varepsilon$ for fixed ε:

- A "classical" numerical scheme

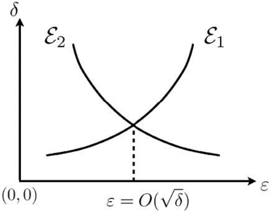
$$\mathcal{E}_1 = \|\mathcal{P}_\varepsilon^\delta - \mathcal{P}_\varepsilon\| = \mathcal{O}(\delta^r/\varepsilon)$$

- An AP scheme

$$\mathcal{E}_2 = \|\mathcal{P}_\varepsilon^\delta - \mathcal{P}_\varepsilon\| \leq \underbrace{\|\mathcal{P}_\varepsilon^\delta - \mathcal{P}^\delta\|}_{\mathcal{O}(\varepsilon)} + \underbrace{\|\mathcal{P}^\delta - \mathcal{P}\|}_{\mathcal{O}(\delta^r)} + \underbrace{\|\mathcal{P} - \mathcal{P}_\varepsilon\|}_{\mathcal{O}(\varepsilon)} = \mathcal{O}(\delta^r + \varepsilon)$$

$$\|\mathcal{P}_\varepsilon^\delta - \mathcal{P}_\varepsilon\| = \min(\mathcal{E}_1, \mathcal{E}_2) = \mathcal{O}(\delta^{r/2})$$

uniformly in ε



[8] Jin, 2012.

Structure Preserving AP Methodology

Steps:

- Formulate the passage $\mathcal{P}_\varepsilon \to \mathcal{P}$, which leads to a change in the type, nature or simply expression of the equations which determine some of the unknowns and their relations.
- Discretize $\mathcal{P}_\varepsilon$ into a scheme $\mathcal{P}_\varepsilon^\delta$ in such a way that the various manipulations, which led from $\mathcal{P}_\varepsilon$ to $\mathcal{P}$ in the continuous case can be performed at the discrete level

Outcome:

- The scheme $\mathcal{P}_\varepsilon^\delta$ appears as a perturbation of a scheme $\mathcal{P}^\delta$, which is consistent with the limit problem $\mathcal{P}$
- Additional properties (conservation, involution constraint, special choices of numerical viscosities, etc) can be imposed on $\mathcal{P}_\varepsilon^\delta$

Remark: We assume that the limit problem $\mathcal{P}$ is well identified and well-posed

Analysis for the Low Froude Number Limit

We plug the formal asymptotic expansions

$$h = h^{(0)} + \varepsilon h^{(1)} + \varepsilon^2 h^{(2)} + \cdots$$
$$u = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \cdots$$
$$v = v^{(0)} + \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \cdots$$

into the SW system:

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0 \\ (hu)_t + \left(hu^2 + \dfrac{1}{\varepsilon^2}\dfrac{h^2}{2}\right)_x + (huv)_y = \dfrac{1}{\varepsilon} fhv \\ (hv)_t + (huv)_x + \left(hv^2 + \dfrac{1}{\varepsilon^2}\dfrac{h^2}{2}\right)_y = -\dfrac{1}{\varepsilon} fhu \end{cases}$$

and then collect the like powers of ε ...

Analysis for the Low Froude Number Limit

The equations for $\mathcal{O}(\varepsilon^{-2})$ and $\mathcal{O}(\varepsilon^{-1})$ terms imply that

$$h_x^{(0)} = 0,\;\; h_y^{(0)} = 0 \; (\Rightarrow h^{(0)} \equiv \text{Const}), \qquad h_x^{(1)} = v^{(0)},\;\; h_y^{(1)} = -u^{(0)},$$

which can be substituted into equations of $\mathcal{O}(1)$ terms to obtain the limit equations:

$$\begin{cases} h_t^{(0)} + (h^{(0)}u^{(0)})_x + (h^{(0)}v^{(0)})_y = 0 \quad \Longrightarrow \quad u_x^{(0)} + v_y^{(0)} = 0 \\ (h^{(0)}u^{(0)})_t + \left[h^{(0)}(u^{(0)})^2\right]_x + (h^{(0)}u^{(0)}v^{(0)})_y + h^{(0)}h_x^{(2)} = h^{(0)}v^{(1)} \\ (h^{(0)}v^{(0)})_t + \left[h^{(0)}(v^{(0)})^2\right]_y + (h^{(0)}u^{(0)}v^{(0)})_x + h^{(0)}h_y^{(2)} = -h^{(0)}u^{(1)} \\ h_t^{(1)} - h^{(0)}\left(h_{xx}^{(1)} + h_{yy}^{(1)}\right)_t = \cdots \\ h_t^{(2)} - \left(h^{(0)}v^{(1)} + h^{(1)}v^{(0)}\right)_{xt} + \left(h^{(0)}u^{(1)} + h^{(1)}u^{(0)}\right)_{yt} = \cdots \end{cases}$$

Goal: To develop an AP numerical methods for the SW system, which yield a consistent approximation of the above limiting equations as $\varepsilon \to 0$

Hyperbolic Flux Splitting[9]

Key idea: Split the stiff pressure term

- We first split the stiff pressure gradient term into two parts, i.e.

$$\frac{1}{\varepsilon^2}\frac{h^2}{2} = \underbrace{\frac{1}{\varepsilon^2}\frac{h^2}{2} - \frac{a(t)h}{\varepsilon^2}}_{non-stiff} + \underbrace{\frac{a(t)h}{\varepsilon^2}}_{stiff}$$

- We then split the flux terms in the continuity equation by introducing a weight parameter α so that we can construct the slow dynamic system as a hyperbolic system:

$$hu = \alpha\, hu + (1-\alpha)hu, \qquad hv = \alpha\, hv + (1-\alpha)hv$$

[9] Haack, Jin, and Liu, 2012.

Hyperbolic Flux Splitting[10]

Key idea: Split the stiff pressure term

$$\begin{cases} h_t + \alpha(hu)_x + \alpha(hv)_y + (1-\alpha)(hu)_x + (1-\alpha)(hv)_y = 0, \\ (hu)_t + \left(hu^2 + \dfrac{\frac{1}{2}h^2 - a(t)h}{\varepsilon^2}\right)_x + (huv)_y + \dfrac{a(t)}{\varepsilon^2}h_x = \dfrac{1}{\varepsilon}hv, \\ (hv)_t + (huv)_x + \left(hv^2 + \dfrac{\frac{1}{2}h^2 - a(t)h}{\varepsilon^2}\right)_y + \dfrac{a(t)}{\varepsilon^2}h_y = -\dfrac{1}{\varepsilon}hu. \end{cases}$$

This system can be written in the following vector form:

$$\boldsymbol{U}_t + \underbrace{\widetilde{\boldsymbol{F}}(\boldsymbol{U})_x + \widetilde{\boldsymbol{G}}(\boldsymbol{U})_y}_{\text{non-stiff terms}} + \underbrace{\widehat{\boldsymbol{F}}(\boldsymbol{U})_x + \widehat{\boldsymbol{G}}(\boldsymbol{U})_y}_{\text{stiff terms}} = \underbrace{\boldsymbol{S}(\boldsymbol{U})}_{\text{source terms}}$$

How to choose parameters α and $a(t)$?

[10] Liu, Chertock, and Kurganov, 2019.

Hyperbolic Flux Splitting

$$\boldsymbol{U}_t + \underbrace{\widetilde{\boldsymbol{F}}(\boldsymbol{U})_x + \widetilde{\boldsymbol{G}}(\boldsymbol{U})_y}_{\text{non-stiff terms}} + \underbrace{\widehat{\boldsymbol{F}}(\boldsymbol{U})_x + \widehat{\boldsymbol{G}}(\boldsymbol{U})_y}_{\text{stiff terms}} = \underbrace{\boldsymbol{S}(\boldsymbol{U})}_{\text{source terms}}$$

nonlinear part (non-stiff terms); linear part (stiff terms and source terms)

Need to ensure: $\boldsymbol{U}_t + \widetilde{\boldsymbol{F}}(\boldsymbol{U})_x + \widetilde{\boldsymbol{G}}(\boldsymbol{U})_y = \boldsymbol{0}$ is both nonstiff and hyperbolic

Eigenvalues of the Jacobians $\partial\widetilde{\boldsymbol{F}}/\partial\boldsymbol{U}$ and $\partial\widetilde{\boldsymbol{G}}/\partial\boldsymbol{U}$:

$$\left\{u \pm \sqrt{(1-\alpha)u^2 + \alpha\frac{h-a(t)}{\varepsilon^2}},\, u\right\}, \quad \left\{v \pm \sqrt{(1-\alpha)v^2 + \alpha\frac{h-a(t)}{\varepsilon^2}},\, v\right\}$$

We then take: $\alpha = \varepsilon^s$ and $a(t) = \min\limits_{(x,y)\in\Omega} h(x,y,t), \quad s \geq 1$

Remark. It is safe to take $\alpha = \varepsilon^2$

Time Discretization of the Split System

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n - \underbrace{\Delta t \widetilde{\boldsymbol{F}}(\boldsymbol{U})_x^n - \Delta t \widetilde{\boldsymbol{G}}(\boldsymbol{U})_y^n}_{\text{nonlinear part, explicit}}$$
$$- \underbrace{\Delta t \widehat{\boldsymbol{F}}(\boldsymbol{U})_x^{n+1} - \Delta t \widehat{\boldsymbol{G}}(\boldsymbol{U})_y^{n+1} + \Delta t \boldsymbol{S}(\boldsymbol{U})^{n+1}}_{\text{linear part, implicit}}$$

- Nonstiff nonlinear part is treated using the second-order central-upwind scheme
- Stiff linear part reduces to a linear elliptic equation for h^{n+1} and straigtforward computations of $(hu)^{n+1}$ and $(hv)^{n+1}$

For simplicity of presentation: First-order accurate in time

In practice: We implement a two-stage second-order globally stiffly accurate IMEX Runge-Kutta scheme ARS(2,2,2) (all the proofs will apply)

Fully Discrete AP Schemes

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^{n} - \Delta t \underbrace{\left[\widetilde{\boldsymbol{F}}(\boldsymbol{U})_x^n + \widetilde{\boldsymbol{G}}(\boldsymbol{U})_y^n\right]}_{\boldsymbol{R}(\boldsymbol{U})^{\,n}} - \Delta t \left[\widehat{\boldsymbol{F}}(\boldsymbol{U})_x^{\,n+1} + \widehat{\boldsymbol{G}}(\boldsymbol{U})_y^{\,n+1} - \boldsymbol{S}(\boldsymbol{U})^{\,n+1}\right]$$

- We use the notation $\boldsymbol{R}^n := (R^{h,n}, R^{hu,n}, R^{hv,n})^\top$ and rewrite the system

$$h^{\,n+1} = h^{\,n} + \Delta t R^{h,n} - \Delta t(1-\alpha)\left[(hu)_x^{n+1} + (hv)_y^{n+1}\right]$$

$$(hu)^{\,n+1} = \frac{1}{K}\left[(hu)^{\,n} + \frac{\Delta t}{\varepsilon}(hv)^{\,n} + \Delta t\left(R^{hu,n} + \frac{\Delta t}{\varepsilon}R^{hv,n}\right) - \frac{a^n \Delta t}{\varepsilon^2}\left(h_x^{n+1} + \frac{\Delta t}{\varepsilon}h_y^{n+1}\right)\right]$$

$$(hv)^{\,n+1} = \frac{1}{K}\left[(hv)^{\,n} - \frac{\Delta t}{\varepsilon}(hu)^{\,n} + \Delta t\left(R^{hv,n} - \frac{\Delta t}{\varepsilon}R^{hu,n}\right) - \frac{a^n \Delta t}{\varepsilon^2}\left(h_y^{n+1} - \frac{\Delta t}{\varepsilon}h_x^{n+1}\right)\right]$$

where $\quad K := 1 + (\Delta t/\varepsilon)^2$

Fully Discrete AP Schemes

- We differentiate equations for $(hu)^{n+1}$ and $(hv)^{n+1}$ with respect to x and y, respectively and substitute them into equation into the first equation and obtain the following elliptic equation for h^{n+1}:

$$h^{n+1} - \frac{a^n(1-\alpha)}{\widetilde{K}}\Delta h^{n+1} = h^n + \Delta t R^{h,n} - \frac{\Delta t(1-\alpha)}{K}\Bigg[(hu)_x^n + (hv)_y^n + \frac{\Delta t}{\varepsilon}\Big((hv)_x^n - (hu)_y^n\Big) + \Delta t\Big(R_x^{hu,n} + R_y^{hv,n}\Big) + \frac{(\Delta t)^2}{\varepsilon}\Big(R_x^{hv,n} - R_y^{hu,n}\Big)\Bigg]$$

where

$$\widetilde{K} := 1 + (\varepsilon/\Delta t)^2$$

- Solve for h^{n+1} and substitute it into the second and third equation to obtain

$$(hu)^{n+1} = \ldots$$

$$(hv)^{n+1} = \ldots$$

Stability of the Proposed AP Scheme

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^{n} - \Delta t \underbrace{\left[\widetilde{\boldsymbol{F}}(\boldsymbol{U})_x^n + \widetilde{\boldsymbol{G}}(\boldsymbol{U})_y^n\right]}_{\boldsymbol{R}(\boldsymbol{U})^{\,n}} - \Delta t \left[\widehat{\boldsymbol{F}}(\boldsymbol{U})_x^{n+1} + \widehat{\boldsymbol{G}}(\boldsymbol{U})_y^{n+1} - \boldsymbol{S}(\boldsymbol{U})^{\,n+1}\right]$$

The stability of the proposed AP scheme is controlled by the CFL condition:

$$\Delta t_{\text{AP}} \le \nu \cdot \min\left(\frac{\Delta x}{\max\limits_{u,h}\left\{|u| + \sqrt{(1-\alpha)u^2 + \alpha\frac{h-a(t)}{\varepsilon^2}}\right\}}, \frac{\Delta y}{\max\limits_{v,h}\left\{|v| + \sqrt{(1-\alpha)v^2 + \alpha\frac{h-a(t)}{\varepsilon^2}}\right\}}\right).$$

The denominators on the RHS are independent of ε (provided $\alpha \sim \varepsilon^s$). Therefore, the use of large time steps of size $\Delta t_{\text{AP}} = \mathcal{O}(\Delta_{\min})$, is sufficient to enforce the stability of the proposed AP scheme.

Proof of Consistency

We consider the asymptotic expansions for the unknowns

$$\overline{h}^n = h^{(0),n} + \varepsilon h^{(1),n} + \varepsilon^2 h^{(2),n} + \dots$$
$$u^n = u^{(0),n} + \varepsilon u^{(1),n} + \varepsilon^2 u^{(2),n} + \dots$$
$$v^n = v^{(0),n} + \varepsilon v^{(1),n} + \varepsilon^2 v^{(2),n} + \dots$$
$$a^n = h^{(0),n} + \varepsilon a^{(1),n} + \varepsilon^2 a^{(2),n} + \dots$$

and assume that the discrete analogs of the first four equations are satisfied at time level $t = t^n$:

$$h^{(0),n} = h^{(0)}, \quad D_x u^{(0),n} + D_y v^{(0),n} = 0$$
$$v^{(0),n} = D_x h^{(1),n} \quad u^{(0),n} = -D_y h^{(1),n}$$

Goal: To obtain the same relations at time $t = t^{n+1}$

Proof of Consistency

- From the elliptic equation for $\overline{h}^{n+1}$, we have

$$\left[I - \frac{a^n(1-\alpha)}{\widetilde{K}}\Delta\right](\overline{h}^{n+1} - h^{(0),n}) = \mathcal{O}(\varepsilon),$$

  where $a^n(1-\alpha)/\widetilde{K} = h^{(0),n} + \mathcal{O}(\varepsilon)$.

  Matrix $I - \frac{a^n(1-\alpha)}{\widetilde{K}}\Delta$ is positive definite and non-singular (with eigenvalues bounded away from zero independently of ε), therefore

$$\overline{h}^{n+1} = h^{(0),n} + \mathcal{O}(\varepsilon)$$

- We also have $\overline{(hu)}^{n+1} = \mathcal{O}(1)$ and $\overline{(hv)}^{n+1} = \mathcal{O}(1)$, which gives

$$u^{n+1} = u^{(0),n+1} + \mathcal{O}(\varepsilon) \quad \text{and} \quad v^{n+1} = v^{(0),n+1} + \mathcal{O}(\varepsilon)$$

Proof of Consistency

We plug the asymptotic expansions

$$h^n = h^{(0),n} + \varepsilon h^{(1),n} + \varepsilon^2 h^{(2),n}, \quad h^{n+1} = h^{(0),n+1} + \varepsilon h^{(1),n+1} + \varepsilon^2 h^{(2),n+1}$$

$$u^n = h^{(0),n} + \varepsilon u^{(1),n} + \varepsilon^2 u^{(2),n}, \quad u^{n+1} = u^{(0),n+1} + \varepsilon u^{(1),n+1} + \varepsilon^2 u^{(2),n+1}$$

$$v^n = v^{(0),n} + \varepsilon v^{(1),n} + \varepsilon^2 v^{(2),n}, \quad v^{n+1} = v^{(0),n+1} + \varepsilon v^{(1),n+1} + \varepsilon^2 v^{(2),n+1}$$

into the implicit-explicit scheme and equate the like powers of ε to obtain the following equations:

$$\mathcal{O}(\varepsilon^{-2}): \qquad h^{(0),n+1} h_x^{(0),n+1} = 0$$

$$h^{(0),n+1} h_y^{(0),n+1} = 0$$

$$\mathcal{O}(\varepsilon^{-1}): \qquad h^{(0),n+1} h_x^{(1),n+1} + h_x^{(0),n+1} h^{(1),n+1} = h^{(0),n+1} v^{(0),n+1}$$

$$h^{(0),n+1} h_y^{(1),n+1} + h_y^{(0),n+1} h^{(1),n+1} = -h^{(0),n+1} u^{(0),n+1}$$

$$\mathcal{O}(1): \qquad \ldots$$

Summary

Theorem. *The proposed hyperbolic flux splitting method coupled with the described fully discrete scheme is asymptotic preserving in the sense that it provides a consistent and stable discretization of the limiting system as the Froude number $\varepsilon \to 0$.*

Remark. In practice, the fully discrete scheme is both second-order accurate in space and time as we increase a temporal order of accuracy to the second one by implementing a two-stage globally stiffly accurate IMEX Runge-Kutta scheme ARS(2,2,2). The proof holds as well.

Remark. The proposed AP scheme is also asymptotically well-balanced in the sense that it preserves geostrophic equilibria in the zero Froude number limit at the discrete level: implies

$$u = -\frac{1}{\varepsilon} h_y, \quad v = \frac{1}{\varepsilon} h_x$$

Numerical Examples

Example — 2-D Stationary Vortex

$$h(r,0) = 1+\varepsilon^2 \begin{cases} \frac{5}{2}(1+5\varepsilon^2)r^2 \\ \frac{1}{10}(1+5\varepsilon^2) + 2r - \frac{1}{2} - \frac{5}{2}r^2 + \varepsilon^2(4\ln(5r) + \frac{7}{2} - 20r + \frac{25}{2}r^2) \\ \frac{1}{5}(1-10\varepsilon+4\varepsilon^2\ln 2), \end{cases}$$

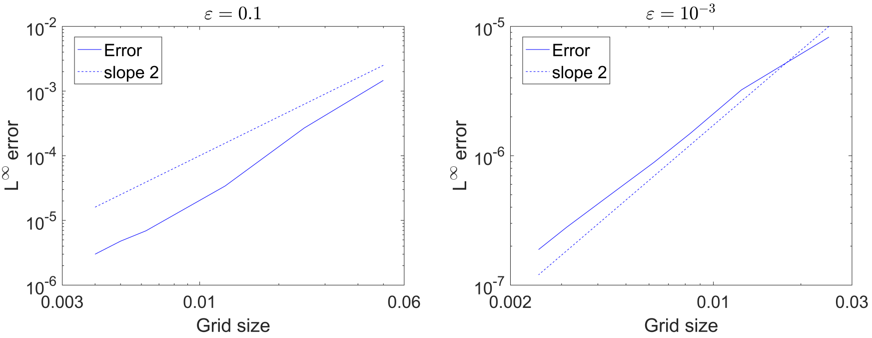
$$u(x,y,0) = -\varepsilon y \Upsilon(r), \quad v(x,y,0) = \varepsilon x \Upsilon(r), \quad \Upsilon(r) := \begin{cases} 5, & r < \frac{1}{5} \\ \frac{2}{r} - 5, & \frac{1}{5} \le r < \frac{2}{5} \\ 0, & r \ge \frac{2}{5}, \end{cases}$$

Domain: $[-1,1] \times [-1,1], \quad r := \sqrt{x^2+y^2}$

Boundary conditions: a zero-order extrapolation in both x- and y-directions[11]

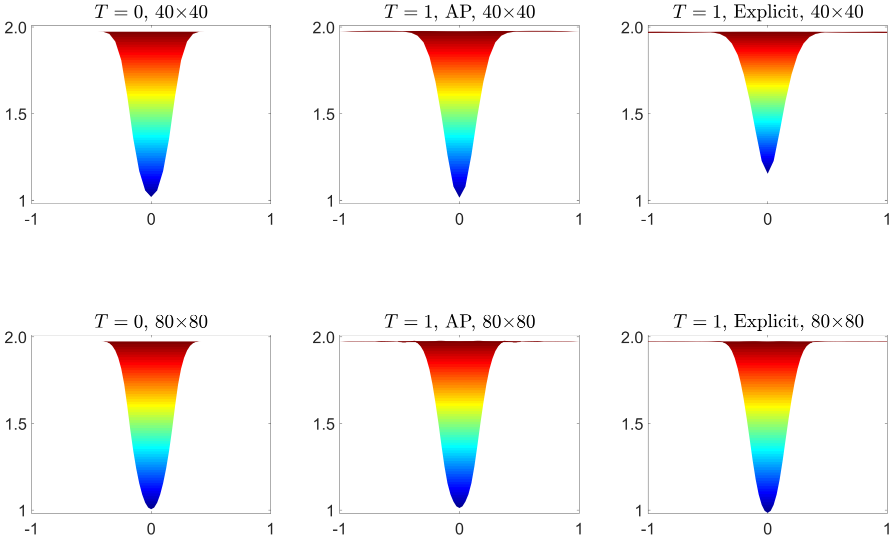
[11] Audusse, Klein, Nguyen, and Vater, 2011.

Experimental order of convergence

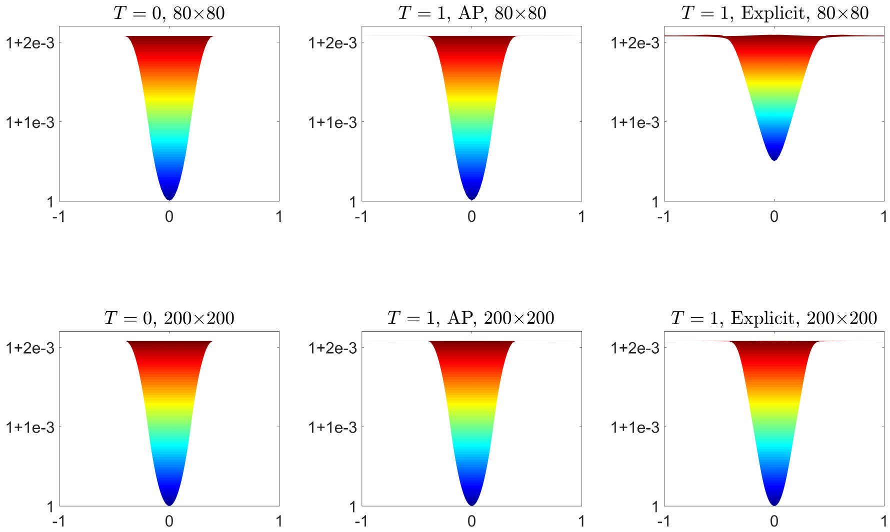


L^∞-errors for h computed using the AP scheme on several different grids for $\varepsilon = 0.1$ (left) and 10^{-3}

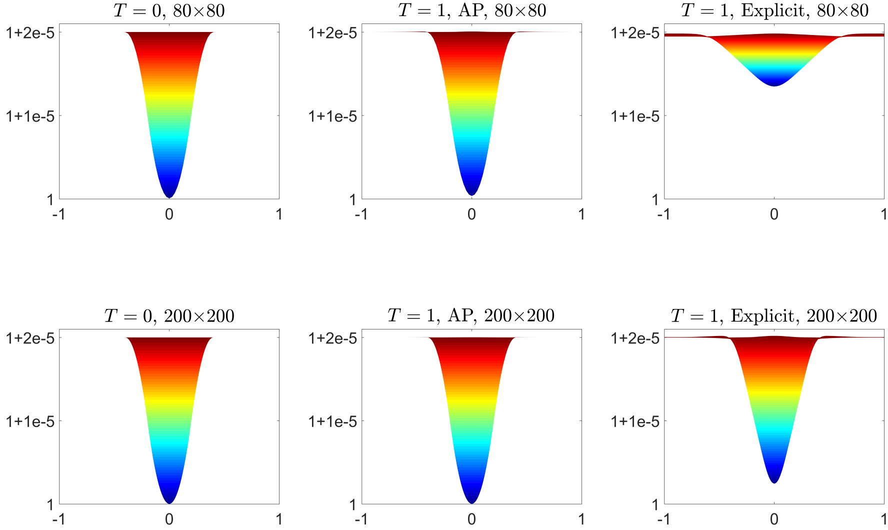
Comparison of non-AP and AP methods, $\varepsilon = 1$



Comparison of non-AP and AP methods, $\varepsilon = 0.1$



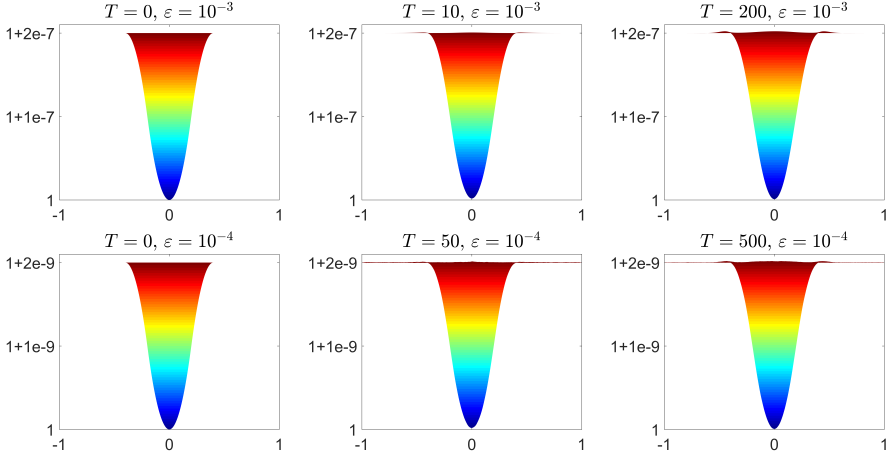
Comparison of non-AP and AP methods, $\varepsilon = 0.01$



Comparison of non-AP and AP methods, CPU times

	$\varepsilon = 1$		$\varepsilon = 0.1$		$\varepsilon = 0.01$	
Grid	AP	Explicit	AP	Explicit	AP	Explicit
40×40	0.18 s	0.16 s	0.06 s	1.25 s	0.03 s	10.53 s
80×80	1.57 s	1.32 s	0.29 s	4.73 s	0.18 s	47.0 s
200×200	24.11 s	21.36 s	5.36 s	163.36 s	3.37 s	804.15 s

Smaller values: $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$



Smaller times: 200×200, larger times: 500×500

Example — 2-D Traveling Vortex

We take $\varepsilon = 10^{-2}$ and simulate a traveling vortex with the same initial water depth profile as in Example 1 but the initial velocities are now modified by adding a constant velocity vector $(15, 15)^\top$:

$$u(x, y, 0) = 15 - \varepsilon y \Upsilon(r), \quad v(x, y, 0) = 15 + \varepsilon x \Upsilon(r)$$

$$\Upsilon(r) := \begin{cases} 5, & r \leq \dfrac{1}{5}, \\ \dfrac{2}{r} - 5, & \dfrac{1}{5} < r \leq \dfrac{2}{5}, \\ 0, & r \geq \dfrac{2}{5}, \end{cases}$$

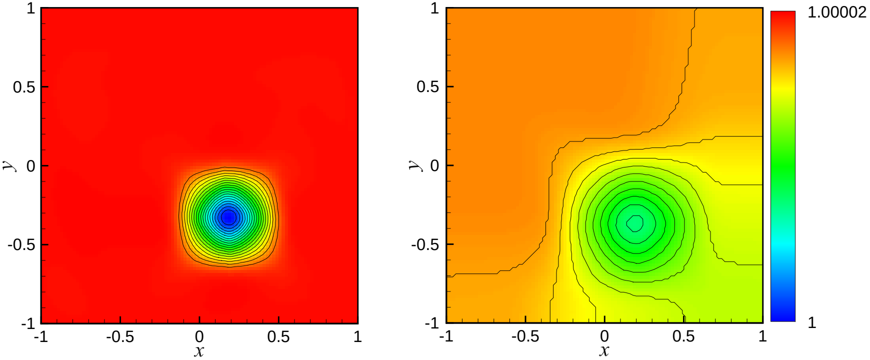
where $r := \sqrt{x^2 + y^2}$.

Domain: $[-1, 1] \times [-1, 1], \quad r := \sqrt{x^2 + y^2}$

Boundary conditions: a zero-order extrapolation in both x- and y-directions

These initial data correspond to a rotating vortex traveling along a circular path

Comparison of non-AP and AP methods, $\varepsilon = 0.01$



100×100

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Thank you!