__Theorem__ (Cauchy Test for alternating series)

If $\{a_n\}$ is positive and strictly decreasing and $\lim_{n\to\infty} a_n = 0$, then $\sum (-1)^n a_n$ converges.

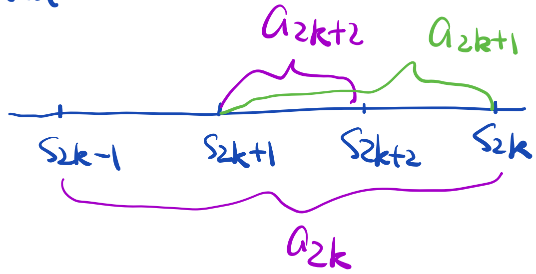
Ex: $\sum (-1)^n \frac{1}{\sqrt{n}}$ converges.

Proof: $S_n = a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n$

$$S_{2k} = S_{2k-1} + a_{2k}$$
$$S_{2k+1} = S_{2k} - a_{2k-1}$$

$a_{2k} > 0 \Rightarrow$ a sequence of intervals $[S_{2k-1}, S_{2k}]$ with length $a_{2k} \to 0$, $k \to \infty$

Nested intervals:



Nested Interval Theorem $\Rightarrow \lim_{k\to\infty} S_{2k-1} = S = \lim_{k\to\infty} S_{2k}$

$\boxed{?} \Rightarrow \lim_{n\to\infty} S_n = S$

$\Rightarrow \sum_{n=0}^{\infty} (-1)^n a_n = S$

$\boxed{?}$: $S_{2k-1} \to S \Rightarrow \forall \varepsilon$, $\exists N_1$ s.t. $|S_{2k-1} - S| < \varepsilon$, $\forall k > N_1$

$S_{2k} \to S \Rightarrow$ for the same ε, $\exists N_2$, $|S_{2k} - S| < \varepsilon$, $\forall k > N_2$.

Question: Can we sum terms in $\sum_{n=0}^{\infty} a_n$ by different orders, and still get the same sum?

Example: $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$, rearrange terms like

$$-1-\frac{1}{3}-\frac{1}{5}+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}-\frac{1}{7}-\frac{1}{9}-\frac{1}{11}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}\cdots$$

do we still have $-\ln 2$?

Excercise: ① (T or F)

If $a_n \geq 0$, then $\sum a_n$ diverges $\Rightarrow \sum a_n = +\infty$.

1) $a_n \geq 0 \Rightarrow S_n \nearrow$; 2) S_n has no upper bound because otherwise $\{S_n\}$ converges

3) have upper bound is "$\exists M, \forall n, S_n \leq M$".

② $1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n+1}+\cdots$ diverge to $+\infty$

$$\frac{1}{2n+1} \geq \frac{1}{2n+2} = \frac{1}{2}\left(\frac{1}{n+1}\right) \Rightarrow S_n \geq \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)$$

<u>Theorem</u> (Rearrangement Theorem)

By rearranging terms in $\sum a_n$, we get a new series

① If $\sum |a_n|$ converges, then the new series converges absolutely, and the sum is the same as $\sum a_n$

Example: $\sum(-1)^n\frac{1}{n^2}$, rearranging in any order, we get same sum.

② If $\sum a_n$ converges conditionally, then by rearranging its terms one can get any real number, or $+\infty, -\infty$.

Example: $\forall L \in \mathbb{R}$, there is one rearrangement of $\sum (-1)^n \frac{1}{n}$

s.t. the new sum is L.

A sketchy proof of ② for $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$:

$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} + \cdots$ diverge to $+\infty$

$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots - \frac{1}{2n} - \cdots$ diverge to $-\infty$

Suppose we want to sum to π:

First, add positive terms until the sum S_0 first exceeds π.

Second, add negative terms to S_0 until the new sum S_1 first becomes less than π

$S_0 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{21} = B_{11}$

$S_2 > \pi$

$S_1 = S_0 - \frac{1}{2} = B_{12}$

$S_3 < \pi$

$S_2 = S_1 + \frac{1}{23} + \frac{1}{25} + \cdots + \frac{1}{101}$

$\vdots$

Let $\sum b_n$ be the new series, and its partial sum is B_n.

Then S_i is a subsequence of B_n

$$S_i = B_{n_i}$$

And $|S_i - \pi| \leq |b_{n_i}|$

$$\Rightarrow \lim_{i \to \infty} S_i = \pi$$

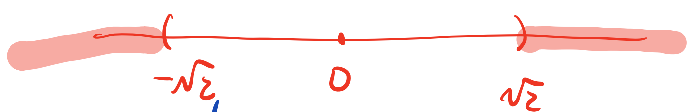
Chapter 8 Power Series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

where a_n are given numbers and x is a variable.

We want to know for what range of x it converges.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n}$



By ratio test, we consider

$$\left|\frac{x^{2n+1}}{2^{n+1}(n+1)}\right| \Big/ \left|\frac{x^{2n}}{2^n n}\right| = \frac{|x|^2}{2} \cdot \frac{n}{n+1} \to \frac{|x|^2}{2}, \quad n \to \infty$$

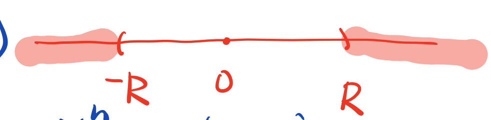
$$\frac{|x|^2}{2} < 1 \iff |x| < \sqrt{2}$$

So ① If $x \in (-\sqrt{2}, \sqrt{2})$, $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n}$ converges absolutely.

② If $|x| > \sqrt{2}$, diverges

③ $x = \sqrt{2}$, or $-\sqrt{2}$, unknown by ratio test.

Theorem (Radius of Convergence)



For each power series $\sum a_n x^n$, there is a unique $R \geq 0$ s.t.

① $\sum a_n x^n$ converges absolutely for $|x| < R$

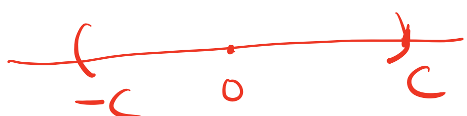
② $\sum a_n x^n$ diverges for $|x| > R$.

R is called radius of convergence

$R = +\infty$ means $\sum a_n x^n$ converges absolutely for any x.

Remark: For $x = R$ or $-R$, convergence needs to be discussed for each series.

Proof: Step I



First, we prove $\sum a_n x^n$ converges for $x = c$

$\Rightarrow \sum |a_n x^n|$ converges for $|x| < |c|$.

$\boxed{\text{Case 1}}$ $c = 1$

Hypothesis is $\sum a_n$ converges

Want to show $\sum |a_n x^n|$ converges for any $x \in (-1, 1)$

$\sum a_n$ converges $\Rightarrow a_n \to 0$

$\Rightarrow |a_n| \to 0 < 1$

Sequence Location Thm $\Rightarrow |a_n| < 1$, $\forall n \geq N$

$\Rightarrow \quad |a_n x^n| < |x^n|, \quad \forall n \geq N$

$$\left(|x|<1 \Rightarrow \sum_{n=0}^{\infty} |x^n| = \frac{1}{1-|x|} \Rightarrow \sum_{n=N}^{\infty} |x^n| \text{ converges} \right)$$

$\Rightarrow \sum_{n=N}^{\infty} |a_n x^n|$ converges

$\Rightarrow \sum_{n=0}^{\infty} |a_n x^n|$ converges.

$\boxed{\text{Case 2}}$ $c \neq 0$ (If $c=0$, nothing to prove)

Hypothesis is $\sum a_n c^n$ converges

Want to show $\sum |a_n x^n|$ converges for any $x \in (-c, c)$

$\sum a_n c^n$ converges

$\Rightarrow \sum a_n c^n u^n$ converges for $u=1$

Case 1 $\Rightarrow \sum |a_n c^n u^n|$ converges for $|u|<1$

$u = \frac{x}{c}$ $\Rightarrow \sum |a_n x^n|$ converges for $|\frac{x}{c}|<1$

or $|x|<|c|$.

<u>Step 2</u> Let $A = \{x : \sum |a_n x^n| \text{ converges}\}$

be the set of points where $\sum |a_n x^n|$ converges

① $x=0$ makes $\sum |a_n x^n|$ converge

$\Rightarrow$ A is not empty

② By step 1, we have

$$c>0,\ c\in A \Rightarrow (-c,c)\subset A$$

because

$$c\in A \Rightarrow \sum |a_n c^n| \text{ converges}$$

$$\Rightarrow \sum a_n c^n \text{ converges}$$

Step 1 $\Rightarrow \sum |a_n x^n|$ converges, $|x|<c$

$$\Rightarrow (-c,c)\subset A$$

③ If $A=(-\infty,\infty)$, define $R=+\infty$.

④ If $A\neq(-\infty,\infty)$, $\exists\, b\notin A$

$$(d\in A \Rightarrow -d\in A)$$

So we may take $b>0$.

Claim b is an upper bound of A

Proof of claim: if not, then $\exists\, c\in A$ s.t. $c>b$.

$$c\in A \Rightarrow (-c,c)\subset A \Rightarrow b\in(-c,c)\subset A$$

$\Rightarrow b \in A$, contradiction.

So A is bounded above

supremum is the smallest upper bound

Completeness Thm for sets $\Rightarrow R = \sup A$ exists

$|x| < R \Rightarrow |x|$ is not an upper bound of A

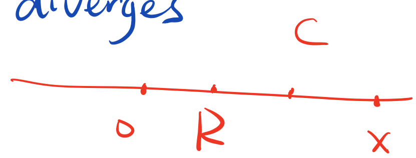
$\Rightarrow |x| < c$ for some $c \in A$

$\Rightarrow x \in (-c, c) \subset A$

$\Rightarrow \sum |a_n x^n|$ converges

Finally, need to show

$|x| > R \Rightarrow \sum a_n x^n$ diverges



Assume $\sum a_n x^n$ converges

$|x| > R \Rightarrow$ we have choose c s.t. $R < c < |x|$.

Step 1 $\Rightarrow \sum |a_n c^n|$ converges, $c < |x|$

$\Rightarrow c \in A$

$\Rightarrow \sup A \geq c > R$ $\square$.

Contradiction