**Theorem**

$$\lim_{x\to x_0} f(x) = L \iff \begin{cases} \lim_{x\to x_0^+} f(x) = L \\ \lim_{x\to x_0^-} f(x) = L \end{cases}$$

Proof: $\lim_{x\to x_0} f(x)=L$: $\forall \varepsilon>0, \exists \delta, \forall x\in (x_0-\delta, x_0)\cup(x_0, x_0+\delta), |f(x)-L|<\varepsilon$

$\lim_{x\to x_0^+} f(x)=L$: $\forall \varepsilon>0, \exists \delta, \forall x\in (x_0, x_0+\delta), |f(x)-L|<\varepsilon$

$\lim_{x\to x_0^-} f(x)=L$: $\forall \varepsilon>0, \exists \delta, \forall x\in (x_0-\delta, x_0), |f(x)-L|<\varepsilon$

Example: ① $\lim_{x\to 0} x\sin\frac{1}{x} = 0$

$$\left.\begin{matrix} |x-0|<\delta \\ \delta \le \varepsilon \end{matrix}\right\} \Rightarrow$$

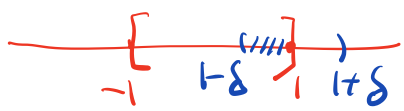
Proof: $\forall \varepsilon>0$, $\left|x\sin\frac{1}{x} - 0\right| = |x|\cdot\left|\sin\frac{1}{x}\right| \le |x| < \varepsilon$

for any $|x|<\varepsilon$, $x\neq 0$

So $\forall \varepsilon>0$, $\exists \delta=\varepsilon$, s.t. $\forall x\in(-\delta,0)\cup(0,\delta)$,

$\left|x\sin\frac{1}{x} - 0\right| < \varepsilon$.

② $\lim_{x\to 1^-} \sqrt{1-x^2} = 0$




Proof: The function $\sqrt{1-x^2}$ is defined for $x\in[-1,1]$.

$\forall \varepsilon>0$, $\left|\sqrt{1-x^2} - 0\right| = \sqrt{1+x}\sqrt{1-x}$

$< \sqrt{2}\sqrt{1-x}$ (for $x<1$)

$< \varepsilon$ (if $1-x<\delta\le\frac{\varepsilon^2}{2}$)

So $\forall \varepsilon>0$, $\exists \delta = \frac{\varepsilon^2}{2}$, $\forall x\in(1-\delta, 1)$

$\left|\sqrt{1-x^2} - 0\right| < \sqrt{2}\sqrt{1-x} < \sqrt{2}\sqrt{\delta} = \varepsilon$.

③ $\lim_{x\to -2} \frac{|x^2-4|}{x+2}$ DNE

$$f(x) = \frac{|x^2-4|}{x+2} = \frac{|x+2|\cdot|x-2|}{x+2} = \begin{cases} |x-2|, & \text{if } x+2>0 \\ -|x-2|, & \text{if } x+2<0 \end{cases}$$

$$\Rightarrow \begin{cases} \lim_{x\to -2^+} f(x) = 4 \\ \lim_{x\to -2^-} f(x) = -4 \end{cases}$$

$$\Rightarrow \lim_{x\to -2} f(x) \text{ DNE}$$

④ Show $\lim_{x\to +\infty} \frac{1}{1+x^2} = 0$ by definition

$$\frac{1}{1+x^2} < \varepsilon \Leftrightarrow 1+x^2 > \frac{1}{\varepsilon} \Leftarrow x > \frac{1}{\sqrt{\varepsilon}}$$

So $\forall \varepsilon > 0$, $\exists M = \frac{1}{\sqrt{\varepsilon}}$, $\forall x > M$, $\frac{1}{1+x^2} < \varepsilon$.

We can also show it by Product/Quotient/Linearity Thm

⑤ Show $\lim_{x\to 1} x^{\frac{1}{n}} = 1$ where $n>0$ is a natural number

Solution : 1) $\lim_{x\to 1+} x^{\frac{1}{n}} = 1$ because

$$x>1 \Rightarrow \quad 1 < x^{\frac{1}{n}} < x$$

Squeeze Thm $\Rightarrow x^{\frac{1}{n}} \to 1$ as $x \to 1+$

2) $\lim_{x\to 1-} x^{\frac{1}{n}} = 1$ because

$0 < x < 1 \Rightarrow \quad x < x^{\frac{1}{n}} < 1$

Squeeze Thm $\Rightarrow x^{\frac{1}{n}} \to 1$ as $x \to 1+$

3) So $\lim\limits_{x \to 1} x^{\frac{1}{n}} = 1$

⑥ $f(x) = \int_1^x \frac{\sqrt{1+t}}{t}\, dt \to +\infty$ as $x \to +\infty$

$\lim\limits_{x \to +\infty} f(x) = +\infty$

$\frac{\sqrt{1+t}}{t} > \frac{\sqrt{t}}{t} = \frac{1}{\sqrt{t}}$ $\quad \forall b > 0, \exists M$ s.t. $\forall x > M$

$f(x) > b$.

$f(x) > \int_1^x \frac{1}{\sqrt{t}} dt = 2\sqrt{t}\Big|_1^x = 2\sqrt{x} - 2 \to +\infty$, as $x \to +\infty$

Squeeze Thm $\Rightarrow f(x) \to +\infty$.

⑦ $f(x) = \int_0^x \frac{dt}{\sqrt{1-t^4}}$. Estimate $\lim\limits_{x \to 1-} f(x)$ from above.

Sol: $x \to 1-$ means $x < 1$

So $0 \leq t < x < 1$

$\Rightarrow \quad t^4 \leq t^2$

$\Rightarrow \quad \sqrt{1-t^4} \geq \sqrt{1-t^2}$

$\Rightarrow \quad \frac{1}{\sqrt{1-t^4}} \leq \frac{1}{\sqrt{1-t^2}}$

$\Rightarrow f(x) \leq \int_0^x \frac{1}{\sqrt{1-t^2}} dt = \sin^{-1} x < \frac{\pi}{2}$, for $0 < x < 1$.

$\Rightarrow \lim\limits_{x \to 1-} f(x) \leq \frac{\pi}{2}$ by Limit Location Thm

Question: can we have $\lim_{x\to 1^-} f(x) < \frac{\pi}{2}$. No

Theorem $f(x)$ is continuous at $x_0$

$$\iff \lim_{x\to x_0} f(x) = f(x_0)$$

Theorem $f(x)$ is continuous at $x_0$, $f(x_0)>0$

$$\Rightarrow f(x)>0 \text{ for } x \approx x_0$$

Proof: Want to show "$\exists \delta>0$, s.t. $\forall x \in (x_0-\delta, x_0+\delta)$, $f(x)>0$".

The assumption is

"$\forall \varepsilon>0$, $\exists \delta>0$ s.t. $\forall x \in (x_0-\delta, x_0+\delta)$, $|f(x)-f(x_0)|<\varepsilon$".

and "$f(x_0)>0$".

$\varepsilon = \frac{f(x_0)}{2}$ $\Rightarrow$ $0 < f(x_0)-\varepsilon < f(x) < f(x_0)+\varepsilon$

So for $\varepsilon = \frac{f(x_0)}{2}$, $\exists \delta_\varepsilon$ s.t.

$\forall x \in (x_0-\delta_\varepsilon, x_0+\delta_\varepsilon)$

$$f(x) > f(x_0)-\varepsilon = \frac{f(x_0)}{2} > 0.$$

Theorem $f(x)$ and $g(x)$ are cont. at $x_0$

$\Rightarrow a f(x) + b g(x)$, $f(x)\cdot g(x)$, $\frac{f(x)}{g(x)}$ are cont. at $x_0$

need to assume $g(x) \neq 0$.

Discontinuity (Four kinds)

① Removable $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

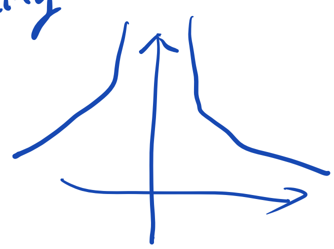
② Jump discontinuity (Two different one-sided limits)

$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

③ Infinite discontinuity

1) $\frac{1}{|x|}$

2) $\frac{1}{x}$



④ Essential (Any other one)

$\sin(\frac{1}{x})$ at $x_0 = 0$

Theorem $f(x)$ and $g(x)$ are continuous

$\Rightarrow f(g(x))$ is continuous.

Theorem $\left.\begin{array}{l} \{a_n\} \to a \\ f(x) \text{ is continuous at } a \end{array}\right\} \Rightarrow \{f(a_n)\} \to f(a)$

Proof: Want to show "$\forall \varepsilon > 0$, $|f(a_n) - f(a)| < \varepsilon$, $n >> 1$"

$f(x)$ is cont. at $a$

$\Rightarrow \forall \varepsilon > 0$, $\exists \delta$ s.t. $\forall x \in (a-\delta, a+\delta)$,
$|f(x) - f(a)| < \varepsilon$.

For $n \gg 1$, $|a_n - a| < \delta$ because $a_n \to a$

so $a_n \in (a-\delta, a+\delta)$, $n \gg 1$

thus $|f(a_n) - f(a)| < \varepsilon$, $n \gg 1$.

Theorem $\left.\begin{array}{l} \{a_n\} \to a \quad a_n \neq a \\ \lim_{x \to a} f(x) = L \end{array}\right\} \Rightarrow f(a_n) \to L$