Chapter 7 Infinite Series

Def ① By an infinite series, we mean

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

② a_n is called n-th term

③ $S_n = a_0 + a_1 + \cdots + a_n$ is the n-th partial sum

④ If $\{S_n\}$ converges, we say the series converges

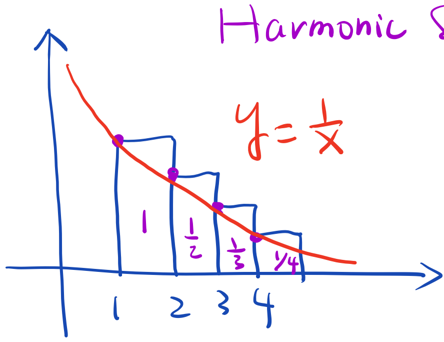
$\{S_n\}$ diverges, we say the series diverges

⑤ If $\lim_{n\to\infty} S_n = S$, then we write $\sum_{n=0}^{\infty} a_n = S$

where $S \in \mathbb{R}$.

Example: ① $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ diverges

Harmonic Series



$$S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} > \int_1^{n+1} \frac{1}{x}\, dx$$

$$= \ln x \Big|_1^{n+1}$$

$$= \ln(n+1) \to +\infty$$

② $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = 2$ Geometric Series

$$S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} = \frac{1-(\frac{1}{2})^{n+1}}{1-\frac{1}{2}} \longrightarrow 2$$

③ $$\sum_{n=0}^{\infty} r^n \begin{cases} = \frac{1}{1-r}, & \text{if } |r|<1 \\ \text{diverges}, & \text{if } |r| \geq 1 \end{cases}$$

Proof of the case $|r| \geq 1$:

If $\{S_n\}$ converges, then $\{S_n\}$ is Cauchy.

$\Rightarrow \forall \varepsilon > 0, \exists N, |S_m - S_n| < \varepsilon, \forall m, n > N$

$\Rightarrow \forall \varepsilon > 0, \exists N, |S_{n+1} - S_n| < \varepsilon, \forall n > N$

$|S_{n+1} - S_n| = |r^{n+1}| = |r|^{n+1} \geq 1$, contradiction.

④ $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots = e$

exponential series

Proof of convergence:

Since S_n is increasing, we only need to show

S_n is bounded above.

$$\frac{1}{n!} = \frac{1}{1 \cdot 2 \cdot 3 \cdots n} < \frac{1}{1 \cdot 2 \cdot 2 \cdots 2} = \frac{1}{2^{n-1}}$$

$$\Rightarrow S_n < 1 + \frac{1}{1!} + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

$$= 1 + \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}}$$

$$< 1 + \frac{1}{1 - \frac{1}{2}} = 3.$$

$\Rightarrow$ The limit exists and we call it e.

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1) Given an infinite series $\sum a_n$, we have a sequence $\{S_n\}$

$$S_n = a_0 + a_1 + \cdots + a_n$$

2) Given a sequence $\{S_n\}$, we can view it as the partial sum of some series $\sum_{n=0}^{\infty} a_n$

where $\begin{cases} a_0 = S_0 \\ a_n = S_n - S_{n-1}, & n \geq 1 \end{cases}$

Example: $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n+1)$, $n \geq 1$

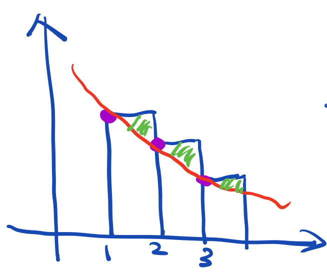
Convert it to an infinite Series

$$a_n = S_n - S_{n-1} = \frac{1}{n} - \ln(n+1) + \ln n$$

$$= \frac{1}{n} - \ln \frac{n+1}{n}$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln \frac{n+1}{n} \right)$$

Proof of convergence: We did one proof in Section 1.5



$$\ln(n+1) = \int_1^{n+1} \frac{1}{x}\, dx$$

Elementary Convergence Tests

<u>Theorem</u> (n-th term Test)

$$\sum a_n \text{ converges} \Rightarrow \lim_{n \to \infty} a_n = 0$$

Proof: $S_n = a_1 + a_2 + \cdots + a_n$

$$\sum a_n \text{ converges} \iff \lim_{n\to\infty} S_n = S \in \mathbb{R}$$

$$a_n = S_n - S_{n-1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1})$$

$$(\text{Linearity Thm}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= S - S = 0$$

Remark : $\lim_{n\to\infty} a_n$ is not $0 \Rightarrow \sum a_n$ diverges

Common Mistake : " $a_n \to 0 \Rightarrow \sum a_n$ converges" is false

Counter Example is Harmonic Series

<u>Theorem</u> (Tail Convergence)

$\sum_{n=N_0}^{\infty} a_n$ converges for some $N_0 \Rightarrow \sum_{n=0}^{\infty} a_n$ converges

$\Rightarrow \sum_{n=N}^{\infty} a_n$ converges, $\forall N \in \mathbb{N}$.

Excercise: Negation of " $\sum_{n=N_0}^{\infty} a_n$ converges for some N_0 "

is "$\forall N_0 \in \mathbb{N}$, $\sum_{n=N_0}^{\infty} a_n$ diverges"

<u>Linearity Theorem</u> c, p, q are real numbers

$\sum a_n$ and $\sum b_n$ converge

$$\Rightarrow \begin{cases} \sum (p a_n + q b_n) = p \sum a_n + q \sum b_n \\ \sum (a_n \pm b_n) = \sum a_n \pm \sum b_n \\ \sum (c a_n) = c \sum a_n \end{cases}$$

<u>Comparison Theorem</u> (Series) Assume $0 \le a_n \le b_n$, $\forall n$.

① $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges

and $\sum a_n \le \sum b_n$

② $\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges

Proof: ① $S_n = a_1 + a_2 + \cdots + a_n$

$a_n \geq 0 \Rightarrow S_n$ is increasing

$\boxed{S_n \le b_1 + \cdots + b_n} \Rightarrow S_n \le b_1 + \cdots + b_n + \cdots + b_k$

Limit Location Thm on the sequence $b_1 + b_2 + \cdots + b_k$ $\Rightarrow S_n \le \sum_{k=1}^{\infty} b_k$ is bounded above

Completeness Thm $\Rightarrow \{S_n\}$ has a limit

Limit Location Thm on $S_n \Rightarrow \lim\limits_{n\to\infty} S_n \le \sum\limits_{n=1}^{\infty} b_n$

Alternating Series

Def

① $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges;

② $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example: ① $\sum (-1)^n \frac{1}{2^n}$ and $\sum (-1)^n \frac{1}{n!}$ are absolutely convergent.

② Any convergent series consisting of positive terms are absolutely convergent.

③ $\sum (-1)^n \frac{1}{n}$ is conditionally convergent

Proof of ③: We know $\sum\limits_{n=1}^{\infty} \left|(-1)^n \frac{1}{n}\right| = +\infty$

To show $\sum (-1)^n \frac{1}{n}$ converges:

$$S_n = -1 + \frac{1}{2} - \frac{1}{3} + \cdots + (-1)^n \frac{1}{n}$$

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$$

$$x = -u$$

$$1-u+u^2+\cdots+(-1)^{n-1}u^{n-1}=\frac{1-(-1)^n u^n}{1+u}$$

$$\int_0^1\left[1-u+u^2+\cdots+(-1)^{n-1}u^{n-1}\right]du=\int_0^1\frac{1-(-1)^n u^n}{1+u}du$$

$$1-\frac{u^2}{2}\Big|_0^1+\frac{u^3}{3}\Big|_0^1+\cdots+(-1)^{n-1}\frac{u^n}{n}\Big|_0^1=\int_0^1\frac{1}{1+u}du-(-1)^n\int_0^1\frac{u^n}{1+u}du$$

$$\underbrace{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{n-1}\frac{1}{n}}_{-S_n}=\underbrace{\int_0^1\frac{1}{1+u}du}_{\ln(1+u)\big|_0^1=\ln 2}-(-1)^n\underbrace{\int_0^1\frac{u^n}{1+u}du}_{b_n}$$

$$0<b_n=\int_0^1\frac{u^n}{1+u}du<\int_0^1\frac{u^n}{1+0}du=\int_0^1 u^n du=\frac{u^{n+1}}{n+1}\Big|_0^1\to 0$$

$$\Rightarrow\quad S_n\to-\ln 2$$

$$\Rightarrow\quad\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}=-\ln 2$$

Theorem $\sum|a_n|$ converges $\Rightarrow\sum a_n$ converges

Proof: Define

$$a_n^+=\begin{cases}|a_n|, & \text{if } a_n>0\\ 0, & \text{otherwise}\end{cases},\qquad a_n^-=\begin{cases}|a_n|, & \text{if } a_n<0\\ 0, & \text{otherwise}\end{cases}$$

Then $a_n = a_n^+ - a_n^-$ and $|a_n| = a_n^+ + a_n^-$

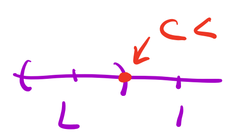
$$\left.\begin{array}{l} 0 \le a_n^+ \le |a_n| \\ 0 \le a_n^- \le |a_n| \\ \text{Comparison Theorem} \end{array}\right\} \Rightarrow \sum a_n^+ \text{ and } \sum a_n^- \text{ converges}$$

$$\left.\begin{array}{l} a_n = a_n^+ - a_n^- \\ \text{Linearity Theorem} \end{array}\right\} \Rightarrow \sum a_n = \sum a_n^+ - \sum a_n^- \text{ converges}$$

__Theorem (Ratio Test)__ Assume $a_n \neq 0$, $n >> 1$

and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

Then $L < 1 \Rightarrow \sum |a_n|$ converges



$L > 1 \Rightarrow \sum a_n$ diverges

$L = 1$: no conclusion. $\begin{cases} ① \sum \frac{1}{n} \text{ diverges} \\ ② \sum (-1)^{n-1} \frac{1}{n} \text{ converges} \end{cases}$

__Theorem (n-th root test)__ Assume $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L$.

Then $L < 1 \Rightarrow \sum |a_n|$ converges

$L > 1 \Rightarrow \sum a_n$ diverges

$L = 1$: no conclusion.

Example : ① $\sum (-1)^n \frac{n}{2^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}} \frac{2^n}{n} = \frac{n+1}{2n} \to \frac{1}{2}$$

Ratio test $\Rightarrow$ absolute convergence of $\sum (-1)^n \frac{n}{2^n}$

② $\sum \frac{1}{n^2}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^2}{(n+1)^2} = \left(\frac{n}{n+1}\right)^2 \to 1$$

Ratio test fails

$$|a_n|^{\frac{1}{n}} = [n^2]^{\frac{1}{n}} = [n^{\frac{1}{n}}]^{-2} = \frac{1}{[n^{\frac{1}{n}}]^2} \to 1$$

because $n^{\frac{1}{n}} \to 1$

$n^{\frac{1}{n}} > 1 \Rightarrow n^{\frac{1}{n}} = 1 + e_n$, $e_n > 0$

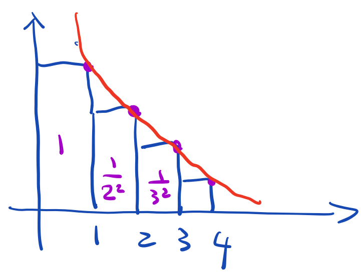
$\Rightarrow n = (1+e_n)^n = 1 + ne_n + \frac{n(n-1)}{2} e_n^2 + \cdots$

$\Rightarrow n > \frac{n(n-1)}{2} e_n^2$

$\Rightarrow e_n \to 0$.

Root test also fails.

To prove convergence:



$$\Rightarrow 0 < 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \int_0^n \frac{1}{x^2} dx$$

$$-\frac{1}{x}\Big|_0^n = +\infty$$

useless

$$0 < 1 + \underbrace{\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}} < \underbrace{1 + \underbrace{\int_1^n \frac{1}{x^2} dx}}$$

$$1 + \left(-\frac{1}{x}\right)\Big|_1^n = 1 - \frac{1}{n} + 1 < 2$$

$\Rightarrow S_n$ is increasing & bounded above.