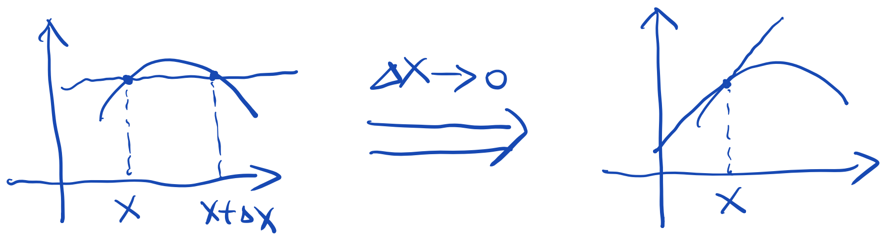
- Analysis is the art of taking limits.
- Where do we need limits?

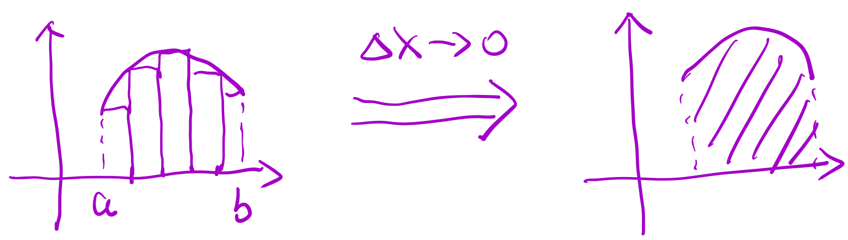
Review of Calculus I:

① Derivative $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

② Integral $\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_i f(x_i)\Delta x$

③ Fundamental Theorem $\int_a^b f'(x)dx = f(b) - f(a)$





Chapter 1

Real numbers as infinite decimals

$1/3$: $0.3333\ldots$

π : $3.1415926\ldots$

$\sqrt[3]{2}$: $1.25992\ldots$

π is the limit of $\underset{a_0}{3}, \underset{a_1}{3.1}, \underset{a_2}{3.14}, \underset{a_3}{3.141}$

$\sqrt[3]{2}$ is the limit of $\underset{b_0}{1}, \underset{b_1}{1.2}, \underset{b_2}{1.25}, \underset{b_3}{1.259}$

$\pi + \sqrt[3]{2}$ is the limit of $4, 4.3, 4.39, 4.400, \ldots$

$a_n + b_n$

$\underbrace{\pi \cdot \sqrt[3]{2}}_{\downarrow}$ is $\underbrace{\text{the limit of } a_0b_0, a_1b_1, a_2b_2, a_3b_3}$

$$\left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) \overset{?}{=} \lim_{n\to\infty} a_n b_n$$

A sequence : $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$

$\{a_n\}, n \geq 0$

Example: ① $\{(-1)^n\}, n \geq 0$

$1, -1, 1, -1, \ldots$

② $\{\frac{1}{n}\}, n \geq 1$

$1, 1/2, 1/3, \ldots$

③ $3, 3.1, 3.14, 3.141, \ldots$

<u>Definition</u> $\{a_n\}$ is
- increasing if $a_n \leq a_{n+1}, \underset{\text{for any}}{\forall n}$
- strictly increasing if $a_n < a_{n+1}, \forall n$
- decreasing if $a_n \geq a_{n+1}, \forall n$
- strictly decreasing if $a_n > a_{n+1}, \forall n$

Definition A number L is the limit of increasing $\{a_n\}$ if, for any given integer k, all the a_n after some place in the sequence agree with L to k decimal places.

$a_0 = 15.34576$
$a_1 = 16.26745$
$a_2 = 16.33654$
$a_3 = 16.34722$
$a_4 = 16.34745$
$a_5 = 16.34747$
$a_6 = 16.34748$

$L = 16.34748$

$L = 0.0000$

$b_1 = 1.1$
$b_2 = 2.3$
$b_3 = 1.0$
$b_4 = 0.9$
$b_5 = 0.001$
$b_6 = 0.0001$
$b_7 = 0.00001$

Def $\{a_n\}$ is bounded above if there exists a number B s.t. $a_n \le B$, $\forall n$.

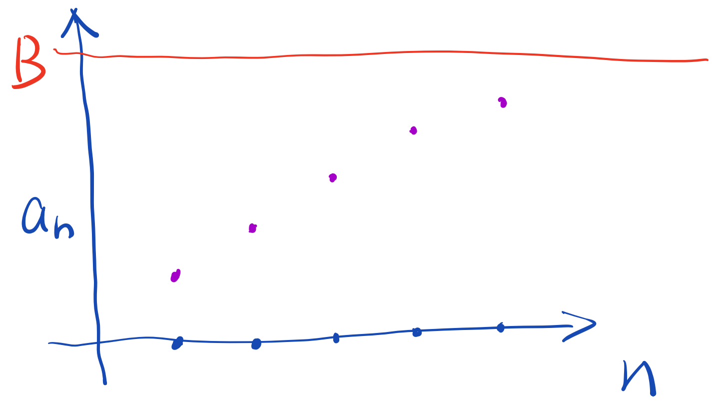
Theorem An increasing and bounded above sequence has a limit.

Step I: find an example

Step II: seek intuition

Step III: convert your intuition into rigorous statement.

$a_0 = 15.34576$
$a_1 = 16.26745$
$a_2 = 16.33654$
$a_3 = 16.34722$
$a_4 = 16.34745$
$a_5 = 16.34747$
$a_6 = 16.34748$
$\vdots$



(Informal) Proof: ① Claim that the integer part will stay the same after some place in the sequence.

Proof of claim:

$\{a_n\}$ is increasing $\Rightarrow$ integer part is increasing.

$\Rightarrow$ integer part is either staying the same or strictly increasing

$\{a_n\}$ is bounded above

$\Rightarrow$ integer part must stop strictly increasing after some place.

② Claim that the first decimal will stay the same after some place in the sequence, because $\begin{cases} \text{integer part is the same after some place} \\ \text{1st decimal place can grow or stay the same.} \end{cases}$

③ Claim that the k-th decimal will stay the same after some place if (k-1)-th decimal stays the same after some place, for similar reasons.