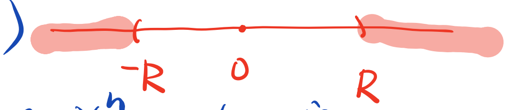
Review

Power Series: $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

where a_n are given numbers and x is a variable.

Theorem (Radius of Convergence)



For each power series $\sum a_n x^n$, there is a unique $R \geq 0$ s.t.

① $\sum a_n x^n$ converges absolutely for $|x| < R$

② $\sum a_n x^n$ diverges for $|x| > R$.

③ The only possible conditional convergence points are $x = \pm R$.

R is called radius of convergence

$R = +\infty$ means $\sum a_n x^n$ converges absolutely for any x.

Theorem (Ratio Test) Assume $a_n \neq 0$, $n >> 1$

and $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$.

Then $L < 1 \Rightarrow \sum |a_n|$ converges

$L > 1 \Rightarrow \sum a_n$ diverges

$L = 1$: no conclusion. $\begin{cases} ① \sum \frac{1}{n} \text{ diverges} \\ ② \sum (-1)^n \frac{1}{n} \text{ converges} \end{cases}$

Theorem (n-th root test) Assume $\lim_{n \to \infty} |a_n|^{\frac{1}{n}} = L$.

Then $L < 1 \Rightarrow \sum |a_n|$ converges

$L > 1 \Rightarrow \sum a_n$ diverges

$L = 1$: no conclusion.

HW#5 Problem: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \Rightarrow \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = L.$

This means that ① if $|a_{n+1}/a_n| \to L$, ratio test is Stronger.
② if $\left|\frac{a_{n+1}}{a_n}\right|$ has no limits, $|a_n|^{\frac{1}{n}}$ can still have one.

Example: $\sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n}$

By ratio test, we consider

$$\left|\frac{x^{2n+1}}{2^{n+1}(n+1)}\right| \Big/ \left|\frac{x^{2n}}{2^n n}\right| = \frac{|x|^2}{2}\cdot\frac{n}{n+1} \to \frac{|x|^2}{2}, \quad n\to\infty$$

$$\frac{|x|^2}{2} < 1 \iff |x| < \sqrt{2}$$

So ① If $x \in (-\sqrt{2}, \sqrt{2})$, $\sum_{n=1}^{\infty}\frac{x^{2n}}{2^n n}$ Converges absolutely.

② If $|x| > \sqrt{2}$, diverges

③ $x = \sqrt{2}$, or $-\sqrt{2}$, unknown by ratio test.

④ What happens at $x = \pm\sqrt{2}$?

$x = \sqrt{2} \Rightarrow \sum \frac{1}{n} = +\infty$

$x = -\sqrt{2} \Rightarrow \sum \frac{1}{n} = +\infty$

Example: The following three series all have radius of convergence 1, test them for convergence at end points ± 1.

$$\left|\frac{x^{n+1}}{(n+1)^2} \Big/ \frac{x^n}{n^2}\right| = |x|\cdot\frac{n^2}{(n+1)^2} \to |x|$$

① $\sum x^n$ ② $\sum \frac{x^n}{n}$ ③ $\sum \frac{x^n}{n^2}$

① $\sum(-1)^n$ and $\sum(1)^n$ diverge

② $\sum(-1)^n\frac{1}{n}$ converges

$\sum\frac{1}{n}=+\infty$

③ $\sum(-1)^n\frac{1}{n^2}$ converges

$\sum\frac{1}{n^2}$ converges

<u>Theorem</u> (Cauchy Test for alternating series)

If $\{a_n\}$ is positive and strictly decreasing

and $\lim_{n\to\infty} a_n=0$, then

$\sum(-1)^n a_n$ converges.

$\sum|a_n(-R)^n|=\sum a_n R^n$

Remark: ① If $a_n\geq 0$, $\sum a_n x^n$ converges at $x=R>0 \Rightarrow \sum a_n x^n$ converges at $x=-R$

Absolute Convergence implies convergence

② If $a_n\geq 0$, and a_n is decreasing to 0,

Cauchy test $\Rightarrow \sum a_n x^n$ converges at $x=-1$.

$\Rightarrow R\geq 1$

<u>Theorem</u> (Linearity Thm)

If $\sum a_n x^n=f(x)$ and $\sum b_n x^n=g(x)$ for $|x|<k$,

then for any real numbers p & q,

$$\sum(pa_n+qb_n)=pf(x)+qg(x), \text{ for } |x|<k.$$

Example: Check Linearity Thm for

$1+x+x^2+x^3+\cdots$

and $1-x+x^2-x^3+\cdots$

$$1+x+x^2+x^3+\cdots+x^n = \frac{1-x^{n+1}}{1-x} \to \frac{1}{1-x}, \quad n\to\infty, \; |x|<1$$

$$\Rightarrow \quad 1+x+x^2+x^3+\cdots = \frac{1}{1-x}, \quad |x|<1 \qquad ①$$

$$1-x+x^2+\cdots+(-x)^n = \frac{1-(-x)^{n+1}}{1-(-x)} \to \frac{1}{1+x}, \quad |x|<1$$

$$\Rightarrow \quad 1-x+x^2-x^3+\cdots = \frac{1}{1+x} \qquad ②$$

Now add two series,

$$2(1+x^2+x^4+\cdots) = \frac{1}{1-x}+\frac{1}{1+x} = \frac{2}{1-x^2}$$

$$\sum_{n=0}^{\infty} 2\cdot(x^2)^n$$

$$2(1+x^2+x^4+\cdots+[x^2]^n) = 2\cdot\frac{1-(x^2)^{n+1}}{1-x^2} \to \frac{2}{1-x^2}, \quad |x|<1.$$

$$(a_0+a_1x+a_2x^2)(b_0+b_1x+b_2x^2) = a_0b_0+(a_0b_1+a_1b_0)x + (a_0b_2+a_1b_1+a_2b_0)x^2 + (*)x^3 + (*)\cdot x^4$$

Theorem (Multiplication - power series)

If $\sum a_n x^n = f(x)$ and $\sum b_n x^n = g(x)$ for $|x|<K$,

then $\sum c_n x^n = f(x)g(x)$, $|x|<K$.

where $c_n = a_0b_n + a_1b_{n-1}+\cdots+a_nb_0 = \sum_{i+j=n} a_ib_j$

$$(\sum a_n x^n)(\sum b_n x^n) = f(x)g(x)$$

$$\sum c_n x^n = f(x)g(x)$$

Theorem (Multiplication of infinite series)

If $\sum a_n$ and $\sum b_n$ converge absolutely to the sums A and B, then $\sum c_n$ converges absolutely to the sum AB

$(\sum a_n)(\sum b_n) = A \cdot B$

$\sum c_n = A \cdot B$

where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum\limits_{i+j=n} a_i b_j$

Proof: ① Assume $a_n \geq 0$ and $b_n \geq 0$, $\forall n$.

$A_n = a_0 + a_1 + \cdots + a_n$

$B_n = b_0 + b_1 + \cdots + b_n$ $\{B_{2n}\}$ is a subsequence of $\{B_n\}$

$C_n = c_0 + c_1 + \cdots + c_n$

$A_{2n} \cdot B_{2n}$

$$\begin{array}{ccccccc}
a_{2n}b_0 & & & \cdots & & & a_{2n}b_{2n} \\
 & a_{2n-1}b_1 & & & & & \vdots \\
\vdots & & \ddots & & & & \vdots \\
a_nb_0 & a_nb_1 & \cdots & a_nb_n & & & \vdots \\
\vdots & \vdots & & \vdots & \ddots & & \vdots \\
a_1b_0 & a_1b_1 & \cdots & a_1b_n & & \ddots & \vdots \\
a_0b_0 & a_0b_1 & \cdots & a_0b_n & \cdots & & a_0b_{2n}
\end{array}$$

$C_{2n} = c_0 + c_1 + \cdots + c_{2n}$ $\quad c_{2n} = a_0 b_{2n} + a_1 b_{2n-1} + \cdots + a_{2n} b_0$

small square = all $a_i b_j$ in $A_n B_n$

lower triangle = all $a_i b_j$ in C_{2n}

big square = all $a_i b_j$ in $A_{2n} B_{2n}$

all numbers are non-negative

$$\Rightarrow \quad \underset{\substack{\downarrow \\ AB}}{A_n B_n} \leq C_{2n} \leq \underset{\substack{\downarrow \\ AB}}{A_{2n} B_{2n}}$$

Squeeze Thm $\Rightarrow$ $C_{2n} \to AB$

$(C_n \geq 0 \Rightarrow C_n \nearrow)$ $\Rightarrow C_n \to AB$

$\Rightarrow \sum C_n = AB$

$\underset{\substack{\downarrow \\ AB}}{C_{2k}} \leq C_{2k+1} \leq \underset{\substack{\downarrow \\ AB}}{C_{2k+2}}$

$\forall \varepsilon > 0$, $|C_{2k} - L| < \varepsilon$, $k > N_1$

$|C_{2k+1} - L| < \varepsilon$, $k > N_2$

② $a^+ = \begin{cases} |a| & , \text{if } a \geq 0 \\ 0 & , \text{if } a < 0 \end{cases}$

$a^- = \begin{cases} 0 & , \text{if } a \geq 0 \\ |a| & , \text{if } a < 0 \end{cases}$

$$\begin{cases} a^+ + a^- = |a| \\ a^+ - a^- = a \end{cases}$$

$$\Leftrightarrow \begin{cases} a^+ = \dfrac{|a| + a}{2} \\ a^- = \dfrac{|a| - a}{2} \end{cases}$$

Absolute convergence of $\sum a_n \Rightarrow \sum |a_n|$ converges

$$\left.\begin{array}{l}\text{Linearity Thm} \\ a^+ = \frac{1}{2}|a| + \frac{1}{2}a\end{array}\right\} \Rightarrow \sum a^+ \text{ converges}$$

Similarly, $\sum a^-$ converges

$a_n = a_n^+ - a_n^-$	$b_n = b_n^+ - b_n^-$
$\sum a_n = \sum a_n^+ - \sum a_n^-$	$\sum b_n = \sum b_n^+ - \sum b_n^-$
$A = A^+ - A^-$	$B = B^+ - B^-$

$$c_n = \sum_{i+j=n} a_i b_j$$

$$= \sum_{i+j=n} (a_i^+ - a_i^-)(b_j^+ - b_j^-)$$

$$= \sum_{i+j=n} (a_i^+ b_j^+ + a_i^- b_j^-) - \sum_{i+j=n} (a_i^- b_j^+ + a_i^+ b_j^-)$$

$$= d_n - e_n$$

$$\left(\sum_{n=0}^{\infty} a_n^+\right)\left(\sum_{n=0}^{\infty} b_n^+\right) = A^+B^+$$

Step ①

$$\sum_{n=0}^{\infty}\left(\sum_{i+j=n} a_i^+ b_j^+\right) = A^+B^+$$

$$d_n = \sum_{i+j=n} a_i^+ b_j^+ + \sum_{i+j=n} a_i^- b_j^- \Rightarrow \sum d_n = A^+B^+ + A^-B^-$$

Similarly, $\sum e_n = A^-B^+ + A^+B^-$

$$|c_n| = |d_n - e_n| \leq |d_n| + |e_n|$$

$$= d_n + e_n$$

$\Rightarrow \sum c_n$ converges absolutely.

$$c_n = d_n - e_n$$

$$\Rightarrow \sum c_n = \sum d_n - \sum e_n$$

$$= A^+ B^+ + A^- B^- - A^- B^+ - A^+ B^-$$

$$= (A^+ - A^-)(B^+ - B^-)$$

$$= A \cdot B.$$